Banach Spaces

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A Banach space is a complete normed linear space

① $\|x\| \geq 0 \quad \forall x \in \mathfrak{X}$
$\|x\| = 0$ iff $x = 0$
② $\|\alpha x\| = |\alpha|\,\|x\| \quad \forall x \in \mathfrak{X} \ \forall \alpha$ in scalers
③ $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathfrak{X}$
④ $\|x_n - x_m\| \to 0$ as $n, m \to \infty \Rightarrow \exists\, x \in \mathfrak{X}$ s.t. $\|x_n - x\| \to 0$
as $n \to \infty$

Examples

① $\ell_p \quad 1 \leq p < \infty \qquad \|(x_n)\| = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$

② ℓ_∞ space of all bounded seq. under norm $\|(x_n)\|_\infty = \sup |x_n|$

③ c_0 subspace of ℓ_∞ consisting of all sequences that tend to 0

[Structure question (solved 1974) Does every Banach space contain a subspace isomorphic to ℓ_1, ℓ_p $(1<p<\infty)$ or c_0? NO]

④ $L_p(\mu) \quad 1 \leq p \leq \infty$

⑤ $C[a,b]$ continuous functions on $[a,b]$ $\quad \|f\| = \max_{t \in [a,b]} |f(t)|$

⑥ $C^2[a,b]$ functions with two continuous derivatives

$$\|f\| = |f(0)| + |f'(0)| + \max_{a\le t\le b} |f''(t)|$$

$$\text{or} \quad \|f\| = \max_t |f(t)| + \max_t |f'(t)| + \max_t |f''(t)|$$

(equivalent norms)

HW/a) Let $(\mathfrak{X}_n)$ be a sequence of Banach spaces. The ℓ_p-sum

$$\left(\sum_{n=1}^{\infty} \mathfrak{X}_n \right)_{\ell_p}$$

is the set of all sequences (x_n) s.t. $x_n \in \mathfrak{X}_n$ and

$$\|(x_n)\|_{\ell_p} := \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p} < \infty$$

Prove this is a Banach space

b) Let Γ be a set and $\mathfrak{X}$ a Banach space. Let $\ell_\infty(\Gamma, \mathfrak{X})$ be all bounded $\mathfrak{X}$-valued functions from Γ. Prove $\ell_\infty(\Gamma, \mathfrak{X})$ is a Banach space with norm

$$\|f\|_\infty = \sup_{t\in\Gamma} \|f(t)\|_{\mathfrak{X}}$$

DEFINITION: Let $\mathfrak{X}$ be a Banach space and $\varepsilon > 0$. Let

$$B_{\mathfrak{X}}(\varepsilon) := \{x \in \mathfrak{X} : \|x\| < \varepsilon\}$$
$$B_{\mathfrak{X}}(\varepsilon, x_0) := \{x \in \mathfrak{X} : \|x - x_0\| < \varepsilon\}$$

DEFINITION: A set in a B-space is open iff it is the union of open balls. A set E of a B-space $\mathfrak{X}$ is bounded if

$$\sup_{x \in E} \|x\| < \infty$$

DEFINITION: A set is nowhere dense if its closure contains no balls. A set is of the first category if it is the union of nowhere dense sets. A set is of second category if it is not of first category

THEOREM: (Baire Category) A B-space is of 2^{ND} category.

HW/ An infinite dimensional B-space is not the union of a sequence of finite dimensional subspaces (You may take for granted that finite dimensional subspaces are closed) Conclude that no infinite dimensional B-space has a countable Hamel basis.

THEOREM: Let $T : \mathfrak{X} \longrightarrow \mathcal{Y}$ be a linear operator. TFAE

(a) T is continuous

(b) T is continuous at a point

(c) $\sup_{\|x\| \leq 1} \|Tx\| < \infty$ (i.e. T is a bounded operator)

(d) $\exists M$ s.t. $\|Tx\| \leq M \|x\| \quad \forall x \in \mathfrak{X}$

Proof. (a) $\Rightarrow$ (b) trivial

(b) $\Rightarrow$ (a). Suppose T is continuous at x_0. Let $x_1 \in \mathfrak{X}$ and $\varepsilon > 0$. Choose $\delta > 0$ s.t.

$$\|x - x_0\| < \delta \Rightarrow \|Tx - Tx_0\| < \varepsilon$$

If $\|x - x_1\| < \delta$, then

$$\delta > \|x - x_1\| = \|(x - x_1 + x_0) - x_0\|$$

$$\overset{\text{cont. at } x_0}{\Longrightarrow} \|T(x - x_1 + x_0) - Tx_0\| < \varepsilon$$

$$\Rightarrow \|Tx - Tx_1\| < \varepsilon$$

Hence T is continuous at x_1.

(c) $\Rightarrow$ (d). Let $\sup\limits_{\|x\| \leq 1} \|Tx\| = M$. Since $\left\| \frac{x}{\|x\|} \right\| = 1$,

$$\left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq M$$

$$\Rightarrow \|T(x)\| \leq M \|x\|$$

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Continuation of proof

(d) $\Rightarrow$ (b) (Will show continuity at 0). Given ε, take $\delta = \varepsilon/M$

(a) $\Rightarrow$ (d) By (a) T is continuous at 0, so $\exists\, \delta > 0$ s.t.

$$\|x\| < \delta \Rightarrow \|Tx\| < 1$$

If $x \neq 0$ is arbitrary in $\mathfrak{X}$, then

$$\left\| \frac{x}{\|x\|} \frac{\delta}{2} \right\| = \frac{\delta}{2} < \delta \Rightarrow \left\| T\left(\frac{x}{\|x\|} \frac{\delta}{2}\right) \right\| < 1$$

$$\Rightarrow \|Tx\| < \frac{2\|x\|}{\delta}$$

Take $M = 2/\delta$.

▨

A moment's glance shows the smallest M that works in (d) is in fact $\sup\limits_{\|x\| \leq 1} \|Tx\|$. This quantity is called the operator norm of T.

FACT: Let $\mathfrak{X}$ be a finite dimensional normed linear space. If $\dim \mathfrak{X} = n$, then $\mathfrak{X}$ is isomorphic (i.e. linearly homeomorphic) to n dimensional ℓ_1 (ℓ_1^n).

Proof. Let $x_1, \ldots, x_n$ be a basis of $\mathfrak{X}$. WLOG we take $\|x_i\| = 1$ $\forall i \le n$. Define $T : \ell_1^n \to \mathfrak{X}$ by

$$T(\alpha) := \sum_{i=1}^{n} \alpha_i x_i$$

Observe T is 1-1, onto, and linear. Also

$$\| T(\alpha) \| \le \sum_{i=1}^{n} |\alpha_i| \, \|x_i\| = \sum_{i=1}^{n} |\alpha_i| = \|\alpha\|$$

Therefore $\|T\| \le 1$. To prove T^{-1} is continuous, it is enough to find a $\delta > 0$ s.t.

$$(*) \qquad \|T\alpha\| \ge \delta \|x\| \quad \forall \alpha \in \ell_1^n$$

(since we can let $\alpha = T^{-1}x$). Suppose no such δ exists. Then there is a sequence (α^m) in ℓ_1^n such that

$$\|\alpha^m\| = 1 \qquad \|T\alpha^m\| \longrightarrow 0$$

Since closed bound sets in ℓ_1^n are compact (Heine-Borel), we can assume $\alpha_m \to \alpha \in \ell_1^n$. Then $\|\alpha\| = 1$ but $\|T\alpha\| = \lim \|T\alpha^m\| = 0$ ↯.

▨

COROLLARY: A finite dimensional subspace of a B-space is closed.

DEFINITION Let $\mathcal{X}$ be a normed linear space and $\mathcal{Y}$ be a Banach space. $B(\mathcal{X}, \mathcal{Y})$ stands for the space of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$.

Notation

$$B(\mathcal{X}) = B(\mathcal{X}, \mathcal{X})$$
$$\mathcal{X}^* = B(\mathcal{X}, \text{scalars})$$

THEOREM: Under the operator norm $B(\mathcal{X}, \mathcal{Y})$ becomes a Banach space.

Proof. That $\|\cdot\|$ satisfies the norm properties 1,2,3 is easy. Suppose (T_n) is a Cauchy sequence in $B(\mathcal{X}, \mathcal{Y})$, i.e.

$$\sup_{\|x\| \le 1} \|T_n x - T_m x\| \to 0 \text{ as } m, n \to \infty$$

Then $(T_n x)$ is a Cauchy sequence in $\mathcal{Y}$ for all $\|x\| \le 1$, and hence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, there is for each $x \in \mathcal{X}$ an element $Tx \in \mathcal{Y}$ s.t. $T_n x \to Tx$. Obviously T is linear. To see T is continuous, observe $(\|T_n\|)$ is bounded. Therefore, if $x \in \mathcal{X}$

$$\|Tx\| = \lim \|T_n x\| \le \overline{\lim} \|T_n\| \|x\| \le M\|x\|$$

for some M. Hence $T \in B(\mathcal{X}, \mathcal{Y})$. To prove $\|T_n - T\| \to 0$ as $n \to \infty$, fix x with $\|x\| \le 1$. Compute

$$\|T_n x - Tx\| \leq \|T_n x - T_m x\| + \|T_m x - Tx\| \qquad \forall x \in \mathfrak{X} \quad \forall m \in \mathbb{N}$$

Select n_0 s.t.

$$m, n \geq n_0 \Rightarrow \sup_{\|x\| \leq 1} \|T_n x - T_m x\| < \varepsilon/2$$

If $\|x\| \leq 1$, select $m(x)$ s.t. $\|T_m x - Tx\| < \varepsilon/2$ and $m > n_0$. Then for $n > n_0$,

$$\|T_n x - Tx\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Hence

$$\sup_{\|x\| \leq 1} \|T_n x - Tx\| \leq \varepsilon \quad \forall n > n_0$$

$\blacksquare$

Spaces and their duals

$$\ell_p^* = \ell_q \quad \text{for } 1 \leq p < \infty \text{ and } 1/p + 1/q = 1$$

$$L_p(\mu)^* = L_q(\mu) \quad \text{for } \sigma\text{-finite } \mu,\ 1 \leq p < \infty \text{ and } 1/p + 1/q = 1$$

$$c_0^* = \ell_1$$

$$C(K)^* = M(K) \text{ (regular finite Borel measures on } K) \quad K \text{ compact } T_2$$

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<u>THEOREM</u> (OPEN MAPPING THEOREM) Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Suppose $T : \mathcal{X} \to \mathcal{Y}$ is continuous, linear, and onto. Then T maps open sets into open sets.

Consequently, if T is also 1-1, then T^{-1} is continuous and hence T is an isomorphism.

Proof. Claim 1: $\overline{T(B_{\mathcal{X}}(\varepsilon))}$ contains an open set. This is an easy consequence of the Baire Category theorem. Since T is onto

$$\mathcal{Y} = \bigcup_{n=1}^{\infty} T(nB_{\mathcal{X}}(\varepsilon))$$

and so

$$\mathcal{Y} = \bigcup_{n=1}^{\infty} \overline{T(nB_{\mathcal{X}}(\varepsilon))}$$

By Baire category $\exists\, n_0$ s.t. $\overline{T(n_0 B_{\mathcal{X}}(\varepsilon)}$ contains a non-empty open set. Hence $\overline{T(B_{\mathcal{X}}(\varepsilon)}$ contains an open set.

Claim 2: $\forall \varepsilon > 0\ \exists \delta > 0$ such that

$$\overline{T(B_{\mathcal{X}}(\varepsilon))} \supset B_{\mathcal{Y}}(\delta)$$

Notice

$$\overline{T(B_{\mathcal{X}}(\varepsilon))} \supseteq \overline{T(B_{\mathcal{X}}(\varepsilon/2) - B_{\mathcal{X}}(\varepsilon/2))} \supset \overline{T(B_{\mathcal{X}}(\varepsilon/2))} - \overline{T(B_{\mathcal{X}}(\varepsilon/2))}$$

$\supseteq$ open set $-$ same open set $=$ open set $\supset \{0\}$

Therefore $\overline{T(B_{\mathfrak{X}}(\varepsilon))}$ contains a neighborhood of the origin.

Claim 3: $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$T(B_{\mathfrak{X}}(\varepsilon)) \supseteq B_{\mathcal{Y}}(\delta)$$

To this end fix $\varepsilon > 0$. Write $\varepsilon = \varepsilon_0$ and pick $\delta = \delta_0 > 0$ s.t.

$$\overline{T(B_{\mathfrak{X}}(\varepsilon_0))} \supseteq B_{\mathcal{Y}}(\delta_0)$$

We'll prove

$$T(B_{\mathfrak{X}}(2\varepsilon_0)) \supseteq B_{\mathcal{Y}}(\delta_0)$$

Select (ε_n) s.t. $\varepsilon_n > 0$ and $\sum \varepsilon_n < \varepsilon_0$. Choose a seq $(\delta_n) \in c_0$ s.t.

$$(*) \qquad \overline{T(B_{\mathfrak{X}}(\varepsilon_n))} \supseteq B_{\mathcal{Y}}(\delta_n)$$

Pick $y_0 \in B_{\mathcal{Y}}(\delta_0)$. By selection of δ_0, $\exists x_0 \in B_{\mathfrak{X}}(\varepsilon_0)$ such that

$$\| y_0 - Tx_0 \| < \delta_1$$

Note that $y_0 - Tx_0 \in B_{\mathcal{Y}}(\delta_1)$. Use $(*)$ to find $x_1 \in B_{\mathfrak{X}}(\varepsilon_1)$ s.t.

$$\| y_0 - Tx_0 - Tx_1 \| < \delta_2$$

Continue this procedure to get a sequence (x_n) in $\mathfrak{X}$ s.t. $\|x_n\| < \varepsilon_n$ and

$$(**) \qquad \left\| y - T\left(\sum_{k=0}^{n} x_k\right) \right\| < \delta_{n+1}$$

Since $\sum_{n=0}^{\infty} \|x_n\| \le \sum_{n=0}^{\infty} \varepsilon_n < 2\varepsilon_0$, we see that the series $\sum_{n=0}^{\infty} x_n$ converges in $\mathfrak{X}$.

Since T is continuous we must $Tx = y$ by $(**)$ and the fact that $\delta_n \to 0$.

Now let $G \subset \mathfrak{X}$ be an open set. Pick $x \in G$. Pick $\varepsilon > 0$ such that $x + B_{\mathfrak{X}}(\varepsilon) \in G$. Choose $\delta > 0$ s.t.

$$T(B_{\mathfrak{X}}(\varepsilon)) \supseteq B_{\mathcal{Y}}(\delta)$$

Then $T(G) \supseteq T(x + B_{\mathfrak{X}}(\varepsilon)) = Tx + T(B_{\mathfrak{X}}(\varepsilon)) \supseteq Tx + B_{\mathcal{Y}}(\delta)$.
Hence $T(G)$ contains a neighborhood of each of its points

$\blacksquare$

<u>COROLLARY</u>: (BANACH) Suppose $\mathfrak{X}$ is a vector space which is a B-space under two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Suppose $I : (\mathfrak{X}, \|\cdot\|_1) \to (\mathfrak{X}, \|\cdot\|_2)$ is continuous. Then $I : (\mathfrak{X}, \|\cdot\|_2) \to (\mathfrak{X}, \|\cdot\|_1)$ is continuous.

Consequently $\exists\, \alpha, \beta > 0$ s.t.

$$\alpha \|x\|_1 \le \|x\|_2 \le \beta \|x\|_1 \qquad \forall x \in \mathfrak{X}$$

DEFINITION: A linear operator $T: \mathfrak{X} \to \mathcal{Y}$ is called a closed operator if its graph is closed in the product space $\mathfrak{X} \times \mathcal{Y}$

i.e. $$\left.\begin{array}{l} x_n \to x \in \mathfrak{X} \\ Tx_n \to y \in \mathcal{Y} \end{array}\right\} \Rightarrow Tx = y$$

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THEOREM: $T : \mathfrak{X} \longrightarrow \mathcal{Y}$, T closed $\Rightarrow$ T continuous

Proof. Note $(\mathfrak{X} \times \mathcal{Y})_{\ell_1}$ is a Banach space and the graph of T is a closed subspace of $(\mathfrak{X} \times \mathcal{Y})_{\ell_1}$. Therefore $Gr(T)$ is a Banach space. The projection $P_{\mathfrak{X}} : (\mathfrak{X} \times \mathcal{Y})_{\ell_1} \longrightarrow \mathfrak{X}$ is continuous because $\|x\| \leq \|x\| + \|y\|$. Notice $P_{\mathfrak{X}}$ is 1-1, continuous and linear onto $Gr(T)$. Hence $P_{\mathfrak{X}}^{-1}$ is continuous from $\mathfrak{X}$ to $Gr(T)$ (by open mapping theorem) Also $P_{\mathcal{Y}} : (\mathfrak{X} \times \mathcal{Y})_{\ell_1} \longrightarrow \mathcal{Y}$ is continuous. But

$$Tx = P_{\mathcal{Y}} P_{\mathfrak{X}}^{-1} x$$

so T is continuous.

COROLLARY: (UNIFORM BOUNDEDNESS PRINCIPLE) Let $\mathfrak{X}$ and $\mathcal{Y}$ be B-spaces and $\{T_\lambda : \lambda \in I\}$ a collection of linear continuous operators from $\mathfrak{X}$ to $\mathcal{Y}$. Suppose for each x in $\mathfrak{X}$ we have

$$\sup_{\lambda} \|T_\lambda x\| < \infty$$

Then

$$\sup_{\lambda} \|T_\lambda\| < \infty$$

Proof. (Dunford) Define $S : \mathfrak{X} \longrightarrow \ell_\infty(I, \mathcal{Y})$ by

$$Sx(\lambda) = T_\lambda x$$

($Sx \in \ell_\infty(I, Y)$ by hypothesis). Claim: S is closed. To this end suppose

$$x_n \longrightarrow x$$
$$Sx_n \longrightarrow f \in \ell_\infty(I, Y)$$

We have to show $Sx = f$. Now

$$Sx_n(\lambda) = T_\lambda(x_n) \longrightarrow T_\lambda(x) = Sx(\lambda)$$

and so $Sx_n \longrightarrow f$ uniformly on I. Hence $Sx = f$.
By the closed graph theorem, S is continuous. But

$$\infty > \|S\| = \sup_{\|x\| \le 1} \|Sx\| = \sup_{\|x\| \le 1} \sup_{\lambda \in I} \|Sx(\lambda)\|$$

$$= \sup_{\lambda} \sup_{\|x\| \le 1} \|T_\lambda x\| = \sup_{\lambda} \|T_\lambda\|$$

$\blacksquare$

HW/ 1) p75 34-36 Note: 31-32

2) Suppose $T: L_2[0,1] \longrightarrow L_2[0,1]$ is cont but $T(L_2[0,1]) \subseteq L_\infty[0,1]$

Prove $T: L_2 \longrightarrow L_\infty$ is cont

<u>COROLLARY</u>: (BANACH - STEINHAUS THEOREM) Suppose (T_n) is a sequence of continuous operators mapping a B-space $\mathfrak{X}$ to a B-space $\mathcal{Y}$ if $\lim T_n x$ exists $\forall x$, then

① $\sup \|T_n\| < \infty$

② the operator $T: \mathfrak{X} \to \mathcal{Y}$ defined by $Tx = \lim T_n x$ is continuous.

