Math 5   Name _____
December 19, 1997
7–10 pm

FINAL EXAMINATION

This exam is nine pages long; check that you have all the pages. Be sure to show all your work on the attached sheets; no credit will be given for unsubstantiated solutions.

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1. Find the derivatives of the following functions. Simplify your answers to a reasonable degree.

(18 points)

(a) $y = \left(\cos \sqrt{x} \right)^4$

(b) $y = x^3 \, e^{2x}$

(c) $y = \ln(4 + x^2)$

1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
9.	
10.	
11.	
12.	
Total	/200

2. Evaluate the following integrals. (12 points)

(a) $\int \cos 7x \, dx$

(b) $\int_{-1}^{3} (x^2 + 3) \, dx$

3. Suppose that you take out a loan of $30,000 that has an annual interest rate of 8%. If you make payments of P every month, then it turns out that the time it will take you to pay off the loan entirely is given by the formula
$$f(P) = -12.5 \ln\left(1 - \frac{200}{P}\right) \quad \text{years.}$$

(16 points)

(a) Without doing any calculation, what can you say about the sign of $f'(P)$? Make sure your reasoning is clear.

(b) Calculate $f(300)$ and $f'(300)$, and interpret what your answers mean in precise, concrete terms.

4. Suppose that g is a smooth function such that $g(5) = 2$ and $g'(5) = 7$. Use a linear (= tangent line) approximation to estimate the value of $g(4.9)$. (15 points)

5. Suppose that for a certain function $f(x)$:

(i) $\displaystyle\int_0^2 f(x)\,dx = 2$ \qquad\qquad (ii) $\displaystyle\int_1^2 f(x)\,dx = -1$ \qquad\qquad (iii) $\displaystyle\int_2^4 f(x)\,dx = 7.$

(15 points)

(a) Evaluate $\displaystyle\int_0^4 3f(x)\,dx.$

(b) Evaluate $\displaystyle\int_0^1 f(x)\,dx.$

(c) Explain clearly why f must be negative somewhere in the interval from $x = 1$ to $x = 2$.

6. A race car starts from rest and moves with increasing acceleration as given in the table below. Estimate as well as you can how fast the car is moving after 8 seconds.

(16 points)

Elapsed time (sec)	Acceleration (ft/sec^2)
0	0
1	5
2	6
4	7
8	8

7. A rectangular plot of land of area 600 ft^2 is to be surrounded by a stone wall and then divided into two equal parts by a fence parallel to one side. If it costs \$10 per foot to build the stone wall and \$2 per foot to build the fence, what should the dimensions of the plot be in order for the total cost to be as small as possible?

8. One indicator of national prosperity is the *per capita* gross domestic product ("per capita" means the average per person). This is computed by taking the total gross domestic product (GDP) and dividing that by the total population. According to figures compiled by the US Department of Commerce, at the beginning of 1995, the total GDP for the United States was 7253.8 billion dollars and was increasing at an instantaneous rate of 245.6 billion dollars per year. At the same time, the total US population was 262.8 million people, increasing at an instantaneous rate of 2.901 million people per year.

(16 points)

(a) Let $f(t)$ be the total GDP of the United States in year t, and let $p(t)$ be the total population of the United States in year t. Write an expression which gives the *per capita* GDP in year t.

(b) Was the per capita GDP in the United States increasing or decreasing at the beginning of 1995? By how much? Be sure to include the units of your answer. (*Reminder:* 1 billion $= 10^9$ and 1 million $= 10^6$.)

