Math 105 **MIDTERM** **March, 2004**

Due in my office 24 hours after you download it, and by 5:00 pm on Sunday 3/28 at the latest. You may use only your text, notes, and old homework. You may not talk to anyone about it, or use any other references. I will not answer any questions other than requests for clarification. Make sure that your answers are clear and legible.

Please note: I will have my usual office hours TW 2-3, but I will be away from Thursday through Sunday.

1. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$. What is the smallest σ-algebra that contains the sets $\{2, 3, 4\}$ and $\{4, 6\}$?

2. Suppose that $X_n \xrightarrow{P} X$, $Y_n \xrightarrow{P} X$, and $\mathbb{P}(X_n \leq Z_n \leq Y_n) = 1$ for all n. Prove that $Z_n \xrightarrow{P} X$.

3. Let Z_n be the number of members of the nth generation of a branching process, where the family sizes of the individuals are independent and distributed identically to the random variable X. Assume that $\mathbb{E}(X) = \mu < 1$. Let $H = Z_0 + Z_1 + \dots$ be the *total* number of individuals who ever live in this process.

(a) Find $\mathbb{E}(H)$ if $Z_0 = 1$.
(b) Find $\mathbb{E}(H)$ if $Z_0 = k$.

4. Consider a Markov chain on the set $S = \{0, 1, 2, \dots\}$ with transition probabilities $p_{i,i+1} = a_i$, $p_{i,0} = 1 - a_i$, $i \geq 0$, where $(a_i : i \geq 0)$ is a sequence of constants which satisfy $0 < a_i < 1$ for all i. Let $b_0 = 1$, $b_i = a_0 a_1 \cdots a_{i-1}$ for $i \geq 1$.

(a) Show that the chain is persistent if and only if $b_i \to 0$ as $i \to \infty$.
(b) Show that the chain is non-null persistent if and only if $\sum_i b_i < \infty$, and find the stationary distribution if this condition holds.

5. **Polya's urn** An urn initially contains one black and one red ball. At each stage, a ball is drawn at random from the urn, its color is noted, and then it is returned to the urn together with a new ball of the same color. Let $X_n = 0$ if the nth drawing results in a black ball, and $X_n = 1$ if it results in a red ball. Show that $\{X_1, X_2, \dots\}$ is *not* a Markov process.

6. A woman continues to have children until she has a boy. Suppose that the probability of having a blond child is p; find the probability that the woman will have k blond children. (Assume that hair color is independent of gender.)