Proof. Note $(T_n x)$ is bounded for each $x \in \mathfrak{X}$ (false for nets). By the uniform boundedness theorem, $\sup \|T_n\| < \infty$. For ②,

$$\|Tx\| = \lim \|T_n x\| \le \lim \|T_n\| \|x\| \le K\|x\|$$

where $K = \sup \|T_n\|$.

Misc. Applications

1. <u>Dunford integral</u>: Suppose (Ω, Σ, μ) is a finite measure space and suppose $f: \Omega \to \mathfrak{X}$ is such that $x^* f \in L_1(\mu)$ $\forall x^* \in \mathfrak{X}^*$. Then

$$\sup_{\|x^*\| \le 1} \|x^* f\|_1 < \infty$$

Proof: Define $S: \mathfrak{X}^* \to L_1(\mu)$ by

$$Sx^* := x^* f$$

We'll show S is closed. Let

$$x_n^* \longrightarrow x^*$$
$$Sx_n^* \longrightarrow g \in L_1(\mu)$$

Notice

$$Sx_n^* = x_n^* f \underset{\text{pointwise}}{\longrightarrow} x^* f = Sx^*$$

But $Sx_n^* \to g$ in $L_1(\mu)$ implies $Sx_{n_k}^* \xrightarrow{\text{(a.e.)}} g$ pointwise (for some subseq). Therefore $g = Sx^*$ a.e.

Hence S is continuous

$$\sup_{\|x^*\| \leq 1} \|x^* f\| = \sup_{\|x^*\| \leq 1} \|Sx^*\| = \|S\| < \infty$$

$\square$

2. <u>Existence of non-differentiable continuous function</u>: There exists $f \in C[0,1]$ such that f is not differentiable at some point

Proof. Let $L_0[0,1]$ = metric space of all measurable functions on $[0,1]$. Assume all functions in $C[0,1]$ are differentiable. Let $(h_n) \subset \mathbb{R}$ tend to 0, and define $T_n : C[0,1] \longrightarrow L_0[0,1]$ by

$$T_n f(x) = \frac{f(x+h_n) - f(x)}{h_n}$$

Then all functions differentiable implies $T_n f \longrightarrow f'$ ptwise $\forall f \in C[0,1]$.
Hence $T_n f \longrightarrow f'$ in measure.

If $L_0[0,1]$ were a Banach space, then Banach-Steinhaus would imply that the operator $f \to f'$ is continuous from $C[0,1]$ to $L_0[0,1]$. For this case, see Dunford-Schwarz. Put

$$f_n(t) = \frac{1}{n}\sin\left(\frac{nt}{2\pi}\right)$$

Then $f_n \longrightarrow 0$ in $C[0,1]$, so $f_n' \longrightarrow 0$ in measure. Therefore

$$\frac{1}{2\pi}\cos\left(\frac{nt}{2\pi}\right) \longrightarrow 0 \quad \text{in measure},$$

which is false.

3. $\exists f \in C[0,2\pi]$ s.t. the Fourier series for f does not converge to f at 0.

Proof. For $f \in C[0,2\pi]$, let

$$S_n(f)(t) = a_0 + \sum_{m=1}^{n} a_n\cos(nt) + b_n\sin(nt)$$

$$= \int_0^{2\pi} \underbrace{\frac{\sin(n+1/2)u}{\sin(u/2)}}_{\text{Dirichlet kernal } D_n} f(t+u)\,du$$

Fact $\|D_n\| = \frac{4}{\pi^2}\log(n) + O(1)$

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(Continuation of proof)

Suppose $f \in C[0,2\pi] \Rightarrow S_n f(0)$ converges. Observe that $S_n(\cdot)(0) \in C[0,2\pi]^*$ because

$$|S_n(f)(0)| = \left| \int_0^{2\pi} D_n(u) f(u)\,du \right| \le \|f\|_\infty \|D_n\|$$

Therefore $\|S_n(\cdot)(0)\| = \|D_n\|$. But $(S_n(f)(0))$ convergent $\forall f \in C[0,2\pi]$ implies

$$\sup_n |S_n f(0)| < \infty \quad \forall f \in C[0,2\pi]$$

Hence by the principle of uniform boundedness,

$$\sup_{\|f\| \le 1} \sup_n |S_n f(0)| < \infty$$

$$\Rightarrow \sup_n \|S_n(\cdot)(0)\| < \infty$$

$$\Rightarrow \sup_n \|D_n\|_1 < \infty \quad \lightning$$

$\boxtimes$

④ Rate of convergence of Fourier coefficients. Work in $L_1[-\pi,\pi]$

Put

$$\hat{f}(n) := \int_{-\pi}^{\pi} f(x) e^{inx}\, dx$$

for all $n \in \mathbb{Z}$. By Riemann-Lebesgue lemma, $\lim_{|n|\to\infty} \hat{f}(n) = 0$

<u>THEOREM</u>: Let (a_k) be any sequence of positive reals in c_0. Then for any subsequence (n_k), there exists $f \in L_1[-\pi,\pi]$ s.t.

$$\sup_k \frac{|\hat{f}(n_k)|}{a_k} = +\infty$$

Proof. Suppose not, i.e. $\exists\ (a_k)$ and (n_k) s.t.

$$(*) \qquad \sup_k \frac{|\hat{f}(n_k)|}{a_k} < \infty$$

for every $f \in L_1[-\pi,\pi]$. Define linear functionals $l_k \in L_1[-\pi,\pi]^*$ by

$$l_k(f) = \frac{\hat{f}(n_k)}{a_k} \quad \forall f \in L_1[-\pi,\pi]$$

The l_k's are continuous because they arise as integration against L_∞ functions. By $(*)$ and the uniform boundedness principle, we see that

$$\sup_k \| l_k \| < \infty$$

i.e.

$$\sup_{\|f\|\le 1} \sup_k \left| \frac{\hat{f}(n_k)}{a_k} \right| < \infty$$

To contradict this consider the sequence $f_m = \frac{m}{\pi} \chi_{[0, \frac{\pi}{m}]}$. Observe that $\|f_m\|_1 = 1$. But now

$$\ell_k(f_m) = \frac{\hat{f}_m(n_k)}{a_k} = \frac{\int_{-\pi}^{\pi} f_m e^{in_k t}\, dt}{a_k}$$

Note $|\hat{f}(n_k)| \le M \|f\| a_k$. Then

$$\hat{f}_m(n_k) = \int_{-\pi}^{\pi} f_m e^{in_k t}\, dt = \frac{m}{\pi} \int_0^{\pi/m} e^{in_k t}\, dt$$

$$= \frac{1}{2\pi} \frac{\sin\left(\frac{n_k}{m}\pi\right)}{n_k/m} - \frac{i}{2\pi} \frac{\cos\left(\frac{n_k}{m}\pi\right) - 1}{\frac{n_k \pi}{m}}$$

$$\xrightarrow[m\to\infty]{} 1/2\pi + 0$$

But $\lim_k |\hat{f}(n_k)| = 0$ unif in $\|f\|_1 \le 1$

$\boxtimes$

Question: Does the operator $T : f \longrightarrow (\hat{f}(n))$ from $L_1[-\pi, \pi]$ to c_0 set up an isomorphism?

① T is continuous

$$\|Tf\|_{c_0} = \sup_n |\hat{f}(n)| = \sup_n \left| \int_{-\pi}^{\pi} f(t) e^{int} dt \right|$$

$$\leq \sup_n \|f\|_1 \|e^{int}\|_\infty = \|f\|_1$$

In fact $\|T\| = 1$.

② T is not onto. If it is onto, then by open mapping its inverse is continuous. Hence $L_1[-\pi,\pi]$ is isomorphic to c_0. But then $L_1[-\pi,\pi]^*$ is isomorphic to c_0^*, i.e. $L_\infty[-\pi,\pi]$ is isomorphic to ℓ_1. However ℓ_1 is separable while $L_\infty[-\pi,\pi]$ is not separable.

⑤ Differential equations Recall $C^2[0,1]$. Let $C_0^2[0,1]$ be the subspace of functions that vanish at 0 and 1. Suppose a_0, a_1, a_2 are functions s.t. for any $f \in C[0,1]$, the differential equation

$$a_0 y'' + a_1 y' + a_2 y = f$$

has a unique solution in $C_0^2[0,1]$. Then it follows that the solution is a continuous function of the forcing function. To see why, define

$$T : C_0^2[0,1] \longrightarrow C^2[0,1]$$

by $Ty := a_0 y'' + a_1 y' + a_2 y$. Then T is 1-1 and onto. Also note that T is continuous because

$$\|Ty\|_C = \sup_t |a_0(t)y''(t) + a_1(t)y'(t) + a_2(t)y(t)|$$

$$\leq M\left(\sup_t |y''(t)| + \sup_t |y'(t)| + \sup_t |y(t)|\right)$$

[where $M = \sup_t \left(|a_0(t)| + |a_1(t)| + |a_2(t)|\right)$]

$$\leq M \|y\|_{C_0^2[0,1]}$$

Therefore T^{-1} is continuous

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DEFINITION: A Banach space $\mathfrak{X}$ has a (Schauder) basis if there exists a sequence (x_n) in $\mathfrak{X}$ s.t. every x in $\mathfrak{X}$ admits a unique expansion

$$x = \sum_{n=1}^{\infty} \alpha_n x_n$$

(norm convergence)

Examples: (1) Complete orthonormal systems in separable Hilbert spaces.
(2) The unit vector basis of ℓ_p $(1 \le p < \infty)$ or c_0

These examples have the property that if $\sum \alpha_n x_n$ is convergent, then so is any subseries. These are called unconditional bases.

In c,

$$(x_n) = ((0, 0, \ldots, 0, 1, 1, 1, \ldots))$$

$\uparrow$ n^{th} slot

is a conditional basis.

THEOREM: Let $\mathfrak{X}$ have a basis (x_n). Set

$$|||x||| := \sup_n \left\| \sum_{k=1}^{n} \alpha_k x_k \right\|$$

for each x. Then $|||\cdot|||$ and $\|\cdot\|$ are equivalent norms in $\mathfrak{X}$.

Proof. Consider $T : (\mathfrak{X}, |||\cdot|||) \longrightarrow (\mathfrak{X}, \|\cdot\|)$ the map $Tx = x$.

Want to show T is a homeomorphism. T is linear, continuous because

$$|||x||| \geq \|x\|$$

Obviously T is 1-1 and onto. By open mapping theorem we will know T^{-1} is continuous once we know $(\mathfrak{X}, |||\cdot|||)$ is a Banach space. Let (y_m) be a $|||\cdot|||$ - Cauchy sequence in $\mathfrak{X}$. Suppose

$$y_m = \sum_{i=1}^{\infty} \alpha_i^m x_i$$

Fix i_0. Then $(\alpha_{i_0}^m)$ is a convergent sequence. For let $\varepsilon > 0$ and choose m_ε s.t.

$$m, n > m_\varepsilon \Rightarrow |||y_m - y_n||| < \varepsilon$$

Notice

$$|\alpha_{i_0}^m - \alpha_{i_0}^n| \, \|x_{i_0}\| = \left\| \sum_{i=1}^{i_0} \alpha_i^m x_i - \sum_{i=1}^{i_0} \alpha_i^n x_i - \sum_{i=1}^{i_0-1} \alpha_i^m x_i + \sum_{i=1}^{i_0-1} \alpha_i^n x_i \right\|$$

$$\leq |||y_m - y_n||| + |||y_m - y_n||| < 2\varepsilon$$

Hence $\lim \alpha_{i_0}^m = \alpha_{i_0}$ exists.

Rest of proof

① Prove $\sum \alpha_i x_i$ converges to x in $\mathfrak{X}$

② Prove $|||y_n - x||| \to 0$

To these ends, observe that every $p \in \mathbb{N}$

$$\left\| \sum_{i=k}^{k+p} \alpha_i^m x_i - \sum_{i=k}^{k+p} \alpha_i^n x_i \right\| \leq 2\varepsilon \quad \text{for } n,m \geq m_\varepsilon$$

Let $n \to \infty$,

$$\left\| \sum_{i=k}^{k+p} \alpha_i^m x_i - \sum_{i=k}^{k+p} \alpha_i x_i \right\| \leq 2\varepsilon \quad \text{for } m \geq m_\varepsilon \qquad (**)$$

$$\Rightarrow \left| \left\| \sum_{i=k}^{k+p} \alpha_i^m x_i \right\| - \left\| \sum_{i=k}^{k+p} \alpha_i x_i \right\| \right| < 2\varepsilon \quad \text{for } m \geq m_\varepsilon$$

Hence $\sum_{i=1}^{r} \alpha_i x_i$ is a Cauchy sequence, so $\sum \alpha_i x_i$ converges, and we also see that $(**)$ implies

$$\sup_p \left\| \sum_1^{1+p} \alpha_i^m x_i - \sum_1^{1+p} \alpha_i x_i \right\| \leq 2\varepsilon \quad \text{for } m \geq m_\varepsilon$$

$$\Rightarrow |||y_m - x||| \leq 2\varepsilon \quad \text{for } m \geq m_\varepsilon$$

$\blacksquare$

<u>COROLLARY</u>: Let $\mathfrak{X}$ have a basis (x_n), i.e.

$$x \in \mathfrak{X} \Rightarrow x = \sum_{i=1}^{\infty} \alpha_i(x) x_i = x$$

for uniquely determined $\alpha_i(x)$ in scalars. Then $\alpha_i(\cdot) \in \mathfrak{X}^* \; \forall i$

Proof. Fix i_0. Obviously $\alpha_{i_0}(\cdot)$ is linear. Now

$$|\alpha_{i_0}(x)| \, \|x_{i_0}\| = \| \alpha_{i_0}(x) x_{i_0} \| \leq \| \sum_{i=1}^{i_0} \alpha_i(x) x_i \| + \| \sum_{i=1}^{i_0-1} \alpha_i(x) x_i \|$$

$$\leq 2 |||x||| \leq 2K \|x\|$$

(where $K = \|T^{-1}\|$) Therefore

$$|\alpha_{i_0}(x)| \leq \frac{2K}{\|x_{i_0}\|} \|x\|$$

and so $\alpha_i(\cdot) \in \mathfrak{X}^*$.

$\blacksquare$

Corollary: If $\mathfrak{X}$ has a basis (x_n), define $P_m : \mathfrak{X} \to \mathfrak{X}$ by

$$P_m\left(\sum_{i=1}^{\infty} \alpha_i x_i \right) = \sum_{i=1}^{m} \alpha_i x_i$$

Then $\sup \|P_m\| = M < \infty$, and moreover

$$\| \sum_{i=1}^{n} \alpha_i x_i \| \leq M \| \sum_{i=1}^{n+p} \alpha_i x_i \| \quad \forall n, p \in \mathbb{N} \ \forall \text{scalars } \alpha_i$$

Proof. Note

$$\| P_m \left(\sum_{i=1}^{\infty} \alpha_i x_i \right) \| = \| \sum_{i=1}^{m} \alpha_i x_i \| \leq ||| \sum_{i=1}^{\infty} \alpha_i x_i |||$$

$$\leq K \| \sum_{i=1}^{\infty} \alpha_i x_i \|$$

Therefore $\sup_m \|P_m\| \le K$. To get the last statement observe

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| = \left\| P_n \left(\sum_{i=1}^{n+p} \alpha_i x_i \right) \right\| \le \|P_n\| \left\| \sum_{i=1}^{n+p} \alpha_i x_i \right\|$$

$$\le M \left\| \sum_{i=1}^{n+p} \alpha_i x_i \right\|$$

▨

THEOREM: Let $\mathfrak{X}$ be a Banach space. Let (x_n) be a sequence in $\mathfrak{X}$. Then (x_n) is a basis for $\mathfrak{X}$ iff

① no x_n is zero

② $\overline{sp}\{(x_n)\} = \mathfrak{X}$

③ $\exists M$ s.t. $\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \le M \left\| \sum_{i=1}^{n+p} \alpha_i x_i \right\|$ $\forall n, p \in \mathbb{N}$, $\forall$ scalars α_i

Proof. ($\Rightarrow$) already done

($\Leftarrow$) Uniqueness of expansion. If $\sum_{i=1}^{\infty} \alpha_i x_i = 0$, want to show all the α_i's are zero. By (3),

$$\|\alpha_1 x_1\| \le M \left\| \sum_{i=1}^{1+p} \alpha_i x_i \right\| \xrightarrow[p\to\infty]{} 0$$

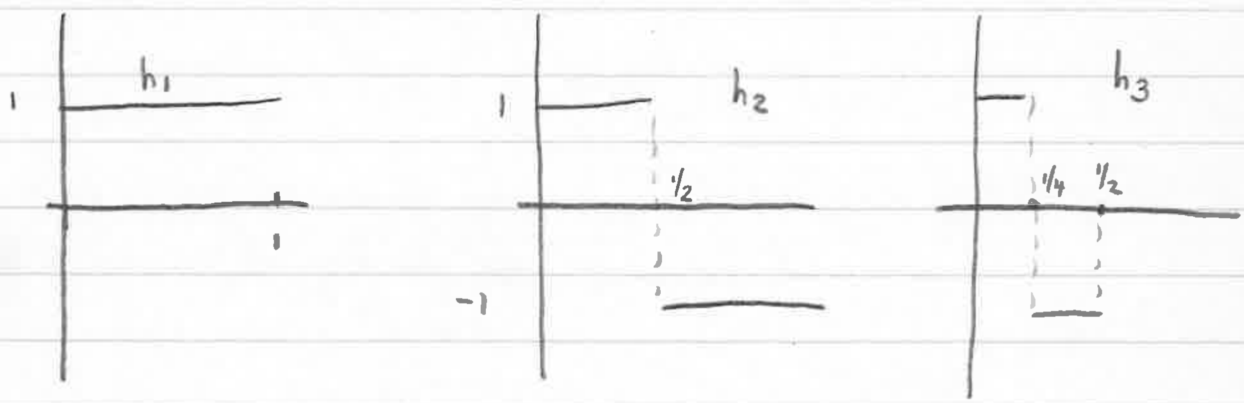
$$\Rightarrow \|\alpha_1 x_1\| = 0 \underset{\text{by ①}}{\Rightarrow} \alpha_1 = 0$$

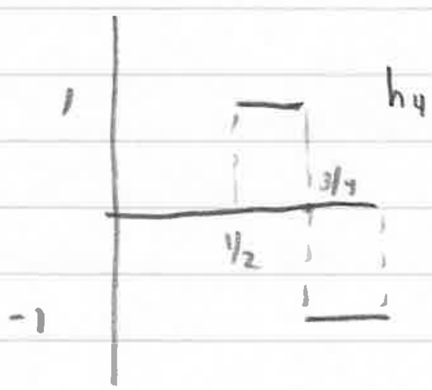
Similarly $\alpha_2 = 0$, $\alpha_3 = 0$, etc.

