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(1) The sequence $\{S_n = X_1 + \frac{X_2}{2} + \dots + \frac{X_n}{n}\}$ is clearly a martingale. To apply the martingale convergence theorem, we must show that $E(S_n^2) < M$ for some M & all n. We observe that

$$E(S_n^2) = E\left(\left(X_1 + \dots + \frac{X_n}{n}\right)^2\right)$$

$$= E\left(X_1^2 + \frac{X_2^2}{2^2} + \dots + \frac{X_n^2}{n^2} + \sum_{i \neq j} \frac{X_i X_j}{ij}\right)$$

$$= 1 + \frac{1}{2^2} + \dots + \frac{1}{n^2} \quad (\text{since } E(X_i^2) = 1$$

$$\& \ E(X_i X_j) = E(X_i)E(X_j) = 0)$$

$$\leq \sum_{i=1}^{\infty} \frac{1}{i^2} = \pi^2/6$$

Thus $\{S_n\}$ converges a.s. and in mean square to a r.v. S. S is not defective (since its L^2 norm exists), so $\mathbb{P}(S = \infty) = 0$, and we're done.

(2a) We'd like to look at $X + Y - Z$, but that could be negative (and so not have a gen. fn.), so we'll look at $X + Y + (n+1-Z)$ instead. Note that $X + Y = Z \iff X + Y + (n+1-Z) = n+1$, and that $n+1-Z$ has the same distrib. as X. Thus

$$G_{X+Y+(n+1-Z)} = G_X \cdot G_Y \cdot G_{(n+1-Z)} = G_X^3$$

$$= \left(\frac{1}{n}\left(s + s^2 + \dots + s^n\right)\right)^3 = \left(\frac{1}{n}\,\frac{s - s^{n+1}}{1-s}\right)^3 = \frac{s^3}{n^3} \cdot (1-s^n)^3 \cdot (1-s)^{-3}$$

We want the coeff. of s^{n+1}, so we apply the binomial theorem twice to get

$$\frac{s^3}{n^3}(1-s^n)^3(1-s)^{-3} = \frac{s^3}{n^3}\left(1 - 3s^n + 3s^{2n} - s^{3n}\right)\left(1 + \binom{3}{1}s + \binom{4}{2}s^2 + \dots \binom{k+2}{2}s^k + \dots\right)$$

So the coeff. of s^{n+1} is $\frac{1}{n^3}\binom{n}{n-2} = \frac{n(n-1)}{n^3 \cdot 2} = \frac{n-1}{2n^2}$

(2b) Since Z and $n+1-Z$ have the same distribution, this is exactly the same as part a).

NOTE: This problem is essentially the same as #5.12.1.

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(3) From the defn. of Poisson process, we have that

$$\mathbb{P}(Q(t+h)=j \mid Q(t)=i) = \begin{cases} \lambda h + o(h) & \text{if } j=i+1 \\ \mu i h + o(h) & \text{if } j=i-1 \\ 1-(\lambda+\mu i)h + o(h) & \text{if } j \neq i+1, i-1 \end{cases}$$

so, by the defn. of birth-death process (p. 268), this is one, with birth rate λ and death rates $\mu_i = i\mu$.

To find the stationary distrib. $(\pi_0, \pi_1, \pi_2, \ldots)$, use formula 2 on p. 265 to find that $\pi_n = \frac{\lambda^n}{n!\mu^n}\pi_0$. We know that $\pi_0 + \pi_1 + \cdots = 1$, so

$$1 = \pi_0 + \frac{1}{1!}\left(\frac{\lambda}{\mu}\right)\pi_0 + \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2\pi_0 + \cdots = \pi_0\left(1 + \frac{\lambda}{\mu} + \frac{\left(\frac{\lambda}{\mu}\right)^2}{2!} + \frac{\left(\frac{\lambda}{\mu}\right)^3}{3!} + \cdots\right)$$

$$= \pi_0 e^{\lambda/\mu}$$

So $\pi_0 = \frac{1}{e^{\lambda/\mu}} = e^{-\lambda/\mu}$, and $\pi_n = \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} e^{-\lambda/\mu}$.

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(4) We know (Th. 13.4.6) that $M(t) := \max\{W(s) : 0 \le s \le t\}$ has the same distribution as $|W(t)|$. We have

$$\mathbb{P}\left(\sup_{0\le s\le t} |W(s)| \ge w\right) \le \mathbb{P}(M(t) \ge w) + \mathbb{P}(\min\{W(s) : 0 \le s \le t\} \le -w)$$

$$= 2\,\mathbb{P}(M(t) \ge w) \quad \text{by symmetry}$$

$$= 2\,\mathbb{P}(|W(t)| \ge w)$$

Chebyshov's inequality gives $\mathbb{P}(|W(t)| \ge w) \le \frac{\mathbb{E}(|W(t)|^2)}{w^2} = \frac{t}{w^2}$.

(5) Intuitively, the original states 1 & 2 must be indistinguishable, ie, $\alpha = \beta$. More formally:

If $\alpha = \beta$, then $Y_0, Y_1, \ldots$ is clearly a Markov chain with transition matrix

$$\begin{pmatrix} 1-\alpha & \alpha \\ 2/3 & 1/3 \end{pmatrix}.$$

Conversely, assume that $Y_0, Y_1, \ldots$ etc. is a Markov chain, Then it must be the case that

$$\mathbb{P}(Y_n = 2 \mid Y_{n-2} = 1, Y_{n-1} = 1) = \mathbb{P}(Y_n = 2 \mid Y_{n-1} = 1)$$

We have that $\mathbb{P}(Y_n = 2 \mid Y_{n-1} = 1) = \frac{1}{2}(\alpha) + \frac{1}{2}(\beta) = \frac{\alpha+\beta}{2}$

Similarly, $\mathbb{P}(Y_n = 2 \mid Y_{n-2} = 1, Y_{n-1} = 1) = \dfrac{\mathbb{P}(Y_n = 2, Y_{n-1} = 1 \mid Y_{n-2} = 1)}{\mathbb{P}(Y_{n-1} = 1 \mid Y_{n-2} = 1)}$

$$= \frac{\frac{1}{2}(1-\alpha)\beta + \frac{1}{2}(1-\beta)\alpha}{\frac{1}{2}(1-\alpha) + \frac{1}{2}(1-\beta)} = \frac{\frac{\alpha+\beta}{2} - \alpha\beta}{1 - \frac{\alpha+\beta}{2}}$$

(unless $\alpha = \beta = 1$, in which case we're done)

So $\dfrac{\alpha+\beta}{2} = \dfrac{\frac{\alpha+\beta}{2} - \alpha\beta}{1 - \frac{\alpha+\beta}{2}}$. Solve to find that $\alpha = \beta$.