(1) (20 points) Find the maximum and minimum *values* of the function $f(x,y) = x^2 + 2y^2$ on the unit disk $x^2 + y^2 \leq 1$.

First, find interior critical pts: set $\nabla f = \vec{0}$

$$\nabla f = (2x, 4y) = (0,0) \Rightarrow (x,y) = \underline{(0,0)}$$

Next, find possible points on the bdy. $g(x,y) = x^2+y^2 = 1$ using Lagrange multipliers: either $\nabla g = \vec{0}$ or $\nabla f = \lambda \nabla g$

- $\nabla g = \vec{0}$: $\nabla g = (2x, 2y) = 0 \Rightarrow (x,y) = (0,0)$ - <u>NOT</u> on $x^2+y^2=1$
- $\nabla f = \lambda \nabla g$

$$\underline{(2x, 4y) = \lambda(2x, 2y)}$$

$$\boxed{\begin{aligned} 2x &= \lambda 2x \\ 4y &= \lambda 2y \end{aligned}} \Rightarrow \lambda = 1 \text{ or } x = 0$$

If $\lambda = 1$, $4y = 2y$, so $y = 0$, & $x = \pm 1$

If $x = 0$, $y = \pm 1$

So possibilities are $(1,0), (-1,0), (0,1), (0,-1)$ & the crit. pt. $(0,0)$

$f(1,0) = 1$

$f(-1,0) = 1$

$f(0,1) = 2$

$f(0,-1) = 2$

$f(0,0) = 0$

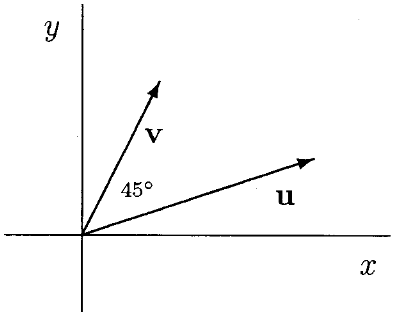
So max. value is 2

min. value is 0

(2) (10 points) Let $\mathbf{u}$ and $\mathbf{v}$ be two 3-dimensional vectors lying in the xy-plane, with an angle of $45°$ between them. If $||\mathbf{u}|| = 3$ and $||\mathbf{v}|| = 2$, find the following:

(a) $\mathbf{u} \cdot \mathbf{v}$

(b) $\mathbf{u} \times \mathbf{v}$



a) $\vec{u} \cdot \vec{v} = ||\vec{u}|| \, ||\vec{v}|| \cos\theta$

$= 3 \cdot 2 \cdot \cos 45° = \frac{6}{\sqrt{2}}$

b) $||\vec{u} \times \vec{v}|| = ||\vec{u}|| \, ||\vec{v}|| \sin\theta$

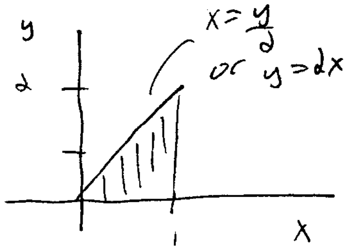
$= 3 \cdot 2 \cdot \sin 45° = \frac{6}{\sqrt{2}}$

direction: by the Right Hand Rule, $\vec{u} \times \vec{v}$ pts. in the pos. $\vec{k}$ direction.

So $\vec{u} \times \vec{v} = \left(0, 0, \frac{6}{\sqrt{2}}\right)$

(3) (10 points) Compute $\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\, dx\, dy$.

Reverse the order of integration:



so $0 \le y \le 2x$
$0 \le x \le 1$

So $\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2} dx\, dy = \int_0^1 \int_0^{2x} e^{x^2} dy\, dx$

$= \int_0^1 2x e^{x^2} dx = e^{x^2}\Big|_0^1 = e^1 - e^0 = e - 1$

(4) (20 points) Verify Gauss's (Divergence) Theorem for the solid unit ball B ($x^2 + y^2 + z^2 \le 1$) and the vector field $\mathbf{F}(x, y, z) = (0, -z, y)$.

Let S be the boundary of B, i.e., the sphere $x^2+y^2+z^2=1$.

Gauss's Theorem says that $\iint_S \vec{F} \cdot d\vec{S} = \iiint_B (\operatorname{div} \vec{F})\, dV$

Check: $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$ On the unit sphere, the normal vector $\vec{n}$ is (x, y, z).

So $= \iint_S \vec{F} \cdot (x,y,z)\, dS$

$= \iint_S (0,-z,y) \cdot (x,y,z)\, dS$

$= \iint_S (-zy + yz)\, dS = \iint_S 0\, dS = 0$

Now, $\iiint_B (\operatorname{div} \vec{F})\, dV = \iiint_B \left(\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(-z) + \frac{\partial}{\partial z}(y) \right) dV$

$= \iiint_B (0+0+0)\, dV = 0$

They are equal!!!

(5) (15 points) State whether each of the following statements is true or false. For each that is false, give an example showing that it's false.

(a) For every C^2 function f, $\nabla \times (\nabla f) = \mathbf{0}$.

True

(b) For every C^2 function f, $\nabla \cdot (\nabla f) = \mathbf{0}$.

False. Ex: $f(x,y) = x^2 + y^2$. Then $\nabla f = (2x, 2y)$,

and $\nabla \cdot (\nabla f) = \nabla \cdot (2x, 2y) = 2 + 2 = 4 \neq 0$

(c) For every C^2 vector field $\mathbf{F}$, $\nabla \cdot (\nabla \times \mathbf{F}) = \mathbf{0}$.

True

(6) (15 points) Give an example of a vector field that is *not* a gradient vector field, and explain how you can be sure that it isn't.

$\vec{F}(x,y,z) = (-y, x, 0)$.

If it were a gradient v.f., its curl would be $\vec{0}$.

But $\text{curl}(\vec{F}) = (0,0,2) \neq \vec{0}$.

(There are lots of other possible answers.)