Positive expansiveness

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Throughout: $f : X \to X$ a continuous map on a locally compact metric/metrizable space.

Sometimes: f a homeomorphism and/or X compact

Expansive, positively expansive, and expanding

f is *expansive* if two different orbits can't stay close for all time (there exists an expansive constant $\rho > 0$ such that for any $x, y \in X$ there exists $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) > \rho$).

f is *positively expansive* (PE) if two different orbits can't stay close for all forward time (there exists $\rho > 0$ such that for any $x, y \in X$ there's an $n \geq 0$ such that $d(f^n(x), f^n(y)) > \rho$).

f is *expanding* if nearby points move farther apart (there exist $\varepsilon > 0$ and $\lambda > 1$ such that whenever $d(x, y) < \varepsilon$, $d(f(x), f(y)) > \lambda d(x, y)$).

On a compact space, expansiveness and positive expansiveness are independent of the choice of metric. Not true for noncompact spaces - second part of talk.

Which spaces admit

- an expansive homeomorphism?
- an expansive map?
- a PE homeomorphism?
- a PE map?

Expansive homeomorphisms

1970 - O'Brien and Reddy: Every compact, orientable surface of positive genus admits an expansive homeomorphism

1979 - Mañe: If a compact space admits an expansive homeomorphism then it is finite dimensional.

1989 - Lewowicz: There are no expansive homeomorphisms on the 2-sphere.

1990 - Hiraide: There are no expansive homeomorphisms on the 2-sphere, the projective plane, or the Klein bottle.

PE maps

1990 - Hiraide: No manifold with boundary admits a positively expansive continous map.

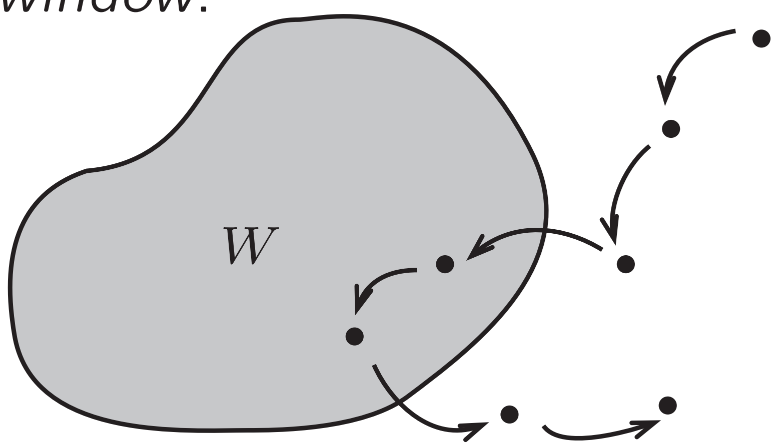
PE homeomorphisms on compact spaces

THEOREM 1 A compact metric space X admits a PE homeomorphism if and only if X is finite.

1952 Schwartzman
1955 Gottschalk and Hedlund
1969 Keynes and Robertson
1990 Hiraide
late 80's/1990/2004 Boyle-Geller-Propp / King / Coven-Keane

Proofs involve: functional topology, generators for topological entropy, metrization lemmas, combinatorics

A dynamical system (homeomorphism, continuous map, flow, semiflow) is said to be *bounded* if there is a compact set W which intersects the forward orbit of every point. The set W is called a *window*.



THEOREM (Fournier, Richeson-W) For a dynamical system on a locally compact space X, the following are equivalent.

1. The dynamical system is bounded.
2. There exists a forward invariant window.
3. There is a compact global attractor Λ (that is, there is an attractor Λ with the property that $\emptyset \neq \omega(x) \subset \Lambda$ for every $x \in X$).

Notation:

Given $f, g : X \to X$, let

$$f \times g : X \times X \to X \times X$$

denote the function

$$(f \times g)(x, y) = (f(x), g(y)).$$

Let $\Delta = \{(x, x) : x \in X\}$ denote the *diagonal* of $X \times X$.

LEMMA Let $f : X \to X$ be a positively expansive homeomorphism on a compact space X. Then Δ is an attractor for

$$f^{-1} \times f^{-1} : X \times X \to X \times X.$$

So if x and y are close enough,

$$d(f^{-n}(x), f^{-n}y) \to 0 \text{ as } n \to \infty.$$

Cover X by a finite number of open sets

$$U_1, \ldots, U_m,$$

each consisting of points that are close enough together. Then

$$\text{diam}(f^{-n}(U_i)) \to 0 \text{ as } n \to \infty.$$

Also, because f is a homeomorphism, the sets $f^{-n}(U_1), \ldots, f^{-n}(U_m)$ cover X. Thus X consists of at most m points.

$\square$

Some other applications of boundedness

COROLLARY Suppose $f : X \to X$ is an volume-preserving map of a noncompact space X. Then f is not bounded. In particular, if $S \subset X$ is any compact set, then there exists a point $x \in X$ such that the forward orbit of x does not intersect S.

COROLLARY (Bernardes 2000, Richeson-W 2001) Let X be a noncompact, locally compact space. There is no forward minimal dynamical system X. That is, there is a point whose forward orbit is not dense in X.

THEOREM (Fournier 1975, Richeson-W 2001) Every bounded dynamical system on $\mathbb{R}^n$ has a fixed point.