9/12 BANACH SPACES

DEEPER EXAMPLES OF BASES

1) Haar system is a basis for every $L_p[0,1]$ $1 \le p < \infty$





$$h_{2^k+\ell}(t) = \begin{cases} +1 & t \in \left[(2\ell-2)2^{-k-1}, (2\ell-1)2^{-k-1}\right) \\ -1 & t \in \left[(2\ell-1)2^{-k-1}, 2\ell\, 2^{-k-1}\right) \end{cases}$$

$k \in \mathbb{N}$

$\ell = 1, 2, \ldots, 2^k$

Why is this a basis

① no $h_i = 0$

② $\overline{sp}(h_n)$ is dense because all functions of the form χ_I for I dyadic are in the linear span of (h_n)

③ Let

$$f = \sum_{i=1}^{n} a_i h_i \qquad g = \sum_{i=1}^{n+1} a_i h_i$$

We'll show $\|f\|_p \le \|g\|_p$. This (by induction) will get the $M=1$ in part (3) of last theorem

One proof - use conditional expectation and Jensen's inequal (Martingales)
2ND proof - Observe that $f=g$ except on a certain interval I

$$\|f\|_p^p = \int_{[0,1)\setminus I} |f|^p \, d\mu + \int_I |f|^p \, d\mu \qquad 1 \le p < \infty$$

$$\|g\|_p^p = \int_{[0,1)\setminus I} |f|^p \, d\mu + \int_I |g|^p \, d\mu \qquad 1 \le p < \infty$$

On I f is constant, i.e. $f\chi_I = b\chi_I$ for some b. But on I,

$$g = \begin{cases} b + a_{n+1} & \text{on } 1^{st} \text{ half of } I \\ b - a_{n+1} & \text{on } 2^{ND} \text{ half of } I \end{cases}$$

Hence

$$\int_I |f|^p \, d\mu = |b|^p \mu(I)$$

$$\int_I |g|^p \, d\mu = |b + a_{n+1}|^p \frac{\mu(I)}{2} + |b - a_{n+1}|^p \frac{\mu(I)}{2}$$

But $|b+x|^p + |b-x|^p \ge 2|b|^p$, so

$$\int_I |g|^p \, d\mu \ge 2|b|^p \frac{\mu(I)}{2} = |b|^p \mu(I)$$

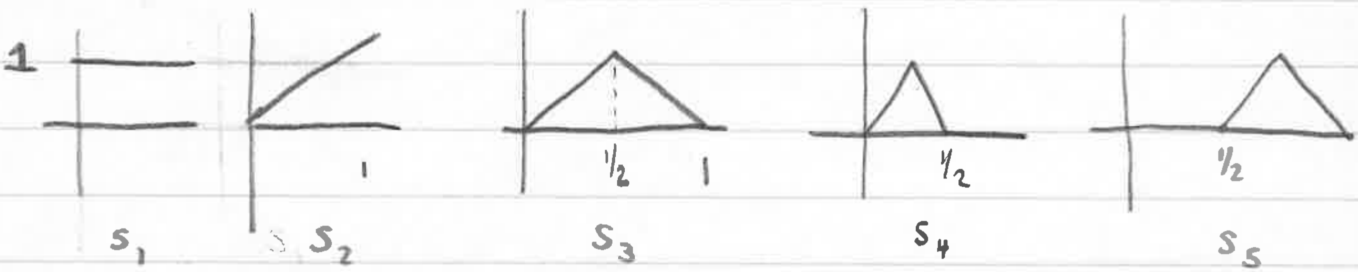
and so

$$\|f\|_p^p \leq \|g\|_p^p$$

$\blacksquare$

HW/ (The Original Schauder basis) A basis for $C[0,1]$

Let $s_1 = 1$ $\quad s_n(t) = \int_0^t h_{n-1}(s)\,ds$



Prove that (s_n) is a basis for $C[0,1]$.

hints: ① linear span of (s_n) = piecewise linear functions with corners at dyadic rationals

② $s_1, \ldots, s_n$ are all linear on $[s_{n+1} \neq 0]$

<u>Another look at basis</u>

Khintchine inequality: Let $r_n(t) := \operatorname{sgn}(\sin(2^n \pi t))$ $\quad 0 \leq t < 1$

(Radamacher functions)

$1 \leq p < \infty \implies \exists$ constants A_p and B_p s.t. $\forall (\alpha_n) \in \ell_2$

$$A_p \left(\sum \alpha_n^2 \right)^{1/2} \leq \left(\int \left| \sum \alpha_n r_n \right|^p d\mu \right)^{1/p} \leq B_p \left(\sum \alpha_n^2 \right)^{1/2}$$

This says that if $\mathfrak{X}_p = \overline{sp}(r_n)$ in $L_p[0,1]$, then $\mathfrak{X}_p$ is isomorphic to ℓ_2 and (r_n) is a basis of $\mathfrak{X}_p$

[Open Problem: Does every B-space have an infinite dimensional subspace with unconditional basis?]

QUOTIENT SPACES

Let $\mathfrak{X}$ be a B-space and let $\mathcal{Y}$ be a (closed) subspace of $\mathfrak{X}$. Let $\mathfrak{X}/\mathcal{Y}$ be the space of cosets. For $x \in \mathfrak{X}$, take $\hat{x} \in \mathfrak{X}/\mathcal{Y}$ and define

$$\|\hat{x}\|_{\mathfrak{X}/\mathcal{Y}} := \inf_{y \in \mathcal{Y}} \|x + y\|_{\mathfrak{X}}$$

THEOREM: $\mathfrak{X}/\mathcal{Y}$ is B-space under the quotient norm.

Proof. ① $\|\cdot\| \geq 0$ and $\|x\| = 0$ iff $x = 0$ coset (since $\mathcal{Y}$ closed)

② $\|\alpha \hat{x}\| = \inf_{y \in \mathcal{Y}} \|\alpha x + y\| = \inf_{y \in \mathcal{Y}} \|\alpha(x+y)\| = |\alpha| \inf_{y \in \mathcal{Y}} \|x+y\|$

③ $$\|\hat{x}_1 + \hat{x}_2\| = \inf_{y_1, y_2 \in \mathcal{Y}} \|x_1 + y_1 + x_2 + y_2\|$$
$$\leq \inf_{y_1, y_2 \in \mathcal{Y}} \left(\|x_1 + y_1\| + \|x_2 + y_2\| \right)$$

$$= \inf_{y_1 \in \mathcal{Y}} \|x_1 + y_1\| + \inf_{y_2 \in \mathcal{Y}} \|x_2 + y_2\|$$

$$= \|\hat{x}_1\| + \|\hat{y}_1\|$$

④ Completeness. Let $(\hat{x}_n)$ be a Cauchy sequence in $\mathcal{X}/\mathcal{Y}$. If we can show (x_n) has a convergent subsequence, we'll be done. WLOG, our original Cauchy seq. satisfies

$$\| \hat{x}_{k+1} - \hat{x}_k \| \leq 1/2^k$$

Select $w_k \in \hat{x}_{k+1} - \hat{x}_k$ s.t. $\|w_k\|_{\mathcal{X}} < 1/2^k$. Pick $u_1 \in \hat{x}_1$ arbitrarily. Pick $u_2 \in \hat{x}_2$ s.t. $w_1 = u_2 - u_1$. Pick $u_3 \in \hat{x}_3$ s.t. $w_2 = u_3 - u_2$. Pick $u_{n+1} \in \hat{x}_{n+1}$ s.t. $w_n = u_{n+1} - u_n$.

Claim: (u_n) is Cauchy in $\mathcal{X}$ and if $\lim u_n = x$, then $\lim \| \hat{x}_n - \hat{x} \| = 0$

To this end, take $m < n$.

$$\| u_n - u_m \| = \| u_n - u_{n-1} + u_{n-1} - u_{n-2} \cdots - u_m \|$$

$$\leq \| u_n - u_{n-1} \| + \| u_{n-1} - u_{n-2} \| + \cdots + \| u_{m+1} - u_m \|$$

$$\leq \sum_{j=m}^{\infty} 2^{-j} \longrightarrow 0 \text{ as } m, n \to \infty$$

Hence (u_n) is Cauchy in $\mathcal{X}$ and hence converges. Moreover

$$\|\hat{x} - \hat{x}_n\| \leq \|x - u_n\| \longrightarrow 0 \quad \text{so } \lim \hat{x}_n = \hat{x}$$

HW/ DS 72 #15, 18

9/14 BANACH SPACES

COROLLARY: Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are B-spaces and $T : \mathfrak{X} \to \mathfrak{Y}$ is a bounded linear operator s.t. $T(\mathfrak{X}) = \mathfrak{Y}$. Then T is isomorphic to $\mathfrak{X}/_{\ker T}$

Proof. Let $B = \ker T$. Then B is closed. Define $\tilde{T} : \mathfrak{X}/_B \to \mathfrak{Y}$ by

$$\tilde{T}(\hat{x}) = Tx$$

Then $\tilde{T}$ is well defined, linear, and continuous since

$$\|\tilde{T}(\hat{x})\| = \|Tx\| \underset{\forall y \in B}{=} \|T(x+y)\| \underset{\forall y \in B}{\leq} \|T\| \|x+y\|$$

Hence $\|\tilde{T}(\hat{x})\| \leq \|T\| \inf_{y \in B} \|x+y\| = \|T\| \|\hat{x}\|$ ▨

HW/ p72 15, 18

DUAL SPACES

THEOREM: (Nikodym) Let $L_0[0,1]$ be the usual metric space of measurable functions on $[0,1]$ under the metric

$$\rho(f,g) = \int_0^1 \frac{|f-g|}{1+|f|+|g|} \, d\mu$$

whose convergent sequence and nets are precisely those that converge in measure. Then if $\ell : L_0 \to \mathbb{R}$ is continuous and linear, then $\ell = 0$

Proof. Suppose $\ell : L_0 \to \mathbb{R}$ is continuous, linear and non-zero. Since L_1 convergence $\Rightarrow$ L_0 convergence, we see that $\ell : L_1 \to \mathbb{R}$ is continuous and linear. Hence $\exists\, g \in L_\infty$ s.t.

$$\ell(f) = \int f g \, d\mu \quad \forall f \in L_1$$

Since L_1 is dense in L_0 under the L_0-topology, and since $\ell \neq 0$, we see that $\|g\|_\infty \neq 0$.

WLOG $\exists\, \alpha > 0$ s.t. $[g > \alpha]$ is of positive Lebesgue measure. Take a function φ in L_0 s.t. φ vanishes outside $[g>\alpha]$ and $\varphi \geq 0$ and $\varphi \notin L_1$. Then

$$\ell(\varphi) = \lim_n \ell(\varphi \overset{\text{inf}}{\wedge} n \chi_{[g>\alpha]})$$

$$= \lim_n \int_{[g>\alpha]} (\varphi \wedge n) g \, d\mu$$

$$\geq \lim \int_{[g>\alpha]} (\varphi \wedge n)\alpha \, d\mu$$

By monotone convergence $\varphi \chi_{[g>\alpha]} = \varphi \in L_1 (\mu)$.

$\blacksquare$

It turns out that B-spaces have rich collections of non-zero continuous linear functionals. This is a consequence of:

<u>THEOREM</u> (HAHN-BANACH) Let $\mathfrak{X}$ be a real vector space. Suppose p is a function from $\mathfrak{X}$ to $\mathbb{R}$ s.t.

① $p(x+y) \leq p(x) + p(y)$

② $p(\alpha x) = \alpha p(x) \quad \forall \alpha \geq 0$

Suppose $\mathcal{Y}$ is a (vector) subspace of $\mathfrak{X}$ and ℓ is a linear functional on $\mathcal{Y}$ that satisfies $\ell(y) \leq p(y) \quad \forall y \in \mathcal{Y}$. Then $\exists$ a linear functional L on $\mathfrak{X}$ s.t.

$$L|_{\mathcal{Y}} = \ell$$

$$L(x) \leq p(x) \quad \forall x \in \mathfrak{X}$$

Proof. Let $\mathcal{E}$ be the family of all legal extensions of ℓ to subspaces $\mathcal{Z} \supseteq \mathcal{Y}$. Partially order $\mathcal{E}$ by saying $f > g$ if f is an extension of g. Zornicate to produce a maximal element L of $\mathcal{E}$. Let $\mathcal{Y}_0$ = domain of L. We'll be done if we can show $\mathcal{Y}_0 = \mathfrak{X}$.

To this end, suppose $\exists\, y_1 \in \mathfrak{X} \setminus \mathcal{Y}_0$. Put $\mathcal{Y}_1 = \mathrm{sp}(\mathcal{Y}_0 \cup \{y_1\})$

Define C on $\mathcal{Y}_1$ by

$$C(y+\alpha y_1) = L(y) + \alpha d$$

for $y \in \mathcal{Y}_0$, $\alpha \in \mathbb{R}$. Then C is a linear extension of L. We must specify d in such a way that $C \in \mathcal{E}$, which will contradict the maximality of L. If $x, y \in \mathcal{Y}_0$, then

$$L(x) - L(y) = L(x-y) \le p(x-y) \le p(x+y_1) + p(-y_1-y)$$

Hence $x, y \in \mathcal{Y}_0 \Rightarrow$

$$\underbrace{-p(-y_1-y) - L(y)}_{\text{independent of } x} \le \underbrace{p(x+y_1) - L(x)}_{\text{independent of } y}$$

It follows that $\exists$ a $d \in \mathbb{R}$ s.t.

$$-p(-y_1-y) - L(y) \le d \le p(x+y_1) - L(x)$$

for all $x, y \in \mathcal{Y}_0$. Now if $\alpha = 0$

$$C(y+\alpha y_1) = L(y) + \alpha d = L(y) \le p(y) = p(y+\alpha y_1)$$

If $\alpha > 0$, then we know $d \le p\left(\frac{x}{\alpha} + y_1\right) - L\left(\frac{x}{\alpha}\right)$ $\forall x \in \mathcal{Y}_0$
Hence $\alpha d \le p(x+\alpha y_1) - L(x)$. Hence

$$C(x+\alpha y_1) = L(x) + \alpha d \leq L(x) + \rho(x+\alpha y_1) - L(x)$$

$$= \rho(x+\alpha y_1)$$

If $\alpha < 0$,

$$-\rho(-y_1 - y/\alpha) - L(y/\alpha) \leq d$$

$$\Rightarrow -\alpha\rho(-y_1 - y/\alpha) - L(y) \geq \alpha d$$

because $\alpha < 0$

$$\Rightarrow \rho(\alpha y_1 + y) - L(y) \geq \alpha d$$

because $-\alpha \geq 0$

Hence $C(y+\alpha y_1) = L(y) + \alpha d \leq \rho(y+\alpha y_1)$

$\blacksquare$

9/17 BANACH SPACES

<u>COROLLARY</u>: (Analytic form of H-B) Let $\mathfrak{X}$ be a B-space and $\mathcal{Y}$ a subspace of $\mathfrak{X}$. If $y^* \in \mathcal{Y}^*$, then $\exists\, x^* \in \mathfrak{X}^*$ s.t.

$$x^*|_{\mathcal{Y}} = y^*$$

$$\|y^*\| = \|x^*\|$$

Proof (Real scalars) Take $y^* \in \mathcal{Y}^*$ and define

$$p(x) = \|y^*\|\,\|x\|$$

Notice $y^*(y) \leq p(y)$ $\forall y \in \mathcal{Y}$. Apply Hahn-Banach to get a linear functional ℓ on $\mathfrak{X}$ s.t.

$$\ell(x) \leq p(x) = \|y^*\|\,\|x\| \quad \forall x \in \mathfrak{X}$$

and $\ell|_{\mathcal{Y}} = y^*$. Then for $x \in \mathfrak{X}$,

$$|\ell(x)| = \ell(\operatorname{sgn} x \cdot x) \leq p(\operatorname{sgn} x \cdot x) = \|y^*\|\,\|\operatorname{sgn} x \cdot x\|$$

$$= \|y^*\|\,\|x\|$$

Hence $|\ell(x)| \leq \|y^*\|\,\|x\|$, and so $\ell \in \mathfrak{X}^*$ and $\|\ell\| \leq \|y^*\|$.
But

$$\|\ell\| = \sup_{\|x\|\le 1} |\ell(x)| \geq \sup_{\substack{\|y\|\le 1 \\ y\in\mathcal{Y}}} |\ell(y)| = \sup_{\substack{\|y\|\le 1 \\ y\in\mathcal{Y}}} |y^*(y)| = \|y^*\|$$

$$\Rightarrow \|\ell\| \geq \|y^*\|$$

Therefore $\|\ell\| = \|y^*\|$ $\blacksquare$

Complex scalars - Theorem true due to a trick found in D-S (Sobczyk-Bohnenblust 1938)

Main consequences of Hahn-Banach (analytic form)

(These are "all" due to Banach)

COROLLARY 1: Let $\mathfrak{X}$ be a B-space and $\mathcal{Y}$ a closed subspace of $\mathfrak{X}$. If $x \in \mathfrak{X}\setminus\mathcal{Y}$, i.e.

$$d = \inf_{y\in\mathcal{Y}} \|x-y\| > 0$$

then $\exists\, x^* \in \mathfrak{X}^*$ s.t. $\|x^*\| = 1$ and $x^* \in \mathcal{Y}^\perp$, $x^*(x) = d$

Consequently, $\mathfrak{X}^*$ is rich enough to separate the points of $\mathfrak{X}$, i.e. if $x, y \in \mathfrak{X}$ and $x \neq y$, then $\exists\, x^* \in \mathfrak{X}^*$ s.t. $x^*(x) \neq x^*(y)$

Proof. Second statement follows from the first. Why? If x and y are linearly independent, apply first statement to x and

$$\mathcal{Y} = \overline{sp}\{y\}$$

If $x \in \overline{sp}\{y\}$, define l on $\overline{sp}\{x\}$ by $l(\alpha y) = \alpha\|y\|$. Then l is continuous and $l(x) \neq l(y)$. Take continuous extension of l to all of $\mathcal{X}$.

<u>1st statement</u>: Set

$$\mathcal{Y}_0 = sp\,(\mathcal{Y} + \{x\})$$

Define z^* on $\mathcal{Y}_0$ by $z^*(y+\alpha x) = \alpha d$. Then z^* is linear and $z^* \in \mathcal{Y}^\perp$. Observe

$$\|y+\alpha x\| = |\alpha|\,\left\|\frac{y}{\alpha} + x\right\| \geq |\alpha| d$$

for all $y \in \mathcal{Y}$. Therefore

$$|z^*(y+\alpha x)| = |\alpha| d \leq \|y + \alpha x\|$$

and so $\|z^*\| \leq 1$. Let x^* be any Hahn-Banach extension. Then $x^* \in \mathcal{Y}^\perp$ because it extends a member of $\mathcal{Y}^\perp$. Also $\|x^*\| \leq 1$ and $x^*(x) = d$.

