1. Let A be a 2×2 matrix with eigenvalues 1 and 2, and B a 2×2 matrix with eigenvalues 0 and -7. Compute the eigenvalues of the following matrices, or prove that you don't have enough information.

(a) AB

Not enough info. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & 0 \\ 0 & -7 \end{bmatrix}$, then AB has evals 0 & -14.

If $B = \begin{bmatrix} -7 & 0 \\ 0 & 0 \end{bmatrix}$, then AB has evals -7 & 0.

(b) A^T $1, 2$

(c) A^{-1} $\frac{1}{1} = 1$ & $\frac{1}{2}$

(d) $A + B$ Not enough info. Same example as (a).

(e) A^2 $1^2 = 1$ & $2^2 = 4$

(f) $I_2 + A$ $1+1 = 2$ & $1+2 = 3$

(If $A\vec{x} = \lambda\vec{x}$, then $(I_2 + A)\vec{x} = \vec{x} + \lambda\vec{x} = (\lambda+1)\vec{x}$)

2. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Show that the product $\langle \vec{x}, \vec{y} \rangle = (A\vec{x}) \cdot (A\vec{y})$ (where $\cdot$ is the usual dot product) is *not* an inner product on $\mathbb{R}^2$.

The problem is that A is singular.

$$< \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} > = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{0} \cdot \vec{0} = 0.$$

But the inner product of a nonzero vector with itself can't be 0.

3. **(a)** Give an example of a vector space V and a linear transformation $T : V \to V$ that is one-one but not onto.

$V = \{(a_1, a_2, a_3, \ldots) \mid a_i \in \mathbb{R}\}$ = all infinite sequences

$T(a_1, a_2, \ldots) = (a_1, 0, a_2, 0, \ldots)$

(There are other answers.)

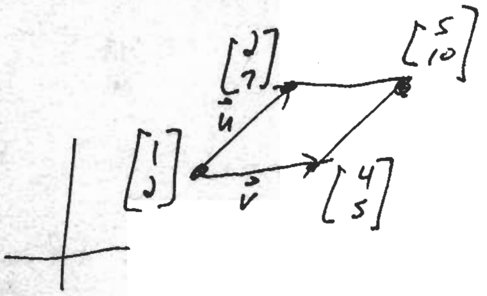
(b) Give an example of a vector space W and a linear transformation $S : W \to W$ that is onto but not one-to-one.

Same V.

$S(a_1, a_2, \ldots) = (a_2, a_3, \ldots)$

(There are other answers.)

4. Find the area of the parallelogram with vertices $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 10 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$.



$$\text{Area} = \left| \det \left[\vec{u} \ \vec{v} \right] \right|$$

$$\vec{u} = \begin{bmatrix} 2 \\ 7 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

So $\text{Area} = \left| \det \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \right| = |3 - 15| = 12.$

5. Let A be a complex $n \times n$ diagonalizable matrix. Show that A has a cube root. (That is, show that there exists a complex $n \times n$ matrix B such that $B^3 = A$.)

$A = SDS^{-1}$, where $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Every complex number has at least one cube root. For each i, let μ_i be a cube root of λ_i (i.e., $\mu_i^3 = \lambda_i$).

Let $M = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix}$. Then $M^3 = D$, and

$$(SMS^{-1})^3 = SM^3S^{-1} = SDS^{-1} = A.$$

So SMS^{-1} is a cube root of A.

6. Let $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$. Because it's nice to be nice, I inform you that the reduced echelon form of A is $\begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find a basis for Im A.

The pivot columns of rref A are the 1st and the third, so the 1st & 3rd columns of A form a basis of the column space of A, which is the same as Im A.

basis: $\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$

(b) Find a basis for Ker A.

Solns. of $A\vec{x} = \vec{0}$ satisfy
$$x_1 - 2x_2 - x_4 + 3x_5 = 0$$
$$x_3 + 2x_4 - 2x_5 = 0,$$

or
$$x_1 = 2x_2 + x_4 - 3x_5$$
$$x_3 = -2x_4 + 2x_5.$$
So solns are of the form

$$\begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}, \text{ or } x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

basis: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

7. As usual, let $\mathcal{M}_{n\times n}$ be the space of $n \times n$ matrices. Define a transformation $T : \mathcal{M}_{n\times n} \to \mathcal{M}_{n\times n}$ by $T(A) = \text{trace}(A)$.

(a) Show that T is linear.

trace $(aA + bB)$ = Sum of diagonal entries of $aA + bB$

= (Sum of diag. entries of aA) + (Sum of diag. entries of bB)

= $a \cdot$ (Sum of diag. entries of A) + $b \cdot$ (Sum of diag. entries of B)

= $a \cdot$ trace (A) + $b \cdot$ trace (B)

(b) What is the dimension of $\ker T$? $T : \mathcal{M}_{n\times n} \to \mathbb{R}$

We know that $\dim(\operatorname{Im} T) + \dim(\ker T) = \dim(\mathcal{M}_{n\times n})$.

