MATH 6B May 2003

Practice Final

Here are a bunch of practice problems, mostly stolen off of other people's old exams. Obviously, your exam will be shorter than this. (The review sections at the end of the chapters in Hughes-Hallett and Gillett are another good source of practice problems.)

1. Find the intervals of convergence (including endpoints!) for the following power series:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^2}$

b) $\displaystyle\sum_{n=0}^{\infty} nx^n$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-7)^{2n}}{4^n}$

2. Determine if the following series converge or diverge.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!n}$

c) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$

d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-1}}$

e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n-3)^2+1}$

3. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges (how do you know?) to a sum S. How many terms would you need for a partial sum to approximate S to within 10^{-4}?

4. Find a power series solution of the form $\sum c_n x^n$ for the differential equation $\frac{dy}{dx} + y = 1$ with $y(0) = 2$.

5. The rhinoceros is now extremely rare. Suppose that enough game preserve land is set aside so that there is no danger of overcrowding. However, if the population is too small, fertile adults have difficulty finding each other when it is time to mate. Write a differential equation that models the rhinoceros population based on these assumptions. (Note that there is more than one reasonable model that fits these assumptions.)

6. Consider the population model

$$\frac{dP}{dt} = 0.3\left(1 - \frac{P}{200}\right)\left(\frac{P}{50} - 1\right)P,$$

where $P(t)$ is the population at time t.

a) For what values of P is the population in equilibrium?

b) For what values of P is the population increasing?

c) For what values of P is the population decreasing?

d) What will happen in the long run if the population now is 150?

7. Determine whether the following statements are true or false.

a) If $\lim_{n\to\infty} b_n = 0$, then the series $\sum(-1)^n b_n$ converges.

b) If f is positive and decreasing, $\int_1^\infty f(x)\,dx$ diverges, and $a_n = f(n)$, then $\sum a_n$ diverges.

c) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and the series $\sum b_n$ diverges, then so must $\sum a_n$.

8. As part of your "New Millennium's Resolution," you decide to invest \$1000 in a savings account every year on January 1st, starting with the year 2000. Presuming that this account earns 5% interest compounded anually, how much money is in your account on December 31, 2029?

(We are supposing that on the last day of the year, you receive the interest for the year (exactly .05 times the amount in the bank), but you have not yet made your new deposit. For instance, at the end of the first year (Dec. 31, 2000), you have \$1000(1.05) in your account.)

Here, by popular demand, are a bunch of Taylor polynomial approximation error questions:

9. a) Find the Maclaurin series for $\frac{1}{1+x^2}$. (There is an easy way and a hard way.)

b) Use (a) to find a Maclaurin series for $\arctan x$. Where does this series converge? (Be sure to check the endpoints...)

c) Recall (?) that $\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$. If we use the series in (b) up *through* the x^{21} term to approximate $\frac{\pi}{6}$, what is an estimate for the error of approximation?[1]

10. Let $f(x) = e^{2x}$.

a) Compute $P_4(x)$ (the fourth-degree Taylor polynomial) at $a = 0$.

[1] Note that if you multiply your approximation by 6, you have a decent approximation of π!

b) Find an upper bound for the error involved in the approximation $e^{.2} \approx P_4(.1)$.

11. Use a Taylor polynomial to approximate the given value with an error of less than .001. (Choose your a wisely, and use Taylor's theorem to justify why your approximation is sufficiently accurate.)

a) $\sin 1$ b) $\ln 1.1$ c) $e^{-.025}$

12. Let f be a function whose fourth derivative satisfies the inequality

$$|f^4(x)| \leq \frac{1}{3+5x^2}$$

for all x, and let $T_3(x)$ be the third degree Taylor polynomial for f at $a = 0$. Estimate the error in using $T_3(2)$ to approximate $f(2)$.

13. For each of the following functions f,

- Approximate f by a Taylor polynomial of degree n at the number a.
- Estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.

a) $f(x) = \dfrac{1}{x}$, $a = 1$, $n = 3$, $.8 \leq x \leq 1.2$

b) $f(x) = e^{x^2}$, $a = 0$, $n = 2$, $|x| \leq .1$

c) $f(x) = x^{\frac{3}{4}}$, $a = 16$, $n = 3$, $15 \leq x \leq 17$

14. Approximate $\sqrt{1.05}$ by using a well-chosen first degree Taylor polynomial. (What is $f(x)$? What is a?) Use Taylor's theorem to estimate the error of your approximation.