To prove $\|x^*\| = 1$, take a sequence (y_n) in $\mathcal{Y}$ s.t. $\|y_n - x\| \to d$. Then

$$d = x^*(x) = x^*(x - y_n) \leq \|x^*\|\,\|x - y_n\| \to \|x^*\| d$$

Hence $1 \leq \|x^*\|$. $\square$

[Weak topology: We say a net (x_α) in $\mathfrak{X}$ converges weakly to x in $\mathfrak{X}$ if

$$\lim x^*(x_\alpha) = x^*(x) \quad \forall x^* \in \mathfrak{X}^*$$]

<u>COROLLARY 2</u>: Let $\mathcal{Y}$ be a subspace of $\mathfrak{X}$. Then $\mathcal{Y}$ is norm closed if and only if $\mathcal{Y}$ is weakly closed.

Proof: Norm convergence implies weak convergence. Hence weakly closed implies norm closed.

Now suppose $\mathcal{Y}$ is norm closed and $x \in \overline{\text{weak}}(\mathcal{Y}) \setminus \mathcal{Y}$. Then $\exists$ a net (x_α) in $\mathcal{Y}$ s.t. $\lim x_\alpha = x$ weakly, i.e.

$$\lim_\alpha x^*(x_\alpha) = x^*(x) \quad \forall x^* \in \mathfrak{X}^*$$

But by corollary 1, $\exists x_0^* \in \mathfrak{X}^*$ s.t. $x_0^*(x) \neq 0$ and $x_0^*(\mathcal{Y}) = 0$. Hence

$$0 \neq x_0^*(x) = \lim_\alpha x^*(x_\alpha) = 0 \quad \lightning$$

$\blacksquare$

<u>COROLLARY 3</u>: Let $x \in \mathfrak{X}$. Then $\exists x^* \in \mathfrak{X}^*$ s.t. $x^*(x) = \|x\|$ and $\|x^*\| = 1$.

Proof. Take the first corollary with $\mathcal{Y} = \{0\}$. Then $d = \|x\|$.

COROLLARY 4: $x \in \mathfrak{X} \Rightarrow \|x\| = \sup_{\|x^*\| \leq 1} |x^*(x)|$ and the sup is attained

Proof. See corollary 3, remembering that

$$\|x^*\| \leq 1 \Rightarrow |x^*(x)| \leq \|x^*\| \|x\| \leq \|x\|$$

COROLLARY 5: $\mathfrak{X}$ is isometric to a subspace of $\mathfrak{X}^{**}$ under a canonical mapping $Q : \mathfrak{X} \to \mathfrak{X}^{**}$

Proof. Define $Q : \mathfrak{X} \to \mathfrak{X}^{**}$ by

$$Qx(x^*) := x^*(x)$$

Then Q is linear. Now

$$\|Qx\| = \sup_{\|x^*\| \leq 1} |Qx(x^*)| = \sup_{\|x^*\| \leq 1} |x^*(x)| = \|x\|$$

Therefore Q is a linear isometry, so Q is 1-1. (Usually we just regard $\mathfrak{X}$ as a closed subspace of $\mathfrak{X}^{**}$)

$\boxtimes$

DEFINITION: If $Q(\mathfrak{X}) = \mathfrak{X}^{**}$, then we say $\mathfrak{X}$ is reflexive

Warning: $\mathfrak{X}$ can be linearly isometric to $\mathfrak{X}^{**}$ without being reflexive (See James' example)

9/19 BANACH SPACES

HW p72 19,20

$$\left(\sum_{n=1}^{\infty} \mathfrak{X}_n\right)^*_{\ell_p} = \left(\sum_{n=1}^{\infty} \mathfrak{X}_n^*\right)_{\ell_q}$$

COROLLARY: Let (x_α) be a family in $\mathfrak{X}$ s.t.

$$\sup_\alpha |x^*(x_\alpha)| < \infty \quad \forall x^* \in \mathfrak{X}^*$$

Then $\sup \|x_\alpha\| < \infty$.

PROOF. Observe

$$\sup |x^*(x_\alpha)| < \infty \Rightarrow \sup_\alpha |(Qx_\alpha)x^*| < \infty \Rightarrow \sup_\alpha \|Qx_\alpha\| < \infty$$

$$\Rightarrow \sup \|x_\alpha\| < \infty$$

uniform bddness

COROLLARY: (Banach-Steinhaus) Let $\{T_\alpha : \alpha \in A\}$ be a family of bounded linear operators from $\mathfrak{X}$ to $\mathfrak{Y}$. TFAE

(a) $\sup \|T_\alpha\| < \infty$

(b) $\sup \|T_\alpha x\| < \infty \quad \forall x \in \mathfrak{X}$

(c) $\sup \|y^* T_\alpha x\| < \infty \quad \forall x \in \mathfrak{X} \quad \forall y^* \in \mathfrak{Y}^*$

Proof. $a \Leftrightarrow b$ follows from uniform boundedness principle

$b \Rightarrow c$ clear

$c \Rightarrow b$ is above corollary with y^* in place of x^*

$\blacksquare$

<u>THEOREM</u>: $\mathfrak{X}^*$ separable $\Rightarrow$ $\mathfrak{X}$ separable

Proof. Let (x_n^*) be a dense sequence in $\mathfrak{X}^*$ with $\|x_n^*\| = 1$. Choose a sequence (x_n) in $\mathfrak{X}$ s.t.

$$|x_n^*(x_n)| \geq \frac{\|x_n^*\|}{2} = \frac{1}{2}$$

and $\|x_n\| \leq 1$. Claim: $\mathfrak{X} = \overline{sp}\{x_n : n \in \mathbb{N}\}$. Suppose not. Then $\exists\, x \in \mathfrak{X}$ such that $x \notin \overline{sp}\{x_n : n \in \mathbb{N}\}$. By separation (H-B) $\exists\, x^* \in \mathfrak{X}^*$ s.t.

$$x^*(x) \neq 0 \qquad x^*(\overline{sp}\{x_n\}) = 0$$

Then

$$\|x^* - x_n^*\| \geq |x^*(x_n) - x_n^*(x_n)| = |x_n^*(x_n)| \geq \frac{\|x_n^*\|}{2} = \frac{1}{2}$$

Hence (x_n^*) is not dense ↯.

$\blacksquare$

The converse is not true since ℓ_1 is separable, but $\ell_\infty = \ell_1^*$ is not separable

<u>COROLLARY</u>: Neither l_1 nor $L_1[0,1]$ are reflexive.

Proof. Let $\mathfrak{X} = l_1$. If $\mathfrak{X}^{**} = l_1$, then $\mathfrak{X}^{**}$ is separable, so $\mathfrak{X}^* = l_\infty$ is separable ↯. Hence l_1 is not reflexive.
Same proof for $L_1[0,1]$. ▨

Fact: $\mathfrak{X}$ reflexive, $x^* \in \mathfrak{X}^* \Rightarrow \exists x \in \mathfrak{X}$ s.t. $\|x\| = 1$ and $x^*(x) = \|x^*\|$.

<u>THEOREM</u> (R.C. JAMES) The above fact characterizes reflexive spaces.

<u>Corollary of fact</u> Consider $g = (1-1, 1-1/2, 1-1/3, 1-1/4, \ldots, 1-1/n, \ldots) \in l_\infty$. Then $\|g\| = 1$. If l_1 is reflexive, then $\exists f \in l_1$ s.t.

$$g(f) = \|g\|_\infty = 1 \quad , \quad \|f\| = 1$$

i.e. if $f = (\alpha_n)$, then $\underbrace{\sum |\alpha_n| = 1 \text{ and } \sum_{n=1}^{\infty} \alpha_n (1 - 1/n) = 1}_{\text{patently incompatible}}$

<u>THEOREM</u>: A closed subspace of a reflexive B-space is also reflexive

Proof. Let $\mathfrak{X}$ be reflexive and $\mathfrak{Y}$ a closed subspace of $\mathfrak{X}$. Let $y^{**} \in \mathfrak{Y}^{**}$. We must find $y \in \mathfrak{Y}$ s.t. $y^{**}(y^*) = y^*(y) \; \forall y^* \in \mathfrak{Y}^*$

Define $T : \mathcal{X}^* \longrightarrow \mathcal{Y}^*$ by

$$Tx^*(y) = x^*(y)$$

(restriction map) Then T is continuous and T is onto by Hahn-Banach.
Define $S : \mathcal{Y}^{**} \longrightarrow \mathcal{X}^{**}$ by

$$Sz^{**} := z^{**} T$$

$\forall z^{**} \in \mathcal{Y}^{**}$. If we can show $S(\mathcal{Y}^{**}) \subseteq \mathcal{Y}$, then we'll be done.
Why? If $y_0^{**} \in \mathcal{Y}^{**}$ and $y^* \in \mathcal{Y}^*$, then

$$\underbrace{y_0^{**}(y^*) = y_0^{**} Tx^*}_{\text{for some } x^*} = S(y_0^{**})x^* = \underbrace{x^* S(y_0^{**})}_{\text{since } S(\mathcal{Y}^{**}) \subset \mathcal{Y}}$$

$$= y^* S(y_0^{**})$$

since definition of T ensures Tx^* and y^* agree on $\mathcal{Y}$.
To prove $S(\mathcal{Y}^{**}) \subset \mathcal{Y}$ $(\subseteq \mathcal{X} = \mathcal{X}^{**})$ suppose not. Then
$\exists y_0^{**}$ s.t. $Sy_0^{**} \notin \mathcal{Y}$. $\exists x^* \in \mathcal{X}^*$ s.t.

$$x^*(Sy_0^{**}) \neq 0 \quad x^* y = 0 \ \forall y \in \mathcal{Y}$$

Then $0 \neq x^* S(y_0^{**}) = S(y_0^{**})(x^*) = y_0^{**} Tx^* = y_0^{**}(0) = 0$ ↯

since Tx^* agrees with x^* on $\mathcal{Y}$

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<u>COROLLARY</u>: $\mathfrak{X}$ is reflexive $\Leftrightarrow$ $\mathfrak{X}^*$ is reflexive

Proof: Take $x_0^{***} \in \mathfrak{X}^{***}$. Our job: Find $x_0^* \in \mathfrak{X}^*$ s.t.

$$x_0^{***}(x^{**}) = x^{**}(x_0^*) \quad \forall x^{**} \in \mathfrak{X}^{**}$$

Fix $x^{**} \in \mathfrak{X}^{**}$. Since $\mathfrak{X}$ is reflexive $\exists x \in \mathfrak{X}$ s.t. $x^{**} = Qx$. Then

$$x_0^{***}(x^{**}) = x_0^{***}(Qx) = (\underbrace{x_0^{***} Q}_{x_0^*})(x) = x_0^*(x)$$

$$= Qx(x_0^*) = x^{**}(x_0^*)$$

Now suppose $\mathfrak{X}^*$ is reflexive. Then $\mathfrak{X}^{**}$ is reflexive by the first part. But $\mathfrak{X}$ is a closed subspace of $\mathfrak{X}^{**}$, so by the theorem $\mathfrak{X}$ is reflexive.

$\blacksquare$

<u>Observations</u>: (1) $\mathfrak{X}$ B-space $\Rightarrow$ $\mathfrak{X}^*$ complemented by a norm one projection on $\mathfrak{X}^{***}$, i.e. $\exists$ projection $P: \mathfrak{X}^{***} \xrightarrow{\text{onto}} \mathfrak{X}^*$ s.t. $\|P\| = 1$.

Proof: Take $P(x^{***}) = x^{***}|_{\mathfrak{X}} = x^{***} \circ Q$

<u>Fact</u> : (Phillips 1940) c_0 is not complemented in $\ell_\infty = c_0^{**}$, i.e. $\exists$ no continuous linear projection from ℓ_∞ onto c_0

(Can show - any projection $P : \ell_\infty \xrightarrow{\text{onto}}$ subspace of c_0 has a finite dimensional range)

Sequences and the Weak Topology

Let $\mathfrak{X}$ be a B-space and let A be a subset of $\mathfrak{X}$. A point $x_0 \in \mathfrak{X}$ is in the weak closure of A $\Leftrightarrow$ for all finite sets $B = \{x_1^*, \dots, x_n^*\} \subset \mathfrak{X}^*$ and $\forall \varepsilon > 0$ $\exists$ a point $a = a(\varepsilon, B) \in A$ s.t.

$$|x_i^*(x_0) - x_i^*(a)| < \varepsilon \quad \forall i$$

(From last time : $x_0 \in$ weak closure of A $\Leftrightarrow$ $\exists$ a net (a_τ) in A s.t. $\lim x^*(a_\tau) = x^*(x_0)$ $\forall x^* \in \mathfrak{X}^*$)

<u>Example</u> (Von Neumann) Sequences do not suffice to describe the weak topology.

Take $\mathfrak{X} = \ell_2$ and recall $\mathfrak{X}^* = \ell_2 = \mathfrak{X}$. Put

$$A = \{ (0, 0, \dots, \underset{\substack{\uparrow \\ m^{th}}}{1}, 0, \dots, \underset{\substack{\uparrow \\ n^{th}}}{m}, 0, \dots) : m < n \}$$

call this point x_{mn}

Take $x^* = (\alpha_k) \in l_2 = \mathfrak{X}^*$. Look at

$$x^*(x_{mn}) = \alpha_m + m\alpha_n$$

Let $\varepsilon > 0$. Choose m_0 s.t. $m \geq m_0 \Rightarrow |\alpha_m| < \varepsilon/2$. Then since $\lim \alpha_k = 0$ we can, for each $m \geq m_0$, find $n = n(m)$ s.t.

$$|m\alpha_n| < \varepsilon/2$$

Hence for each $x^* \in \mathfrak{X}^*$ and $\varepsilon > 0$, $\exists\ x_{m_0 n_0} \in A$ s.t. $|x^*(x_{m_0 n_0})| < \varepsilon$. This proves $0 \in$ weak closure of A

Can there be a sequence (a_n) in A s.t. $\lim a_n = 0$ weakly? i.e. does there exist a sequence (a_n) in A s.t. $x^*(a_n) \to 0$ $\forall x^* \in \mathfrak{X}^*$.

ANSWER: NO. Any infinite sequence (with infinitely many distinct terms) in A must be unbounded or cannot tend to zero weakly. We'll be done once we prove:

Theorem: A weakly Cauchy sequence in a B-space is bounded

(Let (x_n) be weakly Cauchy. Then the set $\{x^*(x_n) : n \in \mathbb{N}\}$ is bounded for each x^*. Apply Banach-Steinhaus)

<u>DEFINITION</u>: A B-space is called weakly sequentially complete (= weakly complete) if every weakly Cauchy sequence converges

THEOREM: Reflexive spaces are weakly sequentially complete

Proof. Let (x_n) be a weak Cauchy seq in a reflexive $\mathfrak{X}$. Define ℓ on $\mathfrak{X}^*$ by $\ell(x^*) = \lim x^*(x_n)$. Then ℓ is linear. Also

$$|\ell(x^*)| = \lim |x^*(x_n)| \leq \varlimsup_n \|x^*\| \|x_n\| \leq \|x^*\| K$$

Hence $\ell \in \mathfrak{X}^{**}$, so $\ell = Qx$ for some $x \in \mathfrak{X}$. Then

$$x^*(x) = \ell(x^*) = \lim x^*(x_n)$$

for all $x^* \in \mathfrak{X}^*$.

$\boxtimes$

Example: c_0 is not weakly seq. complete

Proof. Take $x_n = (1, 1, \ldots, \underset{\uparrow\, n^{th}\text{ slot}}{1}, 0, 0, \ldots)$. For $x^* = (\alpha_k) \in \ell_1 = c_0^*$

$$x^*(x_n) = \sum_{k=1}^{n} \alpha_k \xrightarrow{n \to \infty} \sum_{k=1}^{\infty} \alpha_k$$

So (x_n) is weakly Cauchy. If $x_0 \in c_0$ s.t. $x^*(x_n) \to x^*(x)$ $\forall x^* \in \ell_1$ then $x = (1, 1, 1, \ldots, 1, \ldots) \in \ell_\infty \setminus c_0$.

HW/ Show $\mathfrak{X}$ weakly seq. complete $\Rightarrow$ all closed subspaces are weakly complete and conclude $c_0 \hookrightarrow \mathfrak{X} \Rightarrow \mathfrak{X}$ not weakly complete

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THEOREM: $L_1(\mu)$ is weakly sequentially complete

To prove this we shall use

Rosenthal's Lemma: Let (Ω) be a point set and Σ be a σ-field of subsets of Ω. Let (μ_n) be a sequence of signed measures on Σ s.t.

$$(*) \qquad \sup |\mu_n|(\Omega) < \infty$$

Let (E_n) be a disjoint sequence in Σ. Then $\exists\, n_1 < n_2 < \dots$ s.t.

$$|\mu_{n_j}|\left(\bigcup_{i \neq j} E_{n_i}\right) < \varepsilon$$

Proof of lemma: Let $\mathbb{N} = \bigcup_{p=1}^{\infty} M_p$ where each M_p is infinite and

$$M_p \cap M_q = \phi \quad \text{for } p \neq q$$

Suppose for some p there exists no K in M_p s.t.

$$(**) \qquad |\mu_k|\left(\bigcup_{\substack{j \neq k \\ j \in M_p}} E_j\right) \geq \varepsilon$$

Then we would be done. How? Enumerate $M_p = \{n_1 < n_2 < \dots\}$. If this does not happen, then for each p $\exists\, K_p \in M_p$ s.t.

$$|\mu_{k_p}|\Big(\bigcup_{\substack{j\neq k_p\\ j\in M_p}} E_j\Big) \geq \varepsilon$$

WLOG $\sup_n |\mu_n|(\Omega) = a$. Then for each p,

$$|\mu_{k_p}|\Big(\bigcup_{q=1}^{\infty} E_{k_q}\Big) + |\mu_{k_p}|\Big(\Big(\bigcup_{n=1}^{\infty} E_n\Big)\setminus \bigcup_{q=1}^{\infty} E_{k_q}\Big) \leq a$$

Hence for each p

(***) $$|\mu_{k_p}|\Big(\bigcup_{q=1}^{\infty} E_{k_q}\Big) + \varepsilon \leq a$$

since

$$\bigcup_{\substack{j\neq k_p\\ j\in M_p}} E_j \subseteq \bigcup_{n=1}^{\infty} E_n \setminus \bigcup_{q=1}^{\infty} E_{k_q}$$

Now repeat with the sequence (E_{k_q}) instead of (E_n). We can only do this finitely many times before (***) is no longer possible, whence (**) must hold.