A Poincaré-Birkhoff theorem for the open annulus

THEOREM (Richeson-W 2003) Let A be the open annulus. Suppose $f : A \to A$ is a bounded homeomorphism that preserves orientation, is homotopic to the identity, and has a connected nonwandering set. If there is a lift of f, $\tilde{f} : \tilde{A} \to \tilde{A}$, and points $x, y \in \tilde{A}$ with

$$\lim_{n\to\infty} (\tilde{f}^n(x))_1 = -\infty$$

and

$$\lim_{n\to\infty} (\tilde{f}^n(y))_1 = \infty,$$

then f has a fixed point.

On a compact space, positive expansiveness is independent of the choice of metric, but this is not true for noncompact spaces.

Can we generalize Theorem 1?

First, what's the "right" topological definition for noncompact spaces?

Idea: The diagonal Δ in $X \times X$ should be a repeller for $f \times f$.

DEFN An *expansivity neighborhood* for f is a closed neighborhood $N \subset X \times X$ of Δ such that for any distinct $x, y \in X$ there exists $n \geq 0$ such that $(f \times f)^n(x, y) \notin N$. N is called *overflowing* if $N \subset f(N)$.

- A map $f : X \to X$ is *weakly positively expansive* if it has an expansivity neighborhood.

- A homeomorphism $f : X \to X$ is *strongly positively expansive* if it has an overflowing expansivity neighborhood with compact cross sections.

If X is compact, wPE $\iff$ sPE $\iff$ PE.

Weak positive expansiveness

THEOREM Let $f : X \to X$ be a continuous map on a locally compact metrizable space. Then f is wPE $\iff$ f is PE with respect to some metric compatible with the topology.

So. . .

THEOREM Let X be a **compact** metrizable space and $f : X \to X$ be a continuous map. Then the following are equivalent:

1. f is PE with respect to some compatible metric.

2. f is PE with respect to every compatible metric.

3. f is expanding with respect to some compatible metric.

4. f is wPE.

PROOF Easy, except 1 $\Longrightarrow$ 3 (Reddy, 1982)

If X is compact, then

$$f \text{ is PE } \iff f^n \text{ is PE for all } n.$$

Not true if X is noncompact.

Bryant and Coleman (1966) give an example of a homeomorphism $f : [0, \infty) \to [0, \infty)$ such that f is PE, but f^n isn't for any $n > 1$.

THEOREM Let X be a locally compact metrizable space, $n > 0$. Then

1. f is wPE $\iff$ f^n is wPE, so

2. f is PE with respect to some metric $\iff$ f^n is PE with respect to some (possibly different) metric.

Weakly positively expansive homeomorphisms

Let X be a locally compact metrizable space and let $f : X \to X$ be a wPE homeomorphism (equivalently, f is PE with respect to some compatible metric). Then, for each $x \in X$ one of the following holds:

1. x is a repelling periodic point,
2. $\omega(x) = \emptyset$, or
3. $\omega(x)$ is noncompact.

Strongly positively expansive homeomorphisms

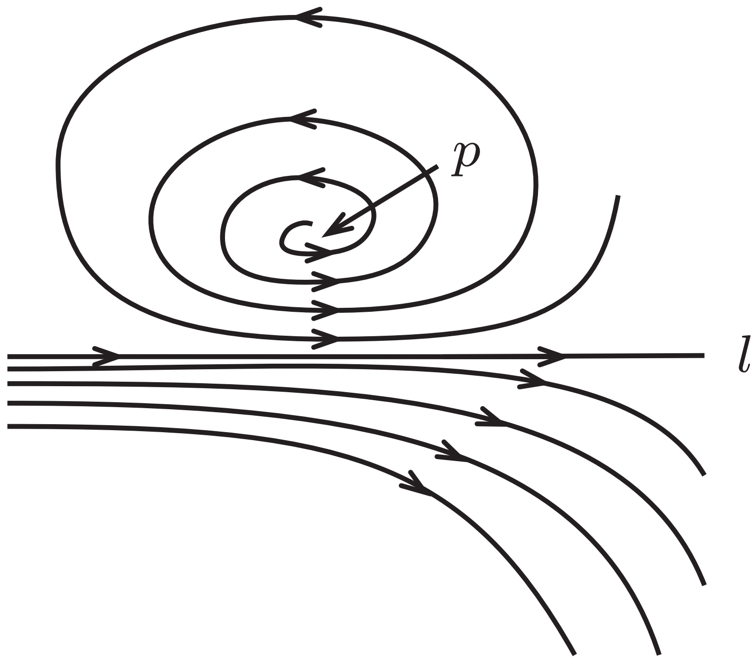
THEOREM X a locally compact metrizable space, $f : X \to X$ a homeomorphism, $n > 0$. Then f is sPE $\iff$ f^n is sPE.

THEOREM Suppose X is a locally compact metrizable space and $f : X \to X$ is an sPE homeomorphism. Then, for each connected component $X' \subset X$, either

1. $\omega(x) = \alpha(x) = \emptyset$ for all $x \in X'$, or

2. X' contains exactly one repelling periodic point y, and for every $x \in X' \setminus \{y\}$, $\omega(x) = \emptyset$ and $\alpha(x)$ is the orbit of y.

EXAMPLE



Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the time-one map of the flow. The map has a repelling fixed point, p, and an invariant line, l. We may assume that the homeomorphism restricted to l moves points to the right at an exponential rate. Thus, f is PE with respect to the usual metric.

Also, notice that although f is PE, f is not sPE.