Math 321 Algebra Practice Final

There will be no questions about the fundamental group on your final.

1. Determine if the following are true or false. No justification is necessary!

(a) Every nontrivial subgroup of an infinite group is infinite.

(b) Every group of order 2 or more has a nontrivial cyclic subgroup.

(c) If n divides the order of G, then G contains an element of order n.

(d) $\mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_6 \simeq \mathbb{Z}_{10} \times \mathbb{Z}_{12}$

(e) If H and K are subgroups of a group G, then $H \cup K$ is a subgroup of G. (Note: $H \cup K = \{x | x \in H \text{ or } x \in K\}$.)

2. Determine if the following are true or false. No justification is necessary!

(a) If x is relatively prime to 9, then $x^8 \equiv 1 \pmod 9$.

(b) If R is a ring and $f(x), g(x) \in R[x]$, then $\deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x))$.

(c) If F is a field and $f(x), g(x) \in F[x]$, then $\deg(f(x) + g(x)) = \max\{\deg(f(x)), \deg(g(x))\}$.

3. Give an example of each of the following. No justification is necessary!

(a) A finite field of order greater than 100.

(b) A subring of a ring that is not an ideal.

(c) A nonabelian group with 14 elements.

(d) An ideal of $Z[x]$ that is not prime.

4. Prove or disprove: Complex conjugation is a field isomorphism of $\mathbb{C}$.

5. Let M_n be the ring of $n \times n$ matrices, and $f : M_n \to M_n$ transposition (that is, $f(A) = A^T$, the transpose of A). Is f a group homomorphism? Is it a ring homomorphism?

6. Consider $\mathbb{R}^3$ with the cross product $\times$. Why is $(\mathbb{R}^3, \times)$ not a group?

7. Show that $\ln : (\mathbb{R}^+, \times) \to (\mathbb{R}, +)$ is a group homomorphism.

8. Let V be an F-vector space. Show that the set $S = \{v_1, \ldots, v_n\}$ is a linearly independent spanning set (that is, a basis) for V if and only if each element of V can be written uniquely as a linear combination of elements of S with coefficients in F.

9. A permutation $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a rigid motion (or isometry) if it preserves distances, that is, if

$$d((x_1, y_1), (x_2, y_2)) = d(f(x_1, y_1), f(x_2, y_2))$$

for all (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$. Prove or disprove: The set of all rigid motions is a group under function composition.

10. (a) Let H be a subgroup of G. Show that the induced operation on the set of cosets G/H is well-defined if H is normal.

(b) Let I be a subring of R. Show that the induced multiplication on the set of cosets R/I is well-defined if I is an ideal.

11. Find the degree and a basis for $\mathbb{Q}(\sqrt[3]{7}, \sqrt{2})$ over $\mathbb{Q}$.

12. Explain why there is no field with exactly six elements.