$\blacksquare$

Example: Let (g_n) be a sequence in $L_1(\mu)$ s.t. (1) the g_n's have disjoint supports and (2) $\exists$ positive constants α, β s.t

$$\beta \leq \|g_n\|_1 \leq \alpha \qquad \forall n$$

Then there exist an isomorphism $T: \ell_1 \longrightarrow L_1(\mu)$ s.t. $T(e_n) = g_n$

Proof. Define $T: \ell_1 \to L_1(\mu)$ by

$$T(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n g_n$$

Observe

$$\| T(\alpha_n) \|_1 \leq \sum_{n=1}^{\infty} |\alpha_n| \|g_n\|_1 \leq \sum_{n=1}^{\infty} |\alpha_n| \alpha = \|(\alpha_n)\|_1 \cdot \alpha$$

This shows $T(\alpha_n) \in L_1(\mu)$ and also that T is continuous with $\|T\| \leq \alpha$.
Observe that $T(e_n) = g_n$. Also

$$\| T(\alpha_n) \| = \int_\Omega \left| \sum_{n=1}^{\infty} \alpha_n g_n \right| d\mu \underset{\text{disjoint support}}{=} \sum_{n=1}^{\infty} \int_\Omega |\alpha_n g_n| \, d\mu$$

$$= \sum_{n=1}^{\infty} |\alpha_n| \|g_n\|_1 \geq \|(\alpha_n)\|_1 \beta$$

Therefore T^{-1} exists and is continuous (from $\mathrm{Rng}(T)$)

DEFINITION: A bounded set K in $L_1(\mu)$ is called uniformly integrable if for each disjoint sequence (E_n) in Σ, then the series

$$\sum_{n=1}^{\infty} \int_{E_n} f \, d\mu$$

are equi-continuous as f varies through K, i.e.

$$\lim_{m} \sup_{f \in K} \sum_{n=m}^{\infty} \int_{E_n} |f| \, d\mu = 0$$

(the indefinite integrals are uniformly countably-additive)

Facts: bdd $K \subset L_1(\mu)$ is uniformly integrable $\Leftrightarrow$

$$\lim_{\mu(E) \to 0} \int_E |f| \, d\mu = 0 \text{ unif. in } f \in K$$

$$\Leftrightarrow \lim_{n \to \infty} \int_{[|f| > n]} |f| \, d\mu = 0 \text{ unif. in } f \in K$$

COROLLARY: If a bounded subset K of $L_1(\mu)$ is not uniformly integrable, then there exists an isomorphism $T: \ell_1 \to L_1(\mu)$ s.t. T maps the unit vectors of ℓ_1 into K. Consequently there exist a sequence in K with no weakly Cauchy subsequence. In particular K is not weakly sequentially compact

[Remark: Rosenthal (1974) $\exists$ an isomorphism $T: \ell_1 \longrightarrow \mathfrak{X} \Rightarrow \mathfrak{X}$ has a bounded sequence with no weakly Cauchy subsequence.]

Proof First observe that the sequence of unit vectors in ℓ_1 has no weakly Cauchy subsequence since ℓ_1 has the Schur property.

Suppose K is not uniformly integrable, i.e.

$$\alpha := \overline{\lim_m} \sup_{f \in K} \sum_{n=m}^{\infty} \int_{E_n} |f| \, d\mu \neq 0$$

for some disjoint sequence (E_n)

Claim: $\exists$ a sequence (f_n) in K and disjoint sets (A_n) in Σ and $\varepsilon > 0$ s.t.

$$\int_{A_m} |f_m| \, d\mu \geq \varepsilon$$

To see this, take $m_1 < m_2 < m_3 < \ldots$ s.t.

$$\sum_{n=m_i}^{\infty} \int_{E_n} |f_i| \, d\mu \geq \alpha/2$$

for some sequence (f_i) in K.

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(Proof continued) Recall $K \subset L_1(\mu)$, K bounded and not uniformly integrable. We had

$$\lim_m \sup_{f \in K} \sum_{n=m}^{\infty} \int_{E_n} |f|\, d\mu = \alpha > 0$$

To find: A disjoint (A_n) in Σ and a sequence (f_n) in K and $\varepsilon > 0$ s.t.

$$\int_{A_n} |f|\, d\mu \geq \varepsilon$$

Take $f_1 \in K$ s.t.

$$\sum_{n=1}^{\infty} \int_{E_n} |f_1|\, d\mu \geq \frac{3\alpha}{4}$$

"Sliding hump": Choose m_1 s.t.

$$\sum_{n=m_1+1}^{\infty} \int_{E_n} |f_1|\, d\mu < \alpha/4$$

Put $A_1 = \bigcup_{n=1}^{m} E_n$ and notice $\int_{A_1} |f_1|\, d\mu \geq \alpha/2$. Now choose $f_2 \in K$

$$\sum_{n=m_1+1}^{\infty} \int_{E_n} |f_2|\, d\mu \geq \frac{3\alpha}{4}$$

Choose $m_2 > m_1+1$ s.t.

$$\sum_{n=m_2+1}^{\infty} \int_{E_n} |f_2| \, d\mu < \alpha/4$$

Put $A_2 = \bigcup_{n=m_1+1}^{m_2} E_n$ and notice $\int_{A_2} |f_2| \, d\mu \geq \frac{\alpha}{2}$. Continue this sliding hump process to produce a disjoint sequence (A_n) and a sequence (f_n) in K s.t.

$$\int_{A_n} |f_n| \, d\mu \geq \frac{\alpha}{2} = \varepsilon$$

Apply Rosenthal's lemma to pass to subsequences (f_{n_j}) and (A_{n_j}) s.t. for each j

$$\int_{A_{n_j}} |f_{n_j}| \, d\mu \geq \varepsilon$$

but $$\int_{\bigcup_{i \neq j} A_{n_i}} |f_{n_j}| \, d\mu < \varepsilon/2$$

Relabel $f_{n_m} = g_m$ and $A_{n_m} = A_m$, so we have

$$\int_{A_m} |g_m| \, d\mu \geq \varepsilon \qquad \int_{\bigcup_{n \neq m} A_n} |g_m| \, d\mu < \varepsilon/2$$

Define $T : \ell_1 \longrightarrow L_1$ by

$$T(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n g_n$$

As before T is continuous because $\sup_n \|g_n\|_1 < \infty$. To see that T^{-1} is continuous, look at

$$\|T(\alpha_n)\|_1 = \left\|\sum_{n=1}^{\infty} \alpha_n g_n\right\|_1 = \int_{\Omega} \left|\sum_{n=1}^{\infty} \alpha_n g_n\right| d\mu$$

$$\geq \int_{\bigcup_{m=1}^{\infty} A_m} \left|\sum_{n=1}^{\infty} \alpha_n g_n\right| d\mu$$

$$= \int_{\Omega} \left|\sum_{n=1}^{\infty}\left[\alpha_n g_n \chi_{A_n} + \alpha_n g_n \chi_{\bigcup_{k\neq n} A_k}\right]\right| d\mu$$

$$\geq \int_{\Omega} \left|\sum_{m=1}^{\infty} \alpha_m g_m \chi_{A_m}\right| d\mu - \int \left|\sum_{n=1}^{\infty} \alpha_n g_n \chi_{\bigcup_{k\neq n} A_k}\right| d\mu$$

$$\geq \sum_{m=1}^{\infty} \int_{A_m} |\alpha_m g_m|\, d\mu - \sum_{n=1}^{\infty} \int_{\Omega} |\alpha_n g_n \chi_{\bigcup_{k\neq n} A_k}|\, d\mu$$

$$\geq \sum_{m=1}^{\infty} |\alpha_m|\, \varepsilon - \sum_{n=1}^{\infty} |\alpha_n|\, \varepsilon/2 = \|(\alpha_n)\|_{\ell_1}\, \varepsilon/2$$

Therefore T^{-1} exists and is continuous. Observe $T(e_n) = g_n \in K$.

$\boxtimes$

<u>Corollary</u>: If a bounded $K \subset L_1(\mu)$ is not uniformly integrable. Then K contains a sequence with no weakly Cauchy subsequence.

Proof. Take the isomorphism T from last theorem. Chose (g_m) in K s.t. $T(e_n) = g_n$. Suppose (g_n) has a weakly Cauchy subsequence (g_{n_j}). Since T is an isomorphism, it follows that (e_{n_j}) is weakly Cauchy in ℓ_1. Hence

$$\lim_j \beta \cdot (e_{n_j}) \text{ exists } \forall \beta = (\beta_n) \in \ell_\infty$$

$$\Longrightarrow \lim_j \beta_{n_j} \text{ exist } \forall \text{ bdd seq } (\beta_n) \ \lightning$$

(Take $\beta_{n_j} = (-1)^j$). $\blacksquare$

<u>Corollary</u>: $L_1(\mu)$ is weakly sequentially complete

Proof. Let (f_n) be a weakly Cauchy sequence in $L_1(\mu)$. Then $K = \{f_n : n \in \mathbb{N}\}$ is bounded and is uniformly integrable (by last corollary). Take $E \in \Sigma$ and notice

$$\lambda(E) := \lim_n \int_E f_n \, d\mu$$

exists (since $\chi_E \in L_\infty(\mu)$). Since K is uniformly integrable, λ is countably additive. By the Radon-Nikodym theorem $\exists f \in L_1(\mu)$ s.t.

$$\lambda(E) = \int_E f\, d\mu \quad \forall E \in \Sigma$$

Therefore

$$\lim_n \int_E f_n\, d\mu = \int_E f\, d\mu \quad \forall E \in \Sigma$$

$$\Rightarrow \lim_n \int f_n g\, d\mu = \int f g\, d\mu \quad \forall \text{simple functions } g$$

$$\Rightarrow \lim_n \int f_n g\, d\mu = \int f g\, d\mu \quad \forall g \in L_\infty(\mu)$$

$$\Rightarrow f_n \to f \text{ weakly}$$

⟦ For bounded g, let $g_n := \sum_{k=-\infty}^{\infty} \frac{k-1}{2^n} \chi_{\left[\frac{k-1}{2^n} \le g \le \frac{k}{2^n}\right]}$. g_n is simple since g is bounded and $\|g - g_n\|_\infty \le 1/2^n$. Hence the simple functions are dense in $L_\infty(\mu)$ ⟧

<u>COROLLARY</u>: Weakly Cauchy sequences in ℓ_1 converge in norm.

Proof. Let $(\alpha_k) = (\alpha_{k,n})_{n=1}^{\infty}$ be a weakly convergent sequence in ℓ_1.

<u>**THEOREM**</u> (Krein-Smulian) The closed convex hull of a weakly compact subset of a B-space is also weakly compact.

Proof. Take a weakly compact subset W of $\mathfrak{X}$. Consider $C(W, \text{weak})$. Its dual consists of all regular Borel measures on W

CLAIM: WLOG $\mathfrak{X}$ is separable

To see why, let (x_n) be a seq. in $\overline{co}(W)$. By Eberlein-Smulian it suffices to extract a weakly convergent subsequence. Each x_n is the norm limit of a sequence from $co(W)$, i.e.

$$x_n = \lim \sum_{p=1}^{k} \alpha_p y_{pn}$$

Hence all the action is taking place inside

$$\underbrace{W \cap sp\{y_{pn}\}}_{\text{separable}}$$

Hence it is enough to show $\overline{co}(W \cap S)$ is weakly compact for every separable subspace S of $\mathfrak{X}$.

Let $f : W \to W$ be the identity function. Hence $x^* f \in C(W, \text{weak})$ $\forall x^* \in \mathfrak{X}^*$

unf. int. $\Rightarrow \lim\limits_{m} \sup\limits_{k} \sum\limits_{n=m}^{\infty} |\alpha_{k,n}| = 0$

Therefore the series $\sum\limits_{n=1}^{\infty} |\alpha_{k,n}|$ are equi-convergent. (*)

$$(\alpha_k) \xrightarrow{w} \alpha \quad \Rightarrow \quad \alpha_{k,n} \longrightarrow \alpha_n$$

$$(*) \Rightarrow \lim_{k} \sum_{n=k}^{\infty} |\alpha_{k,n} - \alpha_n| = 0 \Rightarrow \sum_{n=1}^{\infty} |\alpha_{k,n} - \alpha_n| \longrightarrow 0 \text{ as } k \to \infty$$

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__COROLLARY__: Let (f_n) be a bounded sequence in $L_1(\mu)$ s.t.

$$\lim_n \int_E f_n \, d\mu \quad \text{exists}$$

for every $E \in \Sigma$. Then (f_n) is uniformly integrable.

Proof. By techniques used last time, we see that

$$\lim_n \int_E f_n g \, d\mu \text{ exists } \forall g \in L_\infty(\mu)$$

Hence (g_n) is weakly Cauchy, and therefore is uniformly integrable (If not it would have a subsequence which mimics the ℓ_1 unit basis which is not weakly Cauchy)

Note: This corollary remains true with the L_1 boundedness dropped. The resulting theorem is called the Vitalli-Hahn-Saks-Nikodym theorem with a dash of Lebesgue. This theorem follows from our corollary once we know the Nikodym Boundedness theorem (see Dunford + Schwartz)

$$\sup_n \left| \int_E f_n \, d\mu \right| < \infty \quad \forall E \in \Sigma \Rightarrow \sup_n \int_\Omega |f_n| \, d\mu < \infty$$

__COROLLARY__: ℓ_1 has the Schur property.

Proof. Let (f_n) be a sequence in ℓ_1 that converges weakly to 0.

Write $f_n = (g_{n,m})$. Observe $g_{n,m} \xrightarrow{n\to\infty} 0$ for each fixed m
Write $E_n = \{n\}$. Since (f_n) is weakly convergent, it is uniformly integrable ($\ell_1 = L_1$ (counting measure on $\mathbb{N}$)). Hence

$$\lim_k \sup_n \sum_{m=k}^{\infty} \left| \int_{E_m} f_n \, d\mu \right| = 0$$

$$\Rightarrow \lim_k \sup_n \sum_{m=k}^{\infty} |g_{n,m}| = 0$$

Hence

$$\|f_n\|_{\ell_1} = \sum_{m=1}^{k-1} |g_{n,m}| + \sum_{m=k}^{\infty} |g_{n,m}| \qquad \forall k \ \forall n$$

Let $\varepsilon > 0$ and choose k s.t. $\sup_n \sum_{m=k}^{\infty} |g_{n,m}| < \varepsilon/2$. Choose n_0 s.t.

$$|g_{n,m}| \leq \varepsilon/2(k-1)$$

$\forall n > n_0$ $\forall m \leq k-1$. Then $\|f_n\|_{\ell_1} \leq \varepsilon$ $\forall n > n_0$, so $f_n \to 0$ in norm.

$\blacksquare$

<u>DEFINITION</u>: A subset A of a B-space is relatively weakly sequentially compact if every sequence in A has a weakly convergent subsequence.

Fact: A subset of a reflexive B-space is relatively weakly sequentially compact if and only it is bounded.

Proof. ($\Rightarrow$) Let A be relatively weakly sequentially compact. If A is not bounded, then $\exists$ a sequence (x_n) in A s.t. $\|x_n\| > n$. Select (x_{n_j}) s.t. $\lim x_{n_j}$ exists weakly. Then (x_{n_j}) is a bounded sequence which contradicts $\|x_{n_j}\| > n_j$

($\Leftarrow$) Let A be a bounded set in the reflexive space $\mathfrak{X}$. Let (y_n) be a sequence in A. Let $\mathcal{Y} = \overline{sp(y_n)}$. Then $\mathcal{Y}$ is separable and so $\mathcal{Y}^*$ is separable (since $\mathcal{Y}$ is reflexive). Let (y_n^*) be a sequence in $\mathcal{Y}^*$ s.t.

$$\overline{sp\{y_n^*\}} = \mathcal{Y}^*$$

Use Cantor diagonalization to produce a subsequence (y_{n_j}) of (y_n) s.t. $\lim y_k^*(y_{n_j})$ exists $\forall k$. Since (y_k^*) is norm dense in $\mathcal{Y}^*$, it follows that $\lim y^*(y_{n_j})$ exists $\forall y^* \in \mathcal{Y}^*$. Since $\mathcal{Y}$ is reflexive, it is weakly sequentially complete, and hence $\exists y \in \mathcal{Y}$ s.t.

$$\lim_j y^*(y_{n_j}) = y^*(y)$$

$\forall y^* \in \mathcal{Y}^*$, i.e. $\lim_j y_{n_j} = y$ weakly (in $\mathcal{Y}$ and hence in $\mathfrak{X}$)

$\blacksquare$

HW/ ① Let $\mathfrak{X}$ be a closed subspace of $L_1(\mu)$. Prove $\mathfrak{X}$ is either reflexive or contains an isomorphic copy of ℓ_1 (Use ②)

② Dunford's Th^m : A subset K of $L_1(\mu)$ is weakly seq. compact iff it is bounded and uniformly integrable

10/1 BANACH SPACES

FACT: (Elton - Odell) $\mathfrak{X}$ infinite dimensional B-space $\Rightarrow \exists \varepsilon > 0$ and a sequence (x_n) s.t. $\|x_n\| \le 1$ and $\|x_n - x_m\| \ge 1+\varepsilon \quad \forall m \ne n$

HW/ Without using the above fact prove $\mathfrak{X}$ is finite dimensional iff its closed unit ball is norm compact

Hints: $(\Rightarrow)$ known

$(\Leftarrow)$ Suppose $\exists x_1, \dots, x_n \in B_{\mathfrak{X}}$ s.t. $\overline{B_{\mathfrak{X}}} \subseteq \bigcup_{i=1}^{n} (x_i + \frac{1}{2} B_{\mathfrak{X}})$

Put

$$\mathcal{Y} = \operatorname{sp}\{x_1, \dots, x_n\}$$

Prove $\mathcal{Y} = \mathfrak{X}$ by showing $\pi : \mathfrak{X} \to \mathfrak{X}/\mathcal{Y}$ has norm zero

LINEAR EQUATIONS IN BANACH SPACES

THEOREM (Original form of H-B as Hahn saw it) Let $\mathfrak{X}$ be a normed linear space. Let $\{x_\alpha : \alpha \in A\}$ be a subset of $\mathfrak{X}$ and $\{c_\alpha : \alpha \in A\}$ be a corresponding collection of scalars. Then $\exists x^* \in \mathfrak{X}^*$ s.t. $x^*(x_\alpha) = c_\alpha$ iff $\exists M$ s.t.

$$(*) \qquad \left| \sum_{\alpha \in F} \beta_\alpha c_\alpha \right| \le M \left\| \sum_{\alpha \in F} \beta_\alpha x_\alpha \right\|$$

$\forall$ scalars β_α and $\forall$ finite subsets of A.

Proof. $(\Rightarrow)$

$$\left|\sum_{\alpha\in F}\beta_\alpha c_\alpha\right| = \left|\sum_{\alpha\in F}\beta_\alpha x^*(x_\alpha)\right| = \left|x^*\left(\sum_{\alpha\in F}\beta_\alpha x_\alpha\right)\right|$$

$$\le \|x^*\| \left\|\sum_{\alpha\in F}\beta_\alpha x_\alpha\right\|$$

Set $M = \|x^*\|$.

$(\Leftarrow)$ Let $\mathcal{Y} = \mathrm{sp}\{x_\alpha : \alpha\in A\}$. Define ℓ on $\mathcal{Y}$ by

$$\ell\left(\sum_{\alpha\in F}\beta_\alpha x_\alpha\right) = \sum_{\alpha\in F}\beta_\alpha c_\alpha$$

It follows directly from $(*)$ that ℓ is well defined. Obviously ℓ is linear, and in addition $(*)$ says that ℓ is continuous with $\|\ell\| \le M$. Let x^* be any Hahn-Banach extension.

