Math 105 **FINAL EXAM** **May, 2004**

Due in my office 24 hours after you download it, and by 9:00 am, Sunday, May 16, at the latest. You may use only your text, notes, and old homework. You may not talk to anyone about it, or use any other references. I will not answer any questions other than requests for clarification. Make sure that your answers are clear and legible.

1. Randomized harmonic series We know that the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ diverges, and that the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \ldots$ converges. What if we randomize the series, so that each individual term is positive with probability 1/2 and negative with probability 1/2? More formally, let $\{X_n\}$ be a sequence of independent random variables, with $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2$, and consider the series

$$X_1 + \frac{X_2}{2} + \frac{X_3}{3} + \ldots.$$

Show that this series converges almost surely.

2. Independent random variables X, Y, and Z take integer values $1, 2, \ldots, n$ with equal probabilities $1/n$. Find the following:

(a) $\mathbb{P}(X + Y = Z)$

(b) $\mathbb{P}(X + Y + Z = n + 1)$

3. Customers arrive in a shop in the manner of a Poisson process with parameter λ. There are infinitely many servers, and each service time is exponentially distributed with parameter μ. Show that the number $Q(t)$ of waiting customers at time t constitutes a birth-death process, and find its stationary distribution.

4. Let W be a standard Wiener process. Show that

$$\mathbb{P}\left(\sup_{0 \le s \le t} |W(s)| \ge w\right) \le 2\mathbb{P}(|W(t)| \ge w) \le \frac{2t}{w^2} \quad \text{for } w > 0.$$

5. Let X_0, X_1, X_2, $\ldots$ be a Markov chain with the set of states $S = \{1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1-\alpha & \alpha \\ 1-\beta & 0 & \beta \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Define Y_n by

$$Y_n = \begin{cases} 1 & \text{if either } X_n = 1 \text{ or } X_n = 2, \\ 2 & \text{if } X_n = 3. \end{cases}$$

Under what conditions is the sequence Y_0, Y_1, Y_2, $\ldots$ a Markov chain?