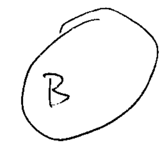
(1) Suppose that the function $f(x, y) = x^2 + 2x + y^2 + 1$ gives the temperature at the point (x, y). Let B be the disk of radius 2 centered at the origin, i.e., $B = \{(x, y) \mid x^2 + y^2 \leq 4\}$.

(a) Find the maximum and minimum values of f restricted to the disk B.



1st, find int. crit. pts. of f. $\nabla f = (2x+2, 2y)$ — always exists, only crit. pt. is $\underline{(-1,0)}$.

2nd, use Lagrange multipliers to find possible extrema on boundary circle $g(x,y) = x^2+y^2 = 4$:
$\nabla g = (2x, 2y)$ — never 0 on $g=4$. So solve $\nabla f = \lambda \nabla g$ on $g = 4$:
$2x + 2 = \lambda 2x$, $\underline{2y = \lambda 2y}$, and $x^2+y^2 = 4$.

$\Downarrow$

$y = 0$ or $\lambda = 1$. If $y=0$, then $x = \pm 2$. If $\lambda = 1$, then $2x+2 = 2x$, or
$2 = 0$ — no soln. So solns. are $\underline{(\pm 2, 0)}$.

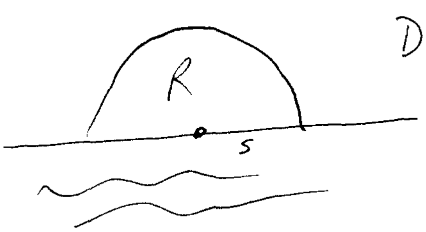
$f(-1,0) = 1-2+0+1 = 0 \leftarrow$ minimum value
$f(2,0) = 4+4+0+1 = 9 \leftarrow$ maximum value
$f(-2,0) = 4-4+0+1 = 1$

(b) If you start at the point (1, 0) and move northeast at unit speed, how fast is the temperature changing?

NE $\nearrow$ $\vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. The answer is

$$f_{\vec{u}} = \nabla f(1,0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$= (2\cdot 1 + 2, 2\cdot 0)\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$= (4,0)\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$= \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

(2) A city occupies a semicircular region of radius 5 km bordering on the ocean. Find the average distance from points in the city to the ocean.



Distance to ocean is y, so the average distance is

$$\frac{\iint_R y\,dA}{\text{area of } R} = \frac{\iint_R y\,dA}{\frac{1}{2}\pi 5^2} = \frac{\iint_R y\,dA}{\frac{25}{2}\pi}$$

This is easier in polar coordinates:

$$\iint_R y\,dA = \int_0^5 \int_0^\pi y \cdot r\,d\theta\,dr = \int_0^5 \int_0^\pi r^2 \sin\theta\,d\theta\,dr \quad (\text{since } y = r\sin\theta)$$

$$= \int_0^5 r^2 \left[-\cos\theta\Big|_0^\pi\right] dr = \int_0^5 r^2\big(-(-1)-(-1)\big)\,dr = \int_0^5 2r^2\,dr = \frac{2}{3}r^3\Big|_0^5 = \frac{2\cdot 125}{3}$$

So the average distance is

$$\frac{\frac{2\cdot 125}{3}}{\frac{25}{2}\pi} = \frac{2\cdot 2\cdot 125}{3\cdot 25\cdot \pi} = \frac{20}{3\pi}$$

We can also do it in rectangular coordinates:

$$\iint_R y\,dA = \int_{-5}^5 \int_0^{\sqrt{25-x^2}} y\,dy\,dx = \int_{-5}^5 \left[\frac{y^2}{2}\Big|_0^{\sqrt{25-x^2}}\right] dx = \int_{-5}^5 \frac{25-x^2}{2}\,dx$$

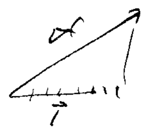
$$= \frac{25x - \frac{1}{3}x^3}{2}\Big|_{-5}^5 = \frac{2\cdot 125}{3},$$ as before

(3) State whether each of the following is true or false. If it's false, explain why or give an example showing that it's false.

(a) If $f(x,y) = k$ for all points (x,y) in a region R, then $\iint_R f\,dA = k \cdot \text{Area}(R)$.

True

(b) There is a function f with $||\nabla f|| = 4$ and $f_{\vec{i}} = 5$ at some point.



False – $f_{\vec i} = \nabla f \cdot \vec i$ = shadow of ∇f on $\vec i$.

If ∇f has length 4, its shadow can't be 5.

(c) There is a function f with $||\nabla f|| = 4$ and $f_{\vec{j}} = -3$ at some point.

True

(d) Let $\rho(x,y)$ be the population density of a city, in people per km^2. If R is a region in the city, then $\iint_R f\,dA$ gives the average number of people per km^2.

False – $\dfrac{\iint_R f\,dA}{\iint_R dA = \text{area of } R}$ is the average

(e) Let $f(x,y,z)$ be a continuous function. If W_1 and W_2 are solid regions with volume(W_1) > volume(W_2), then $\iiint_{W_1} f\,dV > \iiint_{W_2} f\,dV$.

False – If $f<0$ everywhere, then $\iiint_{W_1} f\,dV > \iiint_{W_2} f\,dV$

Solution to exercise 4.4.4 in Colley

If $y = a/x + b$, then the corresponding points on the curve are $(1, a+b)$, $(2, a/2+b)$, $(1/2, 2a+b)$, and $(3, a/3+b)$. We want to minimize the sum of the squares of the vertical distances between these points and our data points, i.e., minimize

$$f(a,b) = (0 - (a+b))^2 + (-1 - (a/2+b))^2 + (1 - (2a+b))^2 + (-1/2 - (a/3+b))^2.$$