▨

How about solving for x: $x_\alpha^*(x) = c_\alpha$. If $\mathcal{X}$ is reflexive then this collapses to above theorem.

$+\left|\sum \beta_\alpha c_\alpha\right| \le m\left\|\sum \beta_\alpha x_\alpha^*\right\|$

If $\mathcal{X}$ is not reflexive, then procedure breaks down. To see why, take $x^{**}\in\mathcal{X}^{**}\setminus\mathcal{X}$ and put $c_\alpha = x^{**}(x_\alpha^*)$ (where (x_α^*) is dense in $\mathcal{X}^*$) If there $\exists x\in\mathcal{X}$ with $x_\alpha^*(x) = c_\alpha$ $\forall\alpha$, then $x^{**} = Qx$ ↯. Note

$$\left|\sum c_\alpha\beta_\alpha\right| \le \|x^{**}\| \left\|\sum \beta_\alpha x_\alpha^*\right\|$$

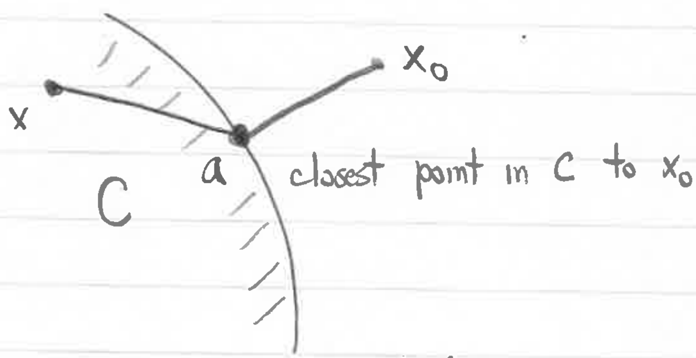
There is something to be saved in this context, and that something is called Helly's theorem. Helly's theorem depends on a separation theorem.

<u>EIDELHEIT SEPARATION THEOREM</u> (finite dimensional case) Let $\mathfrak{X}$ be a finite dimensional B-space. Let C be a closed convex subset of $\mathfrak{X}$. If $x_0 \in \mathfrak{X} \setminus C$, then $\exists x^* \in \mathfrak{X}^*$ s.t.

$$\sup x^*(C) < x^*(x_0)$$

Proof. WLOG $\mathfrak{X} = \ell^2_n$.

Geometric proof



For any $x \in C$

$(x-a)\cdot(x_0-a) \le 0$

$$\Rightarrow (x_0-a)\cdot x \le (x_0-a)\cdot x < (x_0-a)\cdot x_0 \quad \left(\text{since } (x_0-a)\cdot(x_0-a) = \|x_0-a\| > 0\right)$$

Analytic proof: Fix $x \in C$ and let a be as above. Put

$$f(t) = \|(1-t)a + tx - x_0\|^2$$

Then f is minimized over $[0,1]$ at 0. Therefore $0 \le f'(0)$. But

$$f'(t) = 2\left((1-t)a + tx - x_0\right)\cdot(-a+x)$$

Hence

$$0 \leq \frac{f'(0)}{2} = (a - x_0) \cdot (x - a)$$

Now procede as before. ▨

<u>Corollary</u>: Suppose C is a convex set in a finite dimensional B-space. Suppose $x_0 \notin C$. Then $\exists\, x^* \in \mathfrak{X}^*$ s.t.

$$x^*(x_0) \geq \sup x^*(C)$$

Proof. Suppose $x_0 \notin \overline{C}$. Use Eidelheit separation theorem. If $x_0 \in \overline{C}$, then every neighborhood of x_0 contains points outside $\overline{C}$. For each n, choose $x_n \notin \overline{C}$ s.t. $\|x_0 - x_n\| < 1/n$. Choose linear functionals x_n^* s.t. $\|x_n^*\| = 1$ and

$$\sup x_n^*(C) < x_n^*(x_n)$$

Since $\mathfrak{X}$ is isomorphic to ℓ^2_n, so is $\mathfrak{X}^* \simeq \ell^2_n$. WLOG $x_n^* \to x^*$ in norm (unit ball is compact). Then $\|x^*\| = 1$. Evidently

$$\sup x^*(C) \leq x^*(x_0)$$

(since $x_n^*(x_n) \to x^*(x_0)$)

▨

THEOREM (Helly's Theorem) Let $\mathfrak{X}$ be a Banach space and let $x_1^*, \ldots, x_n^*$ be fixed vectors in $\mathfrak{X}^*$. Let $c_1, \ldots, c_n$ be scalars. Then for each $\varepsilon > 0$ there exist $x_\varepsilon \in \mathfrak{X}$ with $\|x_\varepsilon\| \le M + \varepsilon$ and $x_i^*(x_\varepsilon) = c_i$ if and only if

$$(*) \qquad \left| \sum_{i=1}^{n} \beta_i c_i \right| \le M \left\| \sum_{i=1}^{n} \beta_i x_i^* \right\|$$

for all scalars (β_i).

COROLLARY: Let $x_1^*, \ldots, x_n^* \in \mathfrak{X}^*$ and $x^{**} \in \mathfrak{X}^{**}$. Then $\exists x \in \mathfrak{X}$ s.t. $x_i^*(x) = x^{**}(x_i^*)$ $\forall i = 1, \ldots, n$.

Proof. Take $c_i = x^{**}(x_i^*)$ in the theorem

(Remark - There is something called the pincipal of local reflexivity which says that every finite dimensional subspace of $\mathfrak{X}^*$ is "nearly" isometric to a subspace of $\mathfrak{X}$ s.t. under the isomorphism $(*)$ holds for everything in the closed linear span.

10/3 BANACH SPACES

THEOREM: (Helly's theorem) Let $x_1^*, \dots, x_n^* \in \mathfrak{X}^*$. Let $c_1, \dots, c_n$ be scalars. For each $\varepsilon > 0$ $\exists\, x_\varepsilon \in \mathfrak{X}$ s.t. $x_i^*(x_i) = c_i$ with $\|x_\varepsilon\| \le M + \varepsilon$ iff

$$(*) \qquad \left| \sum_{i=1}^{n} \beta_i c_i \right| \le M \left\| \sum_{i=1}^{n} \beta_i x_i^* \right\|$$

for all scalars (β_i)

Proof. ($\Rightarrow$) Suppose $\exists$ solution $x_\varepsilon \in \mathfrak{X}$ with $\|x_\varepsilon\| \le M+\varepsilon$. Then

$$\left| \sum \beta_i c_i \right| = \left| \sum \beta_i x_i^*(x_\varepsilon) \right| \le \|x_\varepsilon\| \left\| \sum_{i=1}^{n} \beta_i x_i^* \right\|$$

$$\le (M+\varepsilon) \left\| \sum_{i=1}^{n} \beta_i x_i^* \right\|$$

Let $\varepsilon \to 0$ to get $(*)$.

($\Leftarrow$) WLOG assume the x_i^* are linearly independent. Define $T: \mathfrak{X} \to \mathbb{R}^n$ by

$$Tx = (x_1^*(x), \dots, x_n^*(x))$$

$T(B(0; M+\varepsilon))$ is a convex set in $\mathbb{R}^n$. Suppose no such x_ε exists. Then $(c_1, c_2, \dots, c_n) \notin T(B_{M+\varepsilon})$. Hence $\exists\, \alpha = (\alpha_1, \dots, \alpha_n) \neq 0$ in $\mathbb{R}^n$ s.t.

$$\sup_{\|x\| \le M+\varepsilon} \alpha \cdot Tx \le \sum_{i=1}^{n} \alpha_i c_i$$

(separation theorem). Since $T(B(0; M+\varepsilon))$ is a symmetric set, so is $\alpha \cdot T(B(0; M+\varepsilon))$. Hence the sup on left is non-negative, and in fact

$$(*) \qquad \sup_{\|x\| \le M+\varepsilon} |\alpha \cdot T(x)| \le \sum_{i=1}^{n} \alpha_i c_i$$

But

$$\sup_{\|x\| \le M+\varepsilon} |\alpha \cdot Tx| = \sup_{\|x\| \le M+\varepsilon} \left| \sum_{i=1}^{n} \alpha_i x^*(x) \right| = \sup_{\|x\| \le M+\varepsilon} \left| \left(\sum_{i=1}^{n} \alpha_i x^* \right)(x) \right|$$

$$= (M+\varepsilon) \left\| \sum_{i=1}^{n} \alpha_i x_i^* \right\|$$

$$\underset{\text{by } (*)}{\le} \sum_{i=1}^{n} \alpha_i c_i \underset{\text{by hypothesis}}{\le} M \left\| \sum_{i=1}^{n} \alpha_i x_i^* \right\|$$

Therefore we must have $\|\sum \alpha_i x_i^*\| = 0$ to avoid contradiction. But this is impossible since the x_i^*'s are independent. Therefore x_ε exists

$\boxtimes$

DEFINITION: We say a net (x_α^*) in $\mathfrak{X}^*$ converges in the weak* topology to $x^* \in \mathfrak{X}^*$ if

$$\lim x_\alpha^*(x) = x^*(x)$$

for all x in $\mathfrak{X}$.

w^* convergence $\not\Rightarrow$ w convergence : $\ell_1 = c_0^*$. Let (e_n) be the usual ℓ_1 basis. Let $(\alpha_k) = \alpha \in c_0$

$$e_n(\alpha) = \alpha_n \longrightarrow 0$$

since $(\alpha_k) \in c_0$. Hence $e_n \to 0$ w^* in ℓ_1. But $e_n \not\to 0$ weakly since it is not even weakly Cauchy.

FACT: Weak convergence $\Rightarrow$ weak* convergence.

Let $x_\alpha^* \to x^*$ weakly in $\mathfrak{X}^*$. Let $x \in \mathfrak{X}$. Then $Qx \in \mathfrak{X}^{**}$ and

$$Qx(x_\alpha^*) \to Qx(x^*)$$

$$\Rightarrow x_\alpha^*(x) \to x^*(x)$$

FACT: If $\mathfrak{X}$ is reflexive, then weak* convergence is the same as weak convergence.

FACT: If $K \subset \mathfrak{X}^*$ is weakly compact, then the weak and weak* topology agrees on K. (HW)

Proof: Consider $T : K(\text{weak}) \to K(\text{weak}^*)$ given by $Tx = x$. Then T is continuous. Since $K(\text{weak})$ is compact, T is a homeomorphism.

THEOREM (Alaoglu) Let $\mathfrak{X}$ be a B-space. Then $B_{\mathfrak{X}^*}$ is relatively compact in the weak* topology.

Proof. Take $x \in \mathfrak{X}$ and put $A_x = [-1,1]$. Set

$$A = \prod_{x \in B_{\mathfrak{X}}} A_x$$

WLOG closed unit ball

A is compact in the product topology. Define $\tau : B_{\mathfrak{X}^*} \to A$ by

$$\tau(x^*) := (x^*(x))_{x \in B_{\mathfrak{X}}}$$

Observe that a net (x^*_α) is weak* convergent to x^* in $\mathfrak{X}^*$ iff $\tau(x^*_\alpha) \to \tau(x^*)$ in the product topology. Therefore τ embeds $B_{\mathfrak{X}^*}$ homeomorphically into A. We shall have shown that $(B_{\mathfrak{X}^*}, \text{weak}^*)$ is compact once we have shown $\tau(B_{\mathfrak{X}^*})$ is closed in A. To this end suppose we have a net (x^*_α) in $B_{\mathfrak{X}^*}$ s.t.

$$\tau(x^*_\alpha) \longrightarrow a \in A$$

Then $\lim x_\alpha^*(x) = a_x$ exists $\forall x \in B_{\mathcal{X}}$, hence $\forall x \in \mathcal{X}$, i.e.

$$a_x = \ell(x) := \lim x_\alpha^*(x) \text{ exists}$$

for all $x \in \mathcal{X}$. Obviously ℓ is linear in $\mathcal{X}$. Also

$$|\ell(x)| = \lim |x_\alpha^*(x)| \leq 1 \cdot \|x\|$$

for all $x \in \mathcal{X}$ since $\|x_\alpha^*\| \leq 1$. Therefore $\ell \in \mathcal{X}^*$, and $\tau(\ell) = a$.
Hence $a \in \tau(B_{\mathcal{X}^*})$.

$\blacksquare$

$\mathcal{X}^*$ is always weak* sequentially complete.

10/5 BANACH SPACES

COROLLARY: A bounded subset of $\mathfrak{X}^*$ is relatively weak*-compact.

Proof. If $A \subseteq B_{\mathfrak{X}^*}$, then since $B_{\mathfrak{X}^*}$ is weak*-compact the weak* closure of A is weak*-compact. If A is bounded, then select K s.t. $1/K\, A \subset B_{\mathfrak{X}^*}$ and see that $\frac{1}{K} A$ has a weak*-compact weak* closure. Hence A has a weak*-compact weak* closure.

$\blacksquare$

COROLLARY: If $\mathfrak{X}$ is reflexive, then $B_{\mathfrak{X}}$ is weakly compact.

Proof $\mathfrak{X} = \mathfrak{X}^{**}$. The weak topology on $\mathfrak{X}$ is precisely the weak* topology of $\mathfrak{X}$ (in $\mathfrak{X}^{**}$).

THEOREM: (Goldstine) Let $\mathfrak{X}$ be a Banach space. Then $B_{\mathfrak{X}}$ is weak* dense in $B_{\mathfrak{X}^{**}}$

Proof. Regard $\mathfrak{X} \subset \mathfrak{X}^{**}$. Note first that the weak* closure in $\mathfrak{X}^{**}$ of $B_{\mathfrak{X}}$ is contained in $B_{\mathfrak{X}^{**}}$. Why? Let x^{**} be the weak* limit of a net (x_α) in $B_{\mathfrak{X}}$. Then

$$|x^{**}(x^*)| = \lim_\alpha |x^*(x_\alpha)| \leq \|x^*\|$$

since $\|x_\alpha\| \leq 1$, so $\|x^{**}\| \leq 1$.

To finish the proof, select $x^{**} \in B_{\mathfrak{X}^{**}}$. We want to find a net $(x_\alpha) \in \mathfrak{X}$ s.t. $\|x_\alpha\| \leq 1$ $\forall \alpha$ and

$$\lim_\alpha x^{**}(x^*) = \lim_\alpha x^*(x_\alpha) \quad \forall x^* \in \mathfrak{X}^*$$

Order the finite subsets of $\mathfrak{X}^*$ by inclusion. For each finite set $F \subset \mathfrak{X}^*$ set

$$c_i = x^{**}(x_i^*) \quad \forall x_i^* \in F$$

Then

$$\left| \sum \beta_i c_i \right| = \left| \sum \beta_i x^{**}(x_i^*) \right| \leq \|x^{**}\| \, \|\sum \beta_i x_i^*\|$$

ε of thm

$$\leq \|\sum \beta_i x_i^*\|$$

Hence this is a Helly theorem set-up, so $\exists\, x_F \in \mathfrak{X}$ with $\|x_F\| \leq 1 + \frac{1}{|F|}$ such that

$$x^{**}(x_i^*) = c_i = x_i^*(x_F) \quad \forall x_i^* \in F$$

Now consider for $x^* \in \mathfrak{X}^*$,

$$x^{**}(x^*) = \lim_F x^*(x_F)$$

To see this, take $F_0 = \{x^*\}$. Then $F \supseteq F_0 \Rightarrow x^*(x_F) = x^{**}(x^*)$.
Hence $\lim x_F = x^{**}$ (weak*), so

$$\lim_F \frac{|F|}{1+|F|} x_F = x^{**} \quad w^*$$

Notice that $\frac{|F|}{1+|F|}\chi_F$ is a net in $B_{\mathfrak{X}}$.

$\blacksquare$

Example. Partially order the finite subsets of $[0,1]$ by inclusion. Then $\{\chi_F : F \text{ a finite subset of } [0,1]\}$ is a bounded net and

$$\lim \chi_F = \chi_{[0,1]}$$

pointwise. But

$$\lim_F \int_{[0,1]} \chi_F \, d\mu = 0 \neq \int_{[0,1]} \lim_F \chi_F \, d\mu$$

Hence bounded convergence theorem fails for nets.

Example: Need $B_{\mathfrak{X}}$ be weak*-sequentially dense in $B_{\mathfrak{X}^{**}}$? If $\mathfrak{X}^*$ is separable, then we shall see that the weak* topology on $B_{\mathfrak{X}^{**}}$ is a metric topology; hence the answer is yes in this case.

Observe: If a sequence (x_n) in $\mathfrak{X}$ is weak*-convergence to $x^{**} \in \mathfrak{X}^{**}$, then (x_n) is weakly Cauchy since

$$\lim_n x^*(x_n) \quad \text{exists}$$

for all x^*. Hence if $\mathfrak{X}$ is weakly sequentially complete (e.g. ℓ_1 or $L_1(\mu)$)

then $x^{**} \in \mathfrak{X}$. Hence

$$\mathfrak{X} \text{ weakly sequentially complete} \Rightarrow \overline{B_{\mathfrak{X}}}^{w^*} = B_{\mathfrak{X}}$$
$$(\text{in } B_{\mathfrak{X}^{**}})$$

FACT: $\mathfrak{X}$ separable, $\ell_1 \not\hookrightarrow \mathfrak{X} \Rightarrow B_{\mathfrak{X}}$ is weak*-seq dense in $B_{\mathfrak{X}^{**}}$

(Let (A_n) be any disjoint sequence of sets of positive measure. Consider

$$\left\langle \frac{\chi_{A_n}}{\mu(A_n)} : n \in \mathbb{N} \right\rangle$$

This is isometric to ℓ_1 inside L_1. Define $T : \ell_1 \to L_1$

$$T(\alpha_n) = \sum \alpha_n \frac{\chi_{A_n}}{\mu(A_n)} \quad)$$

<u>COROLLARY</u>: $\mathfrak{X}$ is reflexive $\Leftrightarrow B_{\mathfrak{X}}$ is weakly compact.

Proof. ($\Rightarrow$) Known

($\Leftarrow$) If $B_{\mathfrak{X}}$ is weakly compact in $\mathfrak{X}$, then it is weak* compact in $B_{\mathfrak{X}^{**}}$. Hence by Goldstine, $B_{\mathfrak{X}} = B_{\mathfrak{X}^{**}}$, so $\mathfrak{X} = \mathfrak{X}^{**}$.

10/8 BANACH SPACES

FACT: A subset of $\mathfrak{X}$ is weakly compact $\Leftrightarrow$ it is weak*-compact in $\mathfrak{X}^{**}$

HW/ COROLLARY OF GOLDSTINE'S THEOREM: Let K be a subset of $\mathfrak{X}$ such that for each $\varepsilon > 0$ $\exists$ a weakly compact set K_ε in $\mathfrak{X}$ s.t.

$$K \subseteq K_\varepsilon + \varepsilon B_{\mathfrak{X}}$$

Then the weak closure of K is weakly compact, i.e. K is relatively weakly compact.