T is onto, so $\operatorname{Im} T = \mathbb{R}$ & $\dim(\operatorname{Im} T) = 1$, and $\dim(\mathcal{M}_{n\times n}) = n^2$.

So $1 + \dim(\ker T) = n^2$

$\dim(\ker T) = n^2 - 1$.

(c) Find a basis for $\ker T$ in the case $n = 2$.

We need $2^2 - 1 = 3$ lin. ind. matrices with trace 0.

One answer: $\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

8. Do **one** of the following problems. Make sure that I can tell which one you're doing.

(a) Determine whether the following statement is true or false. If it's true, prove it; if it's false, give a counterexample.
STATEMENT: Let T be a linear transformation. If the set $\{T(\vec{v}_1), \ldots, T(V_n)\}$ is linearly independent, then so is the set $\{\vec{v}_1, \ldots, \vec{v}_n\}$.

(b) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that there exists a unique $m \times n$ matrix M_T such that $T(\vec{x}) = M_T\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

(c) Let A be an $m \times n$ matrix. Show that $\ker A = \ker A^T A$.

(a) True. Say $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$. Then, by linearity, $c_1 T(\vec{v}_1) + \cdots c_n T(\vec{v}_n) = \vec{0}$. Since the $T(\vec{v}_i)$'s are linearly ind., $c_1 = c_2 = \cdots = c_n = 0$.

(b) Take $M_T = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix}$, where the $\vec{e}_i$'s are the standard basis vectors in $\mathbb{R}^n$. Then $M_T\vec{e}_i = T(\vec{e}_i)$ for all i, and M_T is the only matrix with this property. Since any $\vec{x} \in \mathbb{R}^n$ can be written in terms of the $\vec{e}_i$'s, $M_T\vec{x} = T(\vec{x})$, by linearity.

(c)
$$
\begin{aligned}
\vec{x} \in \ker A^T A &\iff A^T A\vec{x} = \vec{0} \\
&\iff A^T(A\vec{x}) = \vec{0} \\
&\iff A\vec{x} \in (\operatorname{Im} A) \cap (\ker A^T).
\end{aligned}
$$
Since $\operatorname{Im} A$ & $\ker A^T$ are orthogonal complements, their intersection is $\{\vec{0}\}$, so this is true
$$
\begin{aligned}
&\iff A\vec{x} = \vec{0} \\
&\iff \vec{x} \in \ker A.
\end{aligned}
$$

9. Find the equation $y = mx + b$ of the line that best fits the data points $(1,2)$, $(2,1)$, $(3,1)$, and $(4,-1)$.

"Solve" $m \cdot 1 + b = 2$, $m \cdot 2 + b = 1$, $m \cdot 3 + b = 1$, $m \cdot 4 + b = -1$, or

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}}_{\vec{c}}$$

The least-squares soln $\vec{x}^*$ satisfies $A^T A \vec{x}^* = A^T \vec{c}$, or

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

Row reduce $\left[\begin{array}{cc|c} 30 & 10 & 3 \\ 10 & 4 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1/3 & 1/10 \\ 10 & 4 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1/3 & 1/10 \\ 0 & 2/3 & 2 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 0 & -9/10 \\ 0 & 1 & 3 \end{array}\right]$. So $m^* = -9/10$, $b^* = 3$, & the line is $y = -9/10\, x + 3$

(EXTRA CREDIT) An $n \times n$ matrix A is called *skew-symmetric* if $A^T = -A$. Show that all the eigenvalues of a skew-symmetric matrix are pure imaginary. (That is, if $a + bi$ is an eigenvalue, then $a = 0$.)

Say $A\vec{x} = \lambda \vec{x}$ ($\vec{x}$ & λ may be complex) ($\vec{x} \neq \vec{0}$).

Then $\langle \vec{x}, A\vec{x} \rangle = \langle \vec{x}, \lambda \vec{x} \rangle = \bar{\lambda} \langle \vec{x}, \vec{x} \rangle$.

Also, $\langle \vec{x}, A\vec{x} \rangle = \vec{x}^T \overline{(A\vec{x})} = \vec{x}^T \bar{A} \bar{\vec{x}} = \vec{x}^T A \bar{\vec{x}}$ (since A is real)

$= \vec{x}^T (-A^T) \bar{\vec{x}}$ (since A is skew-symmetric)

$= -\vec{x}^T A^T \bar{\vec{x}} = -(A\vec{x})^T \bar{\vec{x}} = -(\lambda \vec{x})^T \bar{\vec{x}} = -\lambda \langle \vec{x}, \vec{x} \rangle$.

So $\bar{\lambda} \langle \vec{x}, \vec{x} \rangle = -\lambda \langle \vec{x}, \vec{x} \rangle$. Since $\vec{x} \neq \vec{0}$, $\langle \vec{x}, \vec{x} \rangle \neq 0$, so

$\bar{\lambda} = -\lambda$, which means that λ is pure imaginary.