Math 30 Spring 2002 Practice Second Midterm

(1) Use power series to find the solution to the ODE $y''+kxy=0$ satisfying the initial conditions $y(0)=0$ and $y'(0)=1$. The "k" denotes a constant. Write your answer using summation notation and clearly show the form of the general term of the series.

Advice: Be careful. Pay attention to the first few terms of the series. I recommend using the initial conditions early in the process and finding the unique solution directly. This will take less computation than finding the general solution and then plugging in the initial conditions.

(2) For each of the following systems of linear ODEs, determine whether the origin $(0,0)$ is a sink, source, saddle, degenerate node, center, spiral source, or spiral sink. If the eigenvalues are real, compute the eigenvectors and sketch the phase portrait (vector field) for the system, clearly labeling the straight-line solutions and the trajectories of the other solutions. What is the long-term behavior of the solutions of each system?

(a) $\begin{aligned} x' &= x+5y \\ y' &= -3y \end{aligned}$ (b) $\begin{aligned} x' &= 4x+y \\ y' &= x+4y \end{aligned}$ (c) $\begin{aligned} x' &= 2x-3y \\ y' &= 5x-4y \end{aligned}$

(3) Find the general solution to each of the following equations.

(a) $y^{(4)}-4y^{(3)}+4y''=0$

(b) $y^{(3)}+y''+y'+y=7$

(4) *(For section 2 only. I will give you the table of transforms from your book on the exam.)* Use the Laplace transform to solve the initial value problem

$$y''-2y'+2y=\cos t, \quad y(0)=0,\ y'(0)=0.$$

(5) *(For section 1 only.)*

(a) For each of the following matrices, find the inverse or prove that none exists.

(i) $\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 3 & 1 & 0 \end{pmatrix}$

(b) Is $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ an eigenvector for the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 6 & 0 & 0 \\ 1 & 1 & -1 & 9 \\ 3 & 3 & 4 & 0 \end{pmatrix}$? If so, what is its eigenvalue?

(6) Find the solution to the system of ODEs $\mathbf{x}'=\mathbf{P}\mathbf{x}$ satisfying $\mathbf{x}(0)=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, given that

$$e^{\mathbf{P}t}=\begin{pmatrix} e^t & -\frac{1}{2}e^t+\frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t}+\frac{1}{2}e^{5t} \\ 0 & e^{3t} & -e^{3t}+e^{5t} \\ 0 & 0 & e^{5t} \end{pmatrix}.$$

(7) Consider the 2nd order, linear ODE $y''+(\sin x)y'+(4x^2-x)y=0$. Without solving, state a minimum value for the radius of convergence for the series solution.

(8) Convert the following 4th order ODE to a system of 1st order ODEs: $3y^{(4)}-5y''+9y=6e^t-2t$.

(9) Find the general solution to the following systems of ODEs.

(a) $\mathbf{x}'=\begin{pmatrix} 0 & 8 \\ 2 & 0 \end{pmatrix}\mathbf{x}+\begin{pmatrix} e^{3t} \\ t \end{pmatrix}$ (b) $\mathbf{x}'=\begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}\mathbf{x}$

(10) We want to analyze the populations in Springfield and Swarthmore, where the populations x of Springfield and y of Swarthmore depend on several factors. The birth rate of Springfield

is a_1, and that of Swarthmore is b_1. Assume that no one ever moves to/from Springfield except from/to Swarthmore, and vice versa. The rate at which people move from Springfield to Swarthmore is a_2, and that at which they move from Swarthmore to Springfield is b_2. Assume that no one ever dies. Then the rate at which Springfield's population changes is

$$\frac{dx}{dt} = a_1x - a_2x + b_2y = (a_1 - a_2)x + b_2y,$$

and the rate at which the population of Swarthmore changes is

$$\frac{dx}{dt} = b_1y - b_2y + a_2x = (b_1 - b_2)y + a_2x.$$

(Why?)

Determine the populations $x(t)$ and $y(t)$ of Springfield and Swarthmore, respectively, at time t if $a_1 = 5$, $a_2 = 4$, $b_1 = 5$, and $b_2 = 3$, and given that $x(0) = 14,000$ and $y(0) = 7000$. What is the long-term value of the ratio of Springfield's population to Swarthmore's (i.e., of $\frac{x(t)}{y(t)}$)?