<u>COROLLARY</u>: If $\mathfrak{X}$ is a Banach space, then $\exists$ a compact Hausdorff space K s.t. $\mathfrak{X}$ is isometric to a subspace of $C(K)$.

Proof. Put $K = B_{\mathfrak{X}^*}$ equipped with the weak* topology. Define $T: \mathfrak{X} \longrightarrow C(K)$ by

$$Tx(x^*) = x^*(x)$$

Then for each $x \in \mathfrak{X}$

$$\|x\| = \sup_{\|x^*\| \le 1} |x^*(x)| = \sup_{t \in K} |Tx(t)| = \|Tx\|_{C(K)}$$

Must also check that Tx is continuous on $B_{\mathfrak{X}^*}$ for the weak* topology. Suppose $x_\alpha^* \longrightarrow x^*$ w^* in $B_{\mathfrak{X}^*}$. Then $x_\alpha^*(x) \longrightarrow x^*(x)$ for each

x, i.e. $Tx(x_\alpha^*) \longrightarrow Tx(x^*)$.

$\blacksquare$

HW/ D+S p438 #36

FACT: If $\mathfrak{X}$ is separable, then the weak*-topology of $B_{\mathfrak{X}^*}$ (and hence for any bounded subset of $\mathfrak{X}^*$) is a norm topology

Proof. Let x_n be dense in $\mathfrak{X}$. It's pretty clear that if (x_α^*) is a net in $B_{\mathfrak{X}^*}$ and $x^* \in B_{\mathfrak{X}^*}$, then

$$\lim x_\alpha^* = x^* \ w^* \iff \lim x_\alpha^*(x_n) = x^*(x_n) \ \forall n$$

$$\iff \lim_\alpha \sum_{n=1}^{\infty} \frac{|(x_\alpha^* - x^*)(x_n)|}{2^n(\|x_n\|+1)} = 0$$

Define $|||\cdot|||$ on $B_{\mathfrak{X}^*}$ by

$$|||x^*||| = \sum_{n=1}^{\infty} \frac{|x^*(x_n)|}{2^n(\|x_n\|+1)}$$

Then $|||\cdot|||$ defines a norm topology on $B_{\mathfrak{X}^*}$ which agrees with the weak* topology.

$\blacksquare$

HW/ Let $\mathfrak{X}$ be separable. Let K be a weakly compact subset of $\mathfrak{X}^*$ s.t. $\overline{\operatorname{sp}(K)} = \mathfrak{X}^*$ (i.e. $\mathfrak{X}^*$ is WCG). Prove $\mathfrak{X}^*$ is norm separable

Hint : Prove weak* and w topology agree on K

FACT : Any compact metric space is the continuous image of the Cantor set.

(See AMM Oct 1976 p646

COROLLARY : If $\mathfrak{X}$ is separable and P is the Cantor set, then $\mathfrak{X}$ is isometric to a subspace of C(P). Consequently, separable B-spaces are isometric to subspaces of C[0,1].

Proof. Let $K = B_{\mathfrak{X}^*}$ in weak*-topology. We know $\exists$ isometry $T: \mathfrak{X} \longrightarrow C(K)$. Let $\varphi : P \longrightarrow K$ be continuous and onto. Define $S : C(K) \longrightarrow C(P)$ by $S(f) = f \circ \varphi$. Since φ is onto

$$\|f\|_{C(K)} = \sup_{t \in K} |f(t)| = \sup_{s \in P} |f(\varphi(s))| = \|Sf\|_{C(P)}$$

Hence $ST : \mathfrak{X} \longrightarrow C(P)$ is an isometry.

To prove the second statement we must find an isometry $R : C(P) \longrightarrow C[0,1]$. Define R as follows. Take $f \in C(P)$ and find a continuous extension Rf to $[0,1]$ by filling in the blanks with linear segments. Then R is linear and

$$\|Rf\|_{C[0,1]} = \|f\|_{C(K)}$$

$\boxtimes$

THEOREM: (Eberlien - Smulian) Any one of the following statement about a subset A of a Banach space $\mathfrak{X}$ implies all the others

① A is relatively weakly compact

② Every sequence in A has a weakly convergent subsequence

③ Every countable subset of A has a weak cluster point

Proof: ② $\Rightarrow$ ③ obvious

① $\Rightarrow$ ② (Smulian 1940)

③ $\Rightarrow$ ① (Eberlien 1947)

10/10 BANACH SPACES

Proof: ① $\Rightarrow$ ② Let (x_n) be a sequence in A. Let $\mathfrak{X}_1 = \overline{\mathrm{sp}(x_n)}$. Let $\Gamma \subset \mathfrak{X}_1^*$ be a countable norming set for $\mathfrak{X}_1$, i.e.

$$x \in \mathfrak{X}_1 \Rightarrow \|x\| = \sup_{x^* \in \Gamma} |x^*(x)|$$

(To get Γ, for each n choose $y_n \in \mathfrak{X}_1^*$ s.t. $y_n(z_n) = \|z_n\|$ and $\|y_n\| = 1$ where (z_n) is a dense sequence in $\mathfrak{X}_1$). Take a subsequence (t_k) of (x_n) s.t.

$$\lim_k y_n^*(t_k) \text{ exists } \forall n$$

using ①

(diagonalization) Let x and $\bar{x}$ be weak cluster points of (t_k). Then

$$\lim_k y_n^*(t_k) = y_n^*(\bar{x}) = y_n^*(x)$$

for every n. Therefore $0 = y_n^*(\bar{x} - x) \ \forall n \Rightarrow$

$$0 = \sup_n |y_n^*(\bar{x} - x)| = \|\bar{x} - x\|$$

$$\Rightarrow \bar{x} = x$$

Hence all cluster points of (t_k) are the same, so (t_k) is weakly convergent.

③ $\Rightarrow$ ① We'll break this down into a series of lemmas.

Lemma: Let $\mathfrak{X}$ be a Banach space, and M a finite dimensional subspace of $\mathfrak{X}^*$. Let $k > 1$. Then there exists a finite set F in the closed unit ball of $\mathfrak{X}$ s.t.

$$x^* \in M \Rightarrow \|x^*\| \leq k \sup_{x \in F} |x^*(x)|$$

Proof. Let $\delta > 0$. Select $\{y_1^*, \dots, y_n^*\}$ such that

$$\bigcup_{i=1}^{n} (y_i^* + \delta B_{\mathfrak{X}^*}) \supseteq \text{surface of } B_{\mathcal{M}}$$

where y_i^* belongs to the surface of the unit ball of M. Pick $x_1, \dots, x_n$ in the unit ball of $\mathfrak{X}$ s.t.

$$|y_i^*(x_i)| > 1 - \delta \quad \forall i$$

Set $F = \{x_1, \dots, x_n\}$. Let $x^* \in M$ and suppose $\|x^*\| = 1$. Pick i_0 such that $\|x^* - y_{i_0}^*\| < \delta$. Then

$$\begin{aligned} |x^*(x_{i_0})| &= |x^*(x_{i_0}) - y_{i_0}^*(x_{i_0}) + y_{i_0}^*(x_{i_0})| \\ &\geq |y_{i_0}^*(x_{i_0})| - |x^*(x_{i_0}) - y_{i_0}^*(x_{i_0})| \\ &\geq |y_{i_0}^*(x_{i_0})| - \|x^* - y_{i_0}^*\| \, \|x_{i_0}\| \end{aligned}$$

$$\geq 1 - \delta - \delta$$

Hence $\sup_{x \in F} |x^*(x)| \geq 1 - 2\delta$ provided $\|x^*\| = 1$ and $x^* \in M$, i.e.

$$\frac{1}{1-2\delta} \sup_{x \in F} |x^*(x)| \geq 1$$

Hence $x^* \in M \implies$

$$\frac{1}{1-2\delta} \sup_{x \in F} \left| \frac{x^*}{\|x^*\|}(x) \right| \geq 1$$

$$\implies \|x^*\| \leq \frac{1}{1-2\delta} \sup_{x \in F} |x^*(x)|$$

Now choose δ at the beginning s.t. $1/(1-2\delta) < k$ to finish the proof.

$\boxtimes$

<u>Day's Lemma</u>: Let $A \subset \mathfrak{X}^*$. Suppose

(1) Every countable subset of A has a weak cluster point

(2) 0 is in the weak* closure of A

Then $\exists$ seq. (x_n^*) in A s.t. $\lim x_n^* = 0$ weakly

Proof. Claim: $\exists$ an increasing sequence (F_n) of finite subsets of the unit ball of $\mathfrak{X}$ and a sequence (y_n^*) in A s.t.

(i) $y^* \in \operatorname{sp}\{y_1^*, \dots, y_n^*\} \implies \|y^*\| \leq 2 \sup_{x \in F_n} |y^*(x)|$

(ii) $\sup_{x \in F_n} |y^*_{n+1}(x)| < \frac{1}{n+1}$

Construction: Chose $y_1^* \in A$ arbitrarily. Chose $x_1 \in B_{\mathfrak{X}}$ s.t.

$$\|y_1^*\| \le 2|y_1^*(x_1)|$$

Set $F_1 = \{x_1\}$. Suppose $F_1 \subset F_2 \subset \cdots \subset F_n$ and $\{y_1^*, \ldots, y_n^*\}$ have been chosen to satisfy (i). We know $\exists$ a net (z_α^*) in A s.t.

$$\lim_\alpha z_\alpha^*(x) = 0 \quad \forall x \in \mathfrak{X}$$

since 0 is a weak* cluster point of A. Therefore

$$\lim_\alpha \sup_{x \in F_n} |z_\alpha^*(x)| = 0$$

Therefore $\exists\, y^*_{n+1} \in A$ s.t.

$$\sup_{x \in F_n} |y^*_{n+1}(x)| < \frac{1}{n+1}$$

Now use the last lemma to find a finite set H_n in the unit ball of $\mathfrak{X}$ s.t.

$$y^* \in \operatorname{sp}\{y_1^*, \ldots, y_{n+1}^*\} \Rightarrow \|y^*\| \le 2 \sup_{x \in H_n} |y^*(x)|$$

Set $F_{n+1} := F_n \cup H_n$

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(Proof continued)

Let $D = \bigcup_{n=1}^{\infty} F$. Then $x^* \in \langle (y_n^*) \rangle$ implies $\|x^*\| \leq 2 \sup_{x \in D} |x^*(x)|$

Hence $x^* \in \overline{\langle (y_n^*) \rangle}$ implies $\|x^*\| \leq 2 \sup_{x \in D} |x^*(x)|$. By the second condition we see that

$$\lim_n y_n^*(x) = 0 \quad \forall x \in D$$

By hypothesis, the sequence (y_n^*) has a weak cluster point y^*. Hence $y^* \in \overline{\langle y_n^* \rangle}$ (Hahn-Banach), and so

$$\|y^*\| \leq 2 \sup_{x \in D} |y^*(x)|$$

Take a subnet (y_α^*) of (y_n^*) such that $\lim y_\alpha^* = y^*$ weakly. Since

$$\lim y_n^*(x) = 0 \quad \forall x \in D$$

and since $(y_\alpha^*(x))$ is a subnet of $(y_n^*(x))$, we see that

$$0 = \lim_\alpha y_\alpha^*(x) \quad \forall x \in D$$

But

$$y^*(x) = \lim_\alpha y_\alpha^*(x) \quad \forall x \in D$$

Therefore $y^*(x) = 0 \quad \forall x \in D$, and so

$$0 \leq \|y^*\| \leq 2 \sup_{x \in D} |y^*(x)| = 0$$

$$\Rightarrow \quad y^* = 0$$

Hence every weak cluster point of (y_n^*) is 0, and so $\lim y_n^* = 0$ weakly

$\boxtimes$

Now for proof of ③ $\Rightarrow$ ①: Let A be a subset of a B-space $\mathfrak{X}$ such that every countable subset of A has a weak cluster point. Then A is relatively weakly compact. In fact, every point in the weak closure of A is the weak limit of a sequence of members in A.

Proof. Regard A as a subset of $\mathfrak{X}^{**}$. A is obviously bounded. Hence its weak* closure $\tilde{A} \subset \mathfrak{X}^{**}$ is bounded. Let x^{**} be a weak* cluster point of A [[$\tilde{A}$ is weak* compact]] Put $C := A - x^{**}$. Then C satisfies the conditions of Day's lemma. An appeal to Day's lemma produces a sequence $(a_n - x^{**})$ in $A - x^{**} = C$ s.t.

$$\lim_n (a_n - x^{**}) = 0 \text{ weakly}$$

$$\Rightarrow \quad \lim_n a_n = x^{**} \text{ weakly in } \mathfrak{X}^{**}$$

Since $\mathfrak{X}$ is a closed subspace of $\mathfrak{X}^{**}$, the Hahn-Banach tells us that $x^{**} \in \mathfrak{X}$. Hence the weak* compact set $\tilde{A}$ is a subset of $\mathfrak{X}$

But inside $\mathfrak{X}$ ($\subset \mathfrak{X}^{**}$) the weak and weak* topology agree on $\mathfrak{X}$. Hence $\tilde{A}$ is weakly compact and = weak closure of A.

$\blacksquare$

<u>COROLLARY</u>: A B-space is reflexive iff each of its bounded sequences has a weakly convergent subsequence.

Proof. A space is reflexive iff the unit ball is weakly compact.

<u>COROLLARY</u>: The relatively weakly compact sets in $L_1(\mu)$ are precisely the bounded uniformly integrable sets.

HW/ ① $\mathfrak{X}$ weakly seq. complete, $\mathfrak{X}^*$ separable $\Rightarrow$ $\mathfrak{X}$ is reflexive

② Let $\mathfrak{X}$ be a reflexive subspace of $L_1(\mu)$ (μ finite) Then on bounded subsets of $\mathfrak{X}$, the L_1- topology agrees with the top of convergence in measure

<u>COROLLARY</u>: Weak compactness in a B-space is separably determined, i.e. A is (relatively) weakly compact $\Leftrightarrow$ $A \cap S$ is (relatively) weakly compact for all separable subspaces S of $\mathfrak{X}$.

<u>COROLLARY</u>: A bounded set A in $C(K)$ is relatively weakly compact iff every sequence in A has a pointwise convergent subsequence.

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LINEAR EQUATIONS

Consider the linear equations

$$x^*(x_i) = c_i \qquad (*)$$

where $x_i \in \mathfrak{X}$, c_i in reals are given. Let $\mathcal{Y} = \langle x_i \rangle_{i=1}^n$. Then $\mathcal{Y}$ is a finite dimensional space and Math 318 tells us that $\exists\, y^* \in \mathcal{Y}^*$ such that $y^*(x_i) = c_i$ for each i provided the original system is consistent. Apply Hahn-Banach to get solutions x^* in $\mathfrak{X}^*$.

Usual manipulations tell us that the solution set for $(*)$ is one fixed solution $+\ \mathcal{Y}^\perp$.

Question: What is the minimum norm solution?

Let x_0^* be fixed solution. Then

$$\|x_0^*\|_{\mathfrak{X}^*/\mathcal{Y}^\perp} = \inf_{y^* \in \mathcal{Y}^\perp} \|x_0 + y^*\| = \text{norm of a min norm solution}$$

(provided min norm solution exists)

Fact 1: Let $x_0^* \in \mathfrak{X}^*$

$$\|x_0^*|_{\mathcal{Y}}\| = \sup\{|x_0^*(y)| : \|y\| \le 1, y \in \mathcal{Y}\} = \inf_{y^* \in \mathcal{Y}^\perp} \|x_0^* + y^*\|$$

(since $\mathcal{Y}^* = \mathcal{X}^*/\mathcal{Y}^\perp$)

Fact 2: The inf on the right is achieved.

Proof. Restrict x_0^* to $\mathcal{Y}$ and let z^* be any norm preserving Hahn-Banach extension of $x_0^*|\mathcal{Y}$ to all of $\mathcal{X}$. Let $y^* = x^* - z^*$. Then $y^* \in \mathcal{Y}^\perp$. Also

$$\|x_0^* - y^*\| = \|z^*\| = \sup_{\substack{y \in \mathcal{Y} \\ \|y\| \le 1}} |x^*(y)|$$

A glance at fact 1 shows that $z^* = x_0^* - y^*$ is a minimum norm solution.

Fact 3: Since $\mathcal{Y}$ is finite dimensional, the sup above is achieved.

Suppose the sup is achieved at $y_0 \in \mathcal{Y}$, $\|y_0\| = 1$. Then

$$\|x^*_{min}\| = x_0^*(y_0) = (x_0^* - y^*)(y_0) = x_{min}(y_0)$$

$$\| \qquad\qquad$$

$$\|x^*_{min}\| \cdot \|y_0\|$$

Fact 4: If sup is achieved at y_0, $\|y_0\|=1$, then

$$x^*_{min}(y_0) = \|x^*_{min}\| \|y_0\|$$

Back to our linear equations

$$x^*(x_i) = c_i \qquad \forall i \leq n$$

To find x^*_{min} = min norm solution, we know

$$\|x^*_{min}\| = \sup_{\substack{y \in \mathcal{Y} \\ \|y\| \leq 1}} \left(x_0^*(y)\right) = \sup_{\|\sum \alpha_i x_i\| \leq 1} \left(x_0^*\left(\sum \alpha_i x_i\right)\right)$$

$$= \sup_{\|\sum \alpha_i x_i\| \leq 1} \left(\sum_{i=1}^{n} \alpha_i c_i\right) = \sum_{i=1}^{n} \bar{\alpha}_i c_i$$

for some $\bar{\alpha}_1, \dots, \bar{\alpha}_n$ with $\|\sum_{i=1}^{n} \bar{\alpha}_i x_i\| = 1$. Once we have $\sum \bar{\alpha}_i x_i$ we can usually find x^*_{min} by specifying its norm (which we know) and forcing

$$x^*_{min}\left(\sum_{i=1}^{n} \bar{\alpha}_i x_i\right) = \|x^*_{min}\|$$

<u>Example</u> : We are to turn a shaft with an electric motor

Want: $\theta(0) = 0 \quad \theta(1) = 1$

$\dot{\theta}(0) = 0 = \dot{\theta}(1)$

Know: $\ddot{\theta} + \dot{\theta} = u$ = current input

Goals : ① $\min_{u} \left(\max_{0\le t\le 1} |u(t)| \right)$ (min current)

② $\min_{u} \int_0^1 u^2(t)\,dt$ (min energy)

Our job : realize this in terms of linear equations in B-space.

