1. Suppose that div $\vec{F} = 4$ everywhere. Compute the following quantities, or state that you don't have enough information.

(a) $\iint_S \vec{F} \cdot d\vec{S}$, where S is the sphere of radius 2 centered at the origin and oriented outward.

(b) $\int_C \vec{F} \cdot d\vec{s}$, where C is the unit circle in the xy-plane, oriented counterclockwise viewed from above.

(c) $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$, where S is the sphere of radius 2 centered at the origin and oriented outward.

a) Gauss's Th. says that $\iint_S \vec{F} \cdot d\vec{S} = \iiint_R (\operatorname{div} \vec{F})\, dV$, where R is the solid ball of radius 2

$= \iiint_R 4\, dV$

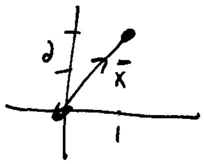
$= 4 \cdot \text{volume}(R) = 4 \cdot \frac{4}{3}\pi 2^3 = \cancel{\frac{64\pi}{3}}\ \frac{128\pi}{3}$

b) Not enough info. You get different answers with $\vec{F} = (-y, x, 4z)$ and $\vec{F} = (0, 0, 4z)$, both of which have $\operatorname{div}\vec{F} = 4$.

c) Stokes' Th. says that $\iint_S (\operatorname{curl} \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$, but $\partial S = \emptyset$, so this is 0.

(OR: Gauss's Th. says that $\iint_S (\operatorname{curl} \vec{F}) \cdot d\vec{S} = \iiint_R \operatorname{div}(\operatorname{curl} \vec{F})\, dV$, where R is as in (a), and $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$.)

2. Let $\vec{F}(x,y) = (y,x)$. Compute the line integral of $\vec{F}$ along the straight line segment from the origin to the point $(1,2)$.



The easiest way to do this is to realize that

$$(y,x) = \text{grad}(xy). \text{ Then } \int_{\vec{x}} \vec{F}\cdot d\vec{s} = xy\big|_{(1,2)} - xy\big|_{(0,0)}$$

$$= 2 - 0 = 2.$$

(You can also do this directly, by parametrizing the line segment.)

3. Let $f(x,y) = \cancel{\frac{x^2-y^2}{x^2+y^2}}$ x^2y^2.

(a) In which direction should you move from the point $(1,1)$ to obtain the maximum rate of increase of f? What is that maximum rate?
(b) Find a direction in which the directional derivative at the point $(1,1)$ is equal to zero.
(c) Suppose that you move along the curve $\vec{x}(t) = (e^{2t}, 2t^3+t+1)$. What is $\frac{d}{dt}f(\vec{x}(t))$ at $t=0$?

a) Move in the direction of the gradient.
$\text{grad} f = (f_x, f_y) = (2xy^2, 2yx^2)$, so
$(\text{grad} f)(1,1) = (2,2)$.
The max rate of change is the magnitude of the gradient,
$\|(2,2)\| = \sqrt{2^2+2^2} = \sqrt{8}$

b) The directional derivative in the direction $\vec{u}$ (unit vector)
is $(\text{grad} f)\cdot\vec{u}$. So we want a $\vec{u}$ that's $\perp$ to $(2,2)$.
Anything of the form $(a,-a)$ will work $(a \neq 0)$.

c) $\frac{d}{dt} f(\vec{x}(t)) = (\text{grad} f)(\vec{x}(t)) \cdot \vec{x}\,'(t)$
$\vec{x}\,'(t) = (2e^{2t}, 6t^2+1)$, so
$\frac{d}{dt} f(\vec{x}(t))$ at $t=0$ is $(\text{grad} f(1,1))\cdot(2,1)$
$= (2,2)\cdot(2,1) = 4+2 = 6.$

(You could also plug in & compute directly, but it's a lot of work.)

4. A bunch of monkeys have \$2000 and want to buy some food. They can get bananas at \$5/pound and doughnuts at \$10/pound. The number M of monkeys who can be fed if they buy x pounds of bananas and y pounds of doughnuts is given by

$$M = x + 2y + \frac{x^2y^2}{2 \cdot 10^8}.$$

What is the maximum number of monkeys that can be fed, and how should they allocate their money?

This is constrained optimization:

maximize $M = x+2y+\frac{x^2y^2}{2\cdot 10^8}$

given the constraint $g(x,y) = 5x+10y = 2000$

Use Lagrange multipliers. grad $g = (5,10)$, which is never $\vec{0}$.

