

**MIDTERM EXAM****October 29, 2002**

Please write your answers in your blue book, and make sure that I can find them. Show your work - correct answers with no justification may receive little or no credit. No calculators, notes, or books are allowed.

- (1) The functions  $x(t)$  and  $y(t)$  give the numbers of two species of animals, rabbits and foxes. Foxes eat only rabbits, while rabbits eat plants, which are essentially unlimited. A system of differential equations modeling these populations is

$$\begin{aligned}x' &= -x + 2xy \\y' &= 3y - xy\end{aligned}$$

- (a) Which function represents which species (i.e., is  $x$  foxes and  $y$  rabbits, or is it the other way around)? How do you know?
- (b) Sketch the nullclines for this system. Without computing any derivatives or eigenvalues, what can you say about how the populations will behave?
- (2) Prove that it's impossible to draw the figure below without lifting your pencil or retracing some lines. Give the statements of any theorems that you use.

- (3) Let  $C_1$  and  $C_2$  be two closed curves in  $\mathbb{R}^3$ . In words (and, if you're so inclined, clear pictures), explain roughly the meanings of  $\text{Lk}(C_1, C_2)$ ,  $\text{Tw}(C_1, C_2)$ , and  $\text{Wr}(C_1)$ . How are the three quantities related?

**(OVER)**

- (4) An analysis of a recent field hockey game between Swarthmore and Haverford gave rise to the following data: If the ball was in the center ( $C$ ), the probabilities that it next entered Haverford territory ( $H$ ) or Swarthmore territory ( $S$ ) were .4 and .6, respectively. From Swarthmore territory, the ball was sent back to the center with probability .95 or into the Swarthmore goal ( $SG$ ) with probability .05 - this means Haverford scored a goal. From Haverford territory, the ball went back to the center with probability .9 and to Haverford's goal ( $HG$ ) with probability .1 - this means Swarthmore scored a goal. Assume that the ball begins in the center.

- (a) Write down and clearly label the state diagram for the 5-state Markov chain that describes action until a goal is scored. What are the absorbing states? Write down the transition matrix - make sure I can tell which states are which.
- (b) What is the probability that Swarthmore will score first?

- (5) (I stole this problem from a textbook from the seventies; I apologize if anyone's offended by it. On the plus side, it's offensive to both men and women.)

Jim Bock, Swarthmore Dean of Admissions (who went to my high school, by the way), is faced with the task of admitting a freshman class of at most 500 students. The typical male applicant can be expected to have a combined SAT score of 1200, contribute \$8000 to the college as an alumnus, cause \$200 damage to dormitory buildings and classrooms over his four years at Swat, and cost \$2400 per year to teach. The typical female applicant can be expected to have a combined SAT score of 1300, contribute \$3000 to the college as an alumna, and cause \$100 in damages in four years. Because of different course selections, she can be educated at a cost of \$2000 per year.

President Bloom demands a freshman class that will eventually contribute at least \$2.5 million to the college, the faculty insists that the average SAT score be 1250 or higher, and the maintenance department can handle up to \$85,000 in damages over the four years. The college treasurer wants to educate the class at the lowest possible cost.

Set up an integer programming problem whose solution will tell the dean of admissions how many men and how many women to admit. Identify the variables and write down the constraints and the function to be optimized. (You don't have to actually solve the problem.)