Math 105 Midterm Solns

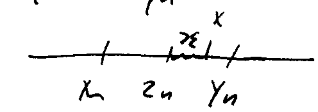
① A σ-algebra (or σ-field) contains ϕ & Ω and is closed under countable unions, countable intersections, and complementation. So take the sets $\{2,3,4\}$ and $\{4,6\}$ and just keep taking unions, intersections, and complements until you quit getting new sets. You end up with

$$\{\phi, \Omega, \{4\}, \{6\}, \{1,5\}, \{2,3\}, \{4,6\}, \{1,4,5\}, \{1,5,6\}, \{2,3,4\}, \{2,3,6\}, \{1,2,3,5\}, \{1,4,5,6\}, \{2,3,4,6\}, \{1,2,3,4,5\}, \{1,2,3,5,6\}\}$$

② $$\mathbb{P}(|Z_n - X| > \varepsilon) = \left(\mathbb{P}(|Z_n - X| > \varepsilon \mid X_n \le Z_n \le Y_n)\right) \cdot \underbrace{\mathbb{P}(X_n \le Z_n \le Y_n)}_{=1} + \mathbb{P}\left(|Z_n - X| > \varepsilon \,\middle|\, \begin{matrix} Z_n < X_n \\ \text{or } Z_n > Y_n \end{matrix}\right) \cdot \underbrace{\mathbb{P}\left(\begin{matrix} Z_n < X_n \\ \text{or } Z_n > Y_n \end{matrix}\right)}_{=0}$$

$$= \mathbb{P}(|Z_n - X| > \varepsilon \mid X_n \le Z_n \le Y_n)$$

$$\le \mathbb{P}(|X_n - X| > \varepsilon) + \mathbb{P}(|Y_n - X| > \varepsilon)$$ (since if X is more than ε from Z_n, it must be more than ε from at least one of X_n & Y_n



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Since $X_n \xrightarrow{P} X$ & $Y_n \xrightarrow{P} X$, both of these terms go to 0 as n goes to infinity, so the same is true of $\mathbb{P}(|Z_n - X| > \varepsilon)$, ie, $Z_n \xrightarrow{P} X$.

③ a) On average, the first individual will have μ offspring, each of whom will have μ offspring of their own. Thus

$$\mathbb{E}(H) = \mathbb{E}(Z_0 + Z_1 + Z_2 + \dots) = \mathbb{E}(Z_0) + \mathbb{E}(Z_1) + \dots = 1 + \mu + \mu^2 + \mu^3 + \dots$$
$$= \frac{1}{1-\mu}$$

b) If $Z_0 = k$, we have k independent copies of the process above, so

$$\mathbb{E}(H) = \frac{k}{1-\mu}.$$

(4) a) The chain is clearly transitive, so it's persistent iff any state is persistent. Look at state 0, and let T be the time of first return to 0, starting from 0. Then 0 is persistent iff $\mathbb{P}(T>n)$ goes to 0 as n goes to ∞. Now, $T>n$ iff at the n^{th} step, the chain is in state n. Thus $\mathbb{P}(T>n) = a_0 a_1 \cdots a_{n-1} = b_n$ $(n \geq 1)$. So the chain is persistent iff $b_n \to 0$.

b) Again, the chain is non-null persistent iff 0 is. The expected time till first return to 0 is $\cancel{\sum_{n=1}^{\infty} n\mathbb{P}(}$

$$\sum_{n=1}^{\infty} n\,\mathbb{P}(T=n) = \sum_{n=0}^{\infty} \mathbb{P}(T>n) = \sum_{n=0}^{\infty} b_n.$$

So the chain is non-null persistent iff $\sum_{n=0}^{\infty} b_n < \infty$

Let $\vec{\pi} = (\pi_0, \pi_1, \ldots)$ be the stationary distribution.

Then $\pi_0 = \sum_{n=0}^{\infty} \pi_n (1-a_n)$ & $\pi_n = \pi_{n-1} a_{n-1}$ (for $n \geq 1$).

So $\pi_1 = a_0 \pi_0 = b_1 \pi_0$, $\pi_2 = a_1 \pi_1 = a_0 a_1 \pi_0 = b_2 \pi_0$, $\ldots$, $\underline{\pi_n = b_n \pi_0}$.

Since $\vec{\pi}$ is a distribution, we have that
$$1 = \pi_0 + \pi_1 + \pi_2 = b_0\pi_0 + b_1\pi_0 + b_2\pi_0 + \cdots = \pi_0(b_0 + b_1 + b_2 + \cdots)$$

So $\pi_0 = \dfrac{1}{\sum_{n=0}^{\infty} b_n}$.

(5) $\mathbb{P}(X_3 = 0 \mid X_1 = 0, X_2 = 0) = 3/4$, but
$\mathbb{P}(X_3 = 0 \mid X_1 = 1, X_2 = 0) = 1/2$. Thus the transition probabilities depend on more than just the current state, so it's not a Markov chain.

(6) Let N be the total # of children; then N is geometric with parameter $1/2$. The number of blond children is $S = X_1 + \dots + X_N$, where the X_i's are independent Bernoulli variables with parameter p. The generating fn. for S is $G_S(t) = G_N(G_X(t)) = G_N(1-p+pt)$

$$= \frac{\frac{1}{2}(1-p+pt)}{1-(1-p+pt)(1-\frac{1}{2})},$$

which simplifies to $\frac{a+bt}{1-bt}$, where $a = \frac{q}{2-q}$, $b = \frac{p}{2-q}$ $(q = 1-p)$.

$$\frac{a+bt}{1-bt} = (a+bt)(1+bt+b^2t^2+\dots)$$

Then p_n, the probability of having n blond children, is the coefficient of t^n in the above expression. We read off:

$$p_0 = a = \frac{a}{2-q} = \frac{1-p}{1+p}, \text{ and}$$

$$p_n = b^n(1+a) = \frac{2p^n}{(1+p)^{n+1}}.$$