So solve grad $M = \lambda(\text{grad } g)$.

$$\left(1+\frac{2xy^2}{2\cdot 10^8},\ 2+\frac{2x^2y}{2\cdot 10^8}\right) = \lambda(5,10)$$

So $1+\frac{xy^2}{10^8} = 5\lambda$, or $\underline{2+\frac{2xy^2}{10^8} = 10\lambda}$

$$\underline{2+\frac{x^2y}{10^8} = 10\lambda}$$

So $2+\frac{2xy^2}{10^8} = 2+\frac{x^2y}{10^8}$, $\frac{2xy^2}{10^8} = \frac{x^2y}{10^8}$, $2xy^2 = x^2y$

So either $x=0$, or $y=0$, or $\cancel{y=x}$ $2y=x$

Plug into the constraint $5x+10y=2000$.

If $x=0$, $y=200$. If $y=0$, $x=400$. If $2y=x$, then $x=200$, $y=100$.

$M(0,200)=400$ $\quad M(400,0)=400$ $\quad M(200,100) = 200+200+\frac{4\cdot 10^8}{2\cdot 10^8} = 402$.

So they should buy 200 lbs. of bananas & 100 lbs of doughnuts to feed 402 monkeys.

5. People commuting to Swarthmore can choose to go either by bus or by train. The number of people who choose either method depends in part upon the price of each. Let $f(P_B, P_T)$ be the number of people who take the bus when P_B is the price of a bus ride and P_T is the price of a train ride. What can you say about the signs of $\partial f/\partial P_B$ and $\partial f/\partial P_T$? Explain.

$\frac{\partial f}{\partial P_B} < 0$: If the price of a bus ride goes up, the number of bus riders goes down

$\frac{\partial f}{\partial P_T} > 0$: If the train gets more expensive, more people will take the bus.

6. Find an equation for the plane tangent to the graph of the function $f(x, y) = \sqrt{x^2 + y^3}$ at the point $(1, 2, 3)$, and use it to estimate $\sqrt{(1.04)^2 + (1.98)^3}$.

tangent plane: $z = f(\vec{a}) + Df(\vec{a}) \cdot (\vec{x} - \vec{a})$

Here, $\vec{a} = (1, 2)$.

$$Df = (f_x, f_y) = \left(\frac{x}{\sqrt{x^2+y^3}}, \frac{3y^2}{2\sqrt{x^2+y^3}} \right), \text{ so}$$

$$Df(1,2) = \left(\frac{1}{3}, \frac{3 \cdot 4}{2 \cdot 3} \right) = \left(\frac{1}{3}, 2 \right).$$

So the tangent plane is $z = 3 + \left(\frac{1}{3}, 2 \right) \cdot (x-1, y-2)$

$$z = 3 + \frac{1}{3}(x-1) + 2(y-2)$$

Then $\sqrt{(1.04)^2 + (1.98)^3} = f(1.04, 1.98)$

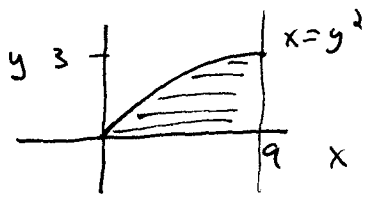
$$\approx 3 + \frac{1}{3}(1.04 - 1) + 2(1.98 - 2)$$

$$= 3 + \frac{1}{3}(0.04) + 2(-0.02)$$

$$= 2.97\overline{3}$$

7. Evaluate $\displaystyle\int_0^3 \int_{y^2}^9 y \sin(x^2)\, dx\, dy$.

Can't antidifferentiate $y \sin x^2$ with respect to x, so try reversing the order of integration



so $0 \le y \le \sqrt{x}$

$0 \le x \le 9$

So $\displaystyle\int_0^3 \int_{y^2}^9 y \sin x^2 \, dx\, dy = \int_0^9 \int_0^{\sqrt{x}} y \sin x^2 \, dy\, dx$

$\displaystyle = \int_0^9 \left(\frac{1}{2} y^2 \sin x^2 \Big|_0^{\sqrt{x}} \right) dx = \int_0^9 \frac{1}{2} x \sin x^2 \, dx$

$\displaystyle = -\frac{1}{4} \cos x^2 \Big|_0^9 = -\frac{1}{4} \cos 81 + \frac{1}{4}$