(1) (10 points) (*This question is for section 2 (with linear algebra) only. If you're in section 1, you've got the wrong version of the exam.*) True or false: When numerically integrating a vector field using Runge-Kutta, we can always get better accuracy by decreasing the step size (i.e., by making Δt smaller). Explain.

(2) (10 points) Find the general solution of the ODE $y'' - 3y' + 2y = 3e^t$.

(3) (15 points) Solve the ODE $\frac{dy}{dt} = ry - ky^2$, $r > 0$ and $k > 0$. (HINT: Make the substitution $v = 1/y$.)

(4) (15 points) Find the general solution of the system $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \mathbf{x}$.

(5) (20 points) Suppose two similar countries Y and Z are engaged in an arms race. Let $y(t)$ and $z(t)$ denote the size of the stockpiles of arms of Y and Z, respectively. We model this situation with the system of differential equations

$$\begin{aligned} y' &= h(y, z) \\ z' &= k(y, z) \end{aligned} .$$

Suppose that all we know about the functions h and k are the two assumptions:

(i) If country Z's stockpile of arms is not changing, then any increase in the size of Y's stockpile of arms results in a decrease in the rate of arms building in country Y. Similarly, if country Y's stockpile of arms is not changing, then any increase in the size of Z's stockpile of arms results in a decrease in the rate of arms building in country Z.

(ii) If either country increases its stockpile, the other responds by increasing its rate of arms production.

(a) What do the assumptions imply about $\partial h/\partial y$ and $\partial k/\partial z$?

(b) What do the assumptions imply about $\partial h/\partial z$ and $\partial k/\partial y$?

(c) What types of equilibrium points are possible for this system? Justify your answer.

(6) (15 points) Consider the system

$$\begin{aligned} x' &= x(x-1) \\ y' &= x^2 - y \end{aligned} .$$

Sketch the x- and y-nullclines. Then find all equilibrium points. Using the direction of the vector field between the nullclines, describe the possible behavior of the solution corresponding to the initial condition $x(0) = -0.5$, $y(0) = 2$. (N.B.: Look at the whole plane, *not* just the first quadrant.)

(7) (10 points) Consider the initial value problem $y'' + y = 0$, $y(0) = y(a) = 0$. For what values of a (if any) will there be more than one solution to the IVP?

(8) (15 points) Let $y(t)$ be the population of fish in a certain lake at time t. Assume that the population is governed by the ODE $y' = y^2 + b$ (where b is a constant) and that initially there are 1000 fish in the lake. What's the long-term behavior of the fish population? (HINT: Your answer will be different for different values of b.)

(9) (20 points) Solve the following ODEs:

(a) $y' = (ty)^2$. (Find the general solution.)

(b) $x + yy' = 0$, $y(0) = -2$. For what x values does the solution exist?

(10) (15 points) Suppose that the solution $y(t)$ of the IVP $y'' + py' + qy = \delta(t)$, $y(0) = y'(0) = 0$ (where p and q are constants) has a Laplace transform $\mathfrak{L}\{y(t)\}$ whose value at $s = 0$ is $1/5$ and whose value at $s = 2$ is $1/17$.

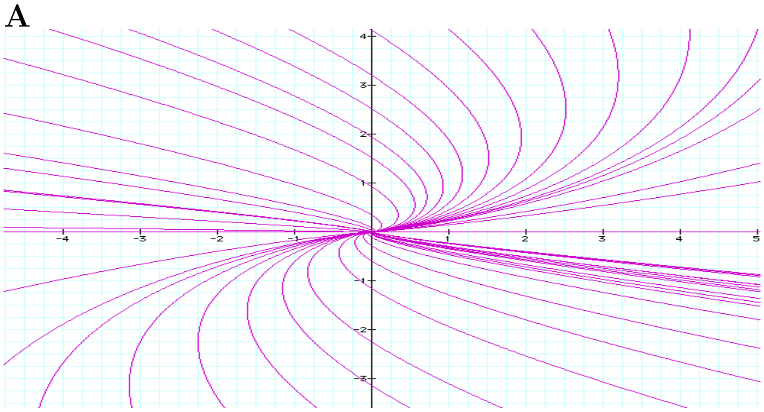
(a) Find p and q.

(b) Find $y(t)$. (If you couldn't solve part (a), use the values $p = 2$ and $q = 5$.)

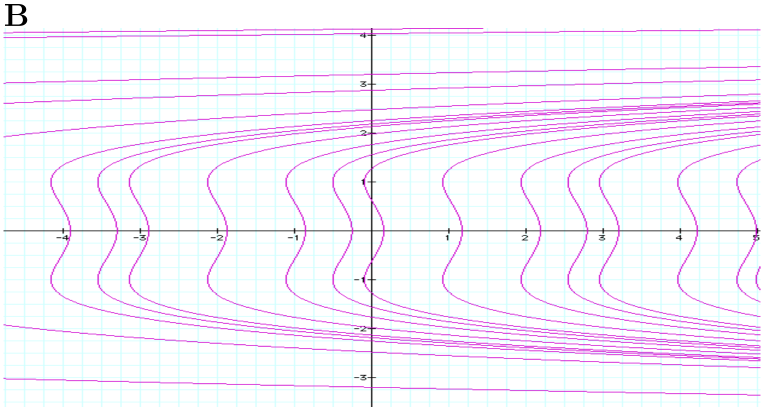
(11) (15 points) Find the first four nonzero terms of the series solution of Airy's equation, $y'' = xy$, with initial conditions $y(0) = 1$, $y'(0) = 0$.

(12) (20 points) The figures below show solutions in the ty-plane of equations of the form $y' = F(t, y)$.

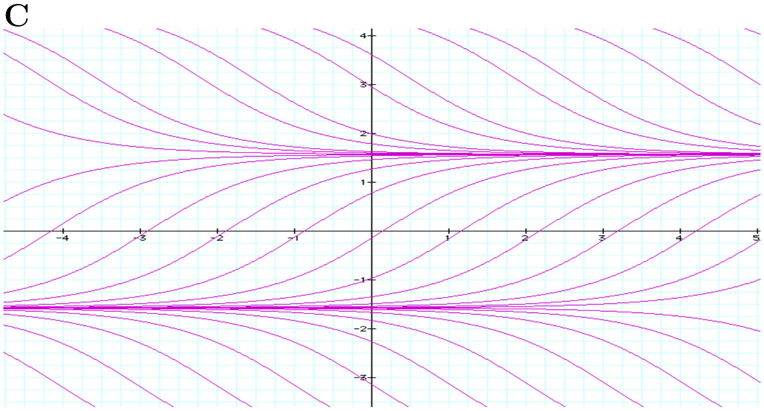
A



B



C



(a) Which systems, if any, have some solutions which are not unique?

(b) Which systems, if any, are autonomous?

(13) (20 points) A 30-gallon tank initially contains 15 gallons of saltwater containing 6 pounds of salt. Suppose saltwater containing 1 pound of salt per gallon is pumped into the top of the tank at the rate of 2 gallons per minute, while a well-mixed solution leaves the bottom of the tank at the rate of 1 gallon per minute. How much salt is in the tank when the tank is full?

EXTRA CREDIT (5 points) True or false: Classification of mathematical problems as linear and nonlinear is like classification of the universe as bananas and non-bananas.