## GROUPS OF ORDER 4

Let's find all (up to isomorphism) the groups of order 4.
(Note: Out of the goodness of my heart, I've checked for you that all the binary tables you'll come up with for this worksheet satisfy associativity, so you don't need to check.)

Let $G=\{e, a, b, c\}$, and consider the following table:

|  | $e$ | $a$ | $b$ | $c$ |
| :---: | :---: | :---: | :---: | :---: |
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $*$ |  |  |
| $b$ | $b$ |  |  |  |
| $c$ | $c$ |  |  |  |

1. There are only three possibilities for the space in the table labeled with *: e, b, c. Why?
2. How many possibilities are there for completing the table if you fill in the ${ }^{*}$ space with $e$ ? Find them/write them out.
3. I claim that, without loss of generality (WLOG), if the * space is not filled with $e$, we may assume that it is filled with $b$. Why? That is, why aren't we losing a potential new group by refusing to fill in that space with $c$ ?
4. How many possibilities are there for completing the table if you fill in the ${ }^{*}$ space with $b$ ? Find them/write them out.
5. Which of the groups are abelian?
6. Which of the group tables that you have found describe groups that are isomorphic to $\mathbb{Z}_{4}$ ? (Or, if you prefer, $U_{4}$ ?) (Hint: Two groups are isomorphic if you can relabel/reorder the elements of the groups so that the resulting group tables are identical. The relabeling function is the isomorphism.)