9. Consider the graph of the function $f(x)$ shown below. (20 points)
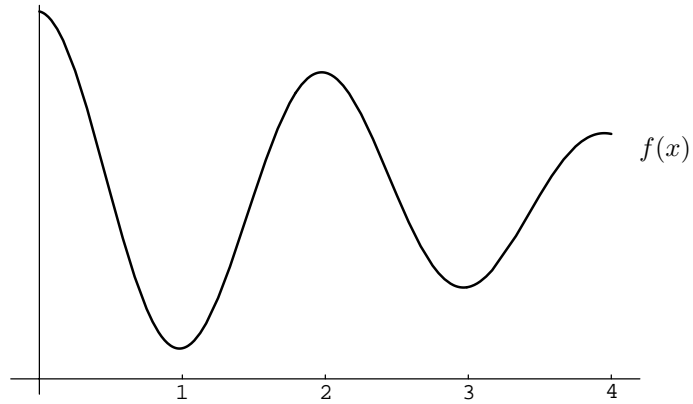
(a) Sketch the graph of a function F such that $F' = f$ and $F(0) = 0$. Where there is enough information to do so, make sure that your sketch reflects accurately where F is positive or negative, increasing or decreasing, and concave up or concave down.
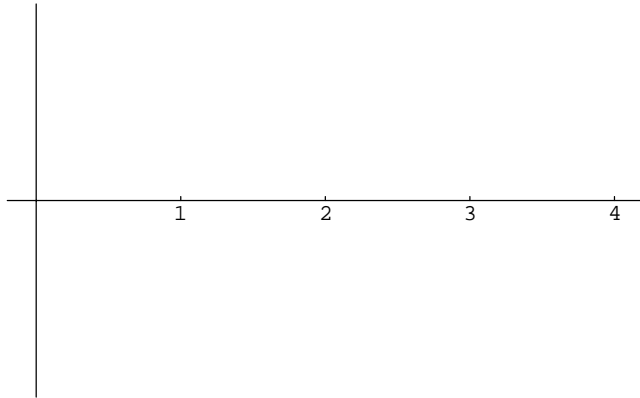
(b) Sketch the graph of f', paying attention, where possible, to the same sort of details as in part (a). In addition, can you determine whether the integral of f' from $x = 0$ to $x = 4$ is positive, negative, or zero? If so, which is it? Or is more information needed before you can tell for sure? Explain clearly.
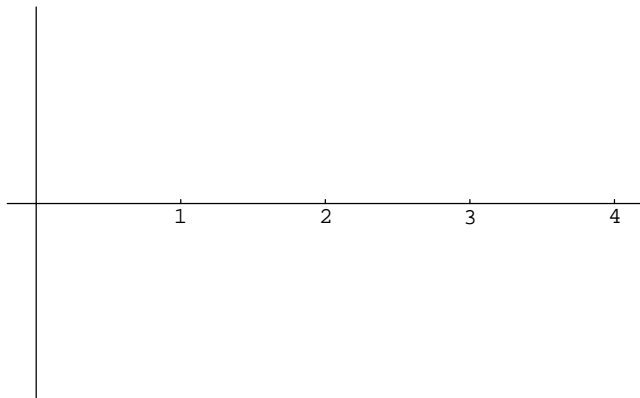
10. Suppose that you deposit $1000 into a savings account that accrues interest continuously so that your account balance grows in a continuous way. Assume that after t years your account balance will be growing at a rate of $40\,e^{0.04t}$ dollars per year. (16 points)

(a) At what rate will your account balance be changing after 4 years?

(b) What will your account balance be after 4 years?

11. Wilbur the pig escapes from his pen and wanders onto a nearby highway. As he is standing there sniffing the air, a car approaches going 60 mph (= 88 ft/sec). The driver slams on the brakes when he is 250 ft from Wilbur, and thus the car decelerates at a constant rate of 16 ft/sec^2. Will the car hit Wilbur? If so, with what velocity? If not, how far will the car be from Wilbur when it stops? (16 points)

12. Do **one and only one** of the following problems. Indicate which problem you have chosen to do by circling the letter (a) or (b) below. If you need more space, continue on the back of the preceding page.
(20 points)

(a) You are on a speedboat in New York harbor traveling straight towards the Statue of Liberty at a speed of 50 ft/sec. The statue stands 300 ft high, including the pedestal, which is 150 ft high. When the boat is 100 ft from the statue, how fast is your viewing angle of the statue changing? (This is the angle labeled θ below.) Is your view of the statue getting better or worse?

(b) Two slightly neurotic people hire you to drive them from Swarthmore to Boston, a distance of 330 miles. Both your employers want to get to Boston efficiently, but they have different ideas about what this means. One of them simply wants to get there as quickly as possible. The other wants to economize on fuel consumption. Suppose that, if you drive the car at a speed of v mph, the car consumes fuel with an efficiency given by the formula $g(v) = 40 - 0.4v$ miles per gallon.* The three of you agree that you will assign one unit of anxiety for every hour you spend on the road and another unit of anxiety for every gallon of gasoline that is consumed. At what speed should you drive if you wish to minimize the total amount of anxiety on the trip? Assume that you drive at constant speed the whole way. (*Hint:* If you need help getting started, suppose you were to drive at 60 mph. How long would it take you to get to Boston? How many gallons of gasoline would you use? Now adapt your thinking to find formulas for the general case of v mph.)

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* (for part (b)) This formula is not too realistic, but we are trying to keep the calculations from getting out of hand.