Observe

$$e^t \ddot{\theta}(t) + e^t \dot{\theta}(t) = e^t u(t)$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^t \dot{\theta}(t)\right) = e^t u(t)$$

$$\Rightarrow \quad e^t \dot{\theta}(t)\Big|_0^1 = \int_0^1 e^s u(s)\,ds$$

$$\Rightarrow \quad 0 = \int_0^1 e^s u(s)\,ds$$

Also

$$\int_0^1 \ddot{\theta}(t)\,dt + \int_0^1 \dot{\theta}(t)\,dt = \int_0^1 1\cdot u(t)\,dt$$

$$\Rightarrow \quad \dot{\theta}(1) - \dot{\theta}(0) + \theta(1) - \theta(0) = \int_0^1 1\cdot u(t)\,dt$$

$$\Rightarrow \quad 1 = \int_0^1 1\cdot u(s)\,ds$$

Hence we want to

$$\min \|u\|_\infty$$

$$\text{s.t. } \int_0^1 e^t u(t)\,dt = 0$$

$$\int_0^1 1 \cdot u(t)\,dt = 0$$

Take $\mathcal{X} = L_1[0,1]$ and $\mathcal{X}^* = L_\infty[0,1]$. Let $\mathcal{Y} = \langle 1, e^t \rangle$ in $L_1[0,1]$
Then

$$\|u_{min}\|_\infty = \sup_{\int_0^1 |\alpha_1 + \alpha_2 e^t|\,dt = 1} (\alpha_1 \cdot 1 + \alpha_2 \cdot 0) = \sup_{\int_0^1 |\alpha_1 + \alpha_2 e^t|\,dt \le 1} \alpha_1$$

It is tedious to solve for $\bar{\alpha}_1, \bar{\alpha}_2$ but not impossible. Suppose this is done
We know

$$\int_0^1 u_{min} (\bar{\alpha}_1 + \bar{\alpha}_2 e^t)\,dt = \|u_{min}\| \cdot 1$$

Hence

$$u_{min} = \operatorname{sgn}(\bar{\alpha}_1 + \bar{\alpha}_2 e^t) \|u_{min}\|$$

$$= \operatorname{sgn}(\bar{\alpha}_1 + \bar{\alpha}_2 e^t)\, \bar{\alpha}_1$$

(Bang-bang solution)

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Fact: $(\mathfrak{X}/\mathcal{Y})^* = \mathcal{Y}^\perp$

Let $x \in \mathfrak{X}$. Then

$$\inf_{y \in \mathcal{Y}} \|x - y\| = \|\hat{x}\|_{\mathfrak{X}/\mathcal{Y}} = \sup_{\substack{\ell \in (\mathfrak{X}/\mathcal{Y})^* \\ \|\ell\| \le 1}} \ell(\hat{x}) = \sup_{\substack{y^* \in \mathcal{Y}^\perp \\ \|y^*\| \le 1}} y^*(x)$$

$$= \sup_{\substack{y^* \in \mathcal{Y}^\perp \\ \|y^*\| \le 1}} y^*(x - y) \quad \forall y \in \mathcal{Y}$$

(By H-B thm)

The sup on the right is achieved at some $y_0^* \in \mathcal{Y}^\perp$ with $\|y_0^*\| = 1$. Suppose the inf on the left is achieved at $y_0 \in \mathcal{Y}$. Then

$$\|x - y_0\| = y_0^*(x - y_0) \quad \left(\|y_0^*\| \le 1 \right)$$

$$\|$$

$$\|x - y_0\| \, \|y_0\|$$

<u>COROLLARY</u> (OF THIS STUFF): Let $\mathfrak{X}$ be a B-space such that there exists $x_0^* \in \mathfrak{X}^*$ with $\|x_0^*\| = 1$ but $|x_0^*(x)| < 1$ for every x in the closed unit ball of $\mathfrak{X}$. Take $x_0 \in \mathfrak{X}$ s.t. $x_0^*(x_0) > 0$ and let $\mathcal{Y}$ = kernel of x_0^*. Then there is no vector in $\mathcal{Y}$ closest to x_0.

Proof. Let $x \in \mathfrak{X}$ and write

$$x = x - \alpha x_0 + \alpha x_0$$

Then $x_0^*(x) = x_0^*(x) - \alpha x_0^*(x_0) + \alpha x_0^*(x_0)$. Setting $\alpha = x^*(x)/x_0^*(x_0)$ gives $x = (x - \alpha x_0) + \alpha x_0$ where $x - \alpha x_0 \in \mathcal{Y}$, i.e.

$$\mathcal{X} = \mathcal{Y} + \langle x_0 \rangle$$

Therefore $\mathcal{X}/\mathcal{Y} = \langle x_0 \rangle$ is one-dimensional. Therefore $(\mathcal{X}/\mathcal{Y})^* = \mathcal{Y}^\perp$ is also one dimensional. Since $x_0^* \in \mathcal{Y}^\perp$, we see

$$\mathcal{Y}^\perp = \langle x_0^* \rangle$$

Suppose $\exists\, y_0 \in \mathcal{Y}$ s.t.

$$(*) \qquad \|x_0 - y_0\| = \inf_{y \in \mathcal{Y}} \|x_0 - y\| = \sup_{\substack{y^* \in \mathcal{Y}^\perp \\ \|y^*\| \le 1}} y^*(x_0) = \underbrace{y_0^*(x_0)}_{\text{some } y_0^* \in \mathcal{Y}^\perp \text{ with } \|y_0^*\| = 1} \overset{\text{since } y_0^* = e^{i\theta} x_0^*}{=} x_0^*(x_0)$$

$$= x_0^*(x_0 - y_0)$$

Hence

$$x_0^*\left(\frac{x_0 - y_0}{\|x_0 - y_0\|}\right) = 1 \qquad \lightning$$

$\blacksquare$

Approximation Theory

THEOREM (Tonelli) Let $f \in C[a,b]$. Let Y = subspace of $C[a,b]$ consisting of polynomials of degree $\le n$. Let p be the polynomial in Y closest to f in $C[a,b]$ norm. Then $\exists$ at least $(n+2)$ distinct points t s.t.

$$|f(t) - p(t)| = \|f - p\|$$

(Why does there exist such a p? Take (y_n) in Y s.t. $\|y_n - f\| \to \inf \|y - f\|$. Then (y_n) is bounded in $C[a,b]$. Since Y is finite dimensional, bounded sets are relatively compact and hence y_n has a convergent subsequence. Its limit p will do the job)

Proof. By what we did before, we know $\exists\, \ell \in Y^{\perp}$, $\|\ell\| = 1$ s.t.

$$\|f - p\| = \ell(f - p)$$

i.e. there exists a regular Borel measure μ s.t

$$\int_{[a,b]} (f-p)\,d\mu = \max_{t \in [a,b]} |f(t) - p(t)| \underbrace{|\mu|[a,b]}_{=1}$$

Therefore μ is fully supported on $\{t \in [a,b] : |f(t) - p(t)| = \sup_{s \in [a,b]} |f(s) - p(s)|\}$

Suppose $\{t \in [a,b] : |f(t) - p(t)| = \|f - p\|\} = \{t_1, \dots, t_k\}$ where $k \le n+1$.
Let t_i be a point in above set s.t. $\mu\{t_i\} \ne 0$ $[\![f \notin Y]\!]$

Put

$$q(t) = \prod_{n \neq i} (t - t_n)$$

Then $q \in \mathcal{Y}$. Hence

$$0 = \int q \, d\mu = q(t_i) \neq 0 \quad \lightning$$

$\blacksquare$

Series in Banach Spaces

① Convergence
② Unconditional convergence, i.e. all of its subseries converges (UC)
③ Absolute convergence ($\sum \|x_n\| < \infty$)

Fact: If $\mathfrak{X}$ is finite dimensional, then a series in $\mathfrak{X}$ is unconditionally convergent if and only if it is absolutely convergent

THEOREM: (Dvoretsky - Rogers) The above fact characterizes finite dimensional spaces.

④ Weakly unconditionally convergent (WUC): $\sum x_n$ is a WUC in $\mathfrak{X}$ if $\sum |x^*(x_n)| < \infty \ \forall x^* \in \mathfrak{X}^*$

FACT 0: Every UC is a WUC

Proof. If every subseries of $\sum x_n$ is norm convergent, then for each $x^* \in \mathfrak{X}^*$ we see that every subsequence of $\sum x^*(x_n)$ is convergent. Hence $\sum |x^*(x_n)| < \infty$.

FACT 1: If $\sum x_n$ is a WUC in $\mathfrak{X}$, then

$$\sup_{\|x^*\| \le 1} \sum |x^*(x_n)| < \infty$$

Proof Define $T: \mathfrak{X}^* \longrightarrow \ell_1$ by $T(x^*) = (x^*(x_n))$. The graph of T is closed, so T is bounded, i.e.

$$\infty > \sup_{\|x^*\| \le 1} \|Tx^*\| = \sup_{\|x^*\| \le 1} \sum_{n=1}^{\infty} |x^*(x_n)|$$

$\blacksquare$

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WUC norm - $\sup_{\|x^*\| \leq 1} \sum |x^*(x_n)|$

<u>Example</u>: Let x_n be the usual unit vector basis of c_0. Then $\sum x_n$ is a WUC but not is convergent

Proof. $\sum x_n$ does not converge since $\lim \|x_n\| = 1 \neq 0$. On the other hand, if $x^* = (\alpha_n)$ is in $\ell_1 = c_0^*$, then

$$\sum |x^*(x_n)| = \sum_{n=1}^{\infty} |\alpha_n| < \infty$$

so $\sum x_n$ is a WUC.

<u>FACT 2</u>: Suppose $\sum x_n$ is not convergent. Then $\exists$ disjoint blocks A_n of positive integers s.t. $A_1 \leq A_2 \leq A_3 \leq \ldots$ and a $\delta > 0$ s.t.

$$\Big\| \sum_{i \in A_n} x_i \Big\| \geq \delta$$

(i.e. the partial sums do not form a Cauchy sequence)

<u>FACT 3</u>: If $\sum x_n$ is a non-convergent WUC, then it has a subseries that can be grouped into a series $\sum y_n$ s.t.

$$\inf \|y_n\| = \delta > 0$$

($y_n = \sum_{i \in A_n} x_i$ from fact 2)

<u>FACT 4:</u> If $\sum y_n$ is as above then $\exists$ a seq (x_n^*) in $\mathfrak{X}^*$ with $\|x_n^*\| = 1$ and $x_n^*(y_n) \geq \delta$ $\forall n$

Define signed measures μ_m on $\mathcal{P}(\mathbb{N})$ by

$$\mu_m(E) := \sum_{n \in E} x_m^*(y_n)$$

Then

$$\sup_m |\mu_m|(\mathbb{N}) = \sup_m \sum_{n=1}^{\infty} |x_m^*(y_n)| \leq \sup_{\|x^*\| \leq 1} \sum_{n=1}^{\infty} |x^*(y_n)|$$

$$\leq \sup_{\|x^*\| \leq 1} \sum_{j=1}^{\infty} |x^*(x_j)| \underbrace{<}_{\text{Fact 1}} \infty$$

Therefore Rosenthal's lemma is applicable. Put $E_n = \{n\}$. Apply Rosenthal's lemma to find $n_1 < n_2 < \ldots$ such that

$$|\mu_{n_j}| \left(\bigcup_{i \neq j} E_{n_i} \right) < \delta/2$$

this delta from fact 3

$$\Rightarrow \sum_{i \neq j} |x_{n_j}^*(y_{n_i})| < \delta/2$$

(Recall $|x_{n_j}^*(y_{n_j})| \geq \delta$

THEOREM (Bessaga-Pelczynski). If $\mathfrak{X}$ has a series $\sum x_n$ that is a WUC but not convergent, then $c_0 \hookrightarrow \mathfrak{X}$

Proof. Let $\sum x_n$ be a non-convergent WUC. Use above facts to find $\sum y_n$, $\delta > 0$, and (x_m^*) in unit ball of $\mathfrak{X}^*$ s.t.

$$|x^*_{m_j}(y_{m_j})| \geq \delta$$

$$\sum_{i \neq j} |x^*_{m_j}(y_{m_i})| < \delta$$

Define $T : c_0 \to \mathfrak{X}$ as follows: if $(\alpha_n) \in c_0$ is finitely non-zero, put

$$T(\alpha_n) = \sum_{j=1}^{\infty} \alpha_j y_{n_j}$$

T is linear on a dense subset of c_0. Let $x^* \in \mathfrak{X}^*$, $\|x^*\| \leq 1$, and note

$$|x^* T(\alpha_n)| \leq \sum_{j=1}^{\infty} |\alpha_j| |x^*(y_{n_j})|$$

$$\leq \|(\alpha_n)\|_{c_0} \sum_{j=1}^{\infty} |x^*(y_{n_j})|$$

$$\leq \|(\alpha_n)\|_{c_0} \sup_{\|x^*\| \leq 1} \sum_{k=1}^{\infty} |x^*(x_k)|$$

$$= W \|(\alpha_n)\|_{c_0}$$

Hence

$$\|T(\alpha_n)\| = \sup_{\|x^*\|\leq 1} |x^* T(\alpha_n)| \leq W \|(\alpha_n)\|_{c_0}$$

Therefore T has a continuous linear extension to all of c_0.

Again let $(\alpha_n) \in c_0$ be finitely non-zero. Consider

$$|x^*_{m_j}(T(\alpha_n))| = \Big| \sum_{i=1}^{\infty} \alpha_i x^*_{m_j}(y_{m_i}) \Big|$$

$$\geq |\alpha_j x^*_{m_j}(y_{m_j})| - \Big| \sum_{i \neq j} \alpha_i x^*_{m_j}(y_{m_i}) \Big|$$

$$\geq |\alpha_j| \, |x^*_{m_j}(y_{m_j})| - \sum_{i\neq j} |\alpha_i| \, |x^*_{m_j}(y_{m_i})|$$

$$\geq |\alpha_j| \delta - \|(\alpha_n)\|_{c_0} \sum_{i \neq j} |x^*_{m_j}(y_{m_i})|$$

$$\geq |\alpha_j| \delta - \|(\alpha_n)\|_{c_0} \frac{\delta}{2}$$

Hence

$$\|T(\alpha_n)\| = \sup_{\|x^*\|\leq 1} |x^* T(\alpha_n)| \geq \sup_j |x^*_{m_j} T(\alpha_n)|$$

$$\geq \|(\alpha_n)\|_{c_0} \delta - \|(\alpha_n)\|_{c_0} \frac{\delta}{2} = \|(\alpha_n)\| \frac{\delta}{2}$$

Therefore T^{-1} exists and is continuous.

$\boxtimes$

COROLLARY: (Bessaga-Pelczynski) $c_0 \not\hookrightarrow \mathfrak{X}$ if and only if all WUC's are UC's.

Proof. If there is a non-convergent WUC we just proved that $c_0 \hookrightarrow \mathfrak{X}$. If $c_0 \xhookrightarrow{T} \mathfrak{X}$ then $\sum T(e_n)$ is a WUC in $\mathfrak{X}$ that is not convergent. Hence

$$c_0 \not\hookrightarrow \mathfrak{X} \iff \text{all WUCs converge}$$

To prove the corollary, notice that any subseries of a WUC is a WUC. If all WUC's converge, then every WUC is an UC.

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<u>COROLLARY</u> (Orlicz 1929) Let $\mathfrak{X}$ be a weakly sequentially complete B-space. Then all WUCs are UCs.

<u>THEOREM</u> (Orlicz-Pettis) Let $\sum x_n$ be a series in $\mathfrak{X}$ s.t. each of its subseries is weakly convergent. Then $\sum x_n$ is convergent (and consequently is an UC.)

Proof. Evidently $\sum x_n$ is a WUC. Since $\sum x^*(x_n)$ and all of its subseries converge, $\sum x^*(x_n)$ is absolutely convergent, so

$$\sum |x^*(x_n)| < \infty$$

This holds for each $x^* \in \mathfrak{X}^*$. Suppose $\sum x_n$ is not norm convergent and pass to the series $\sum y_n$ as before. Find $m_1 < m_2 < \dots$ and a sequence $(x^*_{m_j})$ in the closed unit ball of $\mathfrak{X}^*$ and a $\delta > 0$ such that

$$x^*_{m_j}(y_{m_j}) > \delta$$

$$\sum_{i \neq j} |x^*_{m_j}(y_{m_i})| < \delta/2$$

Define $T : l_\infty \longrightarrow \mathfrak{X}$ as follows. First recall the finitely valued sequences $\sum \alpha_i \chi_{A_i}$ are dense in l_∞ $(A_i \subset \mathbb{N},\ A_i \cap A_j = \emptyset)$
Write

$$T\left(\sum_{i=1}^{n}\alpha_i \chi_{A_i}\right) = \sum_{i=1}^{n}\alpha_i \left(\underset{\text{weak limit}}{w}\text{-}\sum_{j\in A_i} y_{m_j}\right)$$

Notice that T is linear and densely defined. Also, if $x^* \in \mathfrak{X}^*$ and $\|x^*\| \le 1$, then

$$\left| x^* T\left(\sum \alpha_i \chi_{A_i}\right)\right| = \left|\sum_{i=1}^{n}\alpha_i \sum_{j\in A_i} x^* y_{m_j}\right|$$

$$\le \sum_{i=1}^{n}\sum_{j\in A_i} |\alpha_i|\, |x^*(y_{m_j})|$$

$$\le \left\|\sum_{i=1}^{n}\alpha_i \chi_{A_i}\right\|_{\ell_\infty} \sum_{i=1}^{n}\sum_{j\in A_i} |x^*(y_{m_j})|$$

$$\le \left\|\sum_{i=1}^{n}\alpha_i \chi_{A_i}\right\|_{\ell_\infty} \left(\text{WUC norm of } \sum x_n\right)$$

Therefore T is bounded.

On the other hand, let $\sum_{i=1}^{n}\alpha_i\chi_{A_i}$ be a finitely valued sequence in ℓ_∞. Suppose $|\alpha_1| =$ norm of this sequence. Pick $p \in A_1$. Then

$$\left| x^*_{m_p} T\left(\sum_{i=1}^{n}\alpha_i \chi_{A_i}\right)\right| = \left|\sum_{i=1}^{n}\sum_{j\in A_i}\alpha_i\, x^*_{m_p}(y_{m_j})\right|$$

$$= \left|\sum_{j\in A_1}\alpha_1 x^*_{m_p}(y_{m_j}) + \sum_{i=2}^{n}\sum_{j\in A_i}\alpha_i x^*_{m_p}(y_{m_j})\right|$$

$$\geq |\alpha_1 x^*_{m_p}(y_{m_p})| - |\text{everything else}|$$

$$\geq |\alpha_1|\delta - |\text{everything else}|$$

$$\geq |\alpha_1|\delta - \sum_{i \neq j} x^*_{m_p}(y_{m_i}) \cdot |\alpha_1|$$

$$\geq |\alpha_1|\delta - |\alpha_1|\delta/2 = |\alpha_1|\delta/2$$

$$= \left\| \sum_{i=1}^{n} \alpha_i \chi_{A_i} \right\| \delta/2$$

Hence

$$\left\| T\left(\sum_{i=1}^{n} \alpha_i \chi_{A_i}\right) \right\| \geq \frac{\delta}{2} \left\| \sum_{i=1}^{n} \alpha_i \chi_{A_i} \right\|$$

and so T is invertible and its inverse is continuous. Therefore has an extension to an isomorphism from l_∞ to $\mathfrak{X}$.

But where is the range of T? It is in the weak closure of $\langle x_n \rangle$ = norm closure of $\langle x_n \rangle$ = separable set ↯, since l_∞ is not separable.

$\boxed{/\!/}$

HW/ ① Let $\sum x_n$ be a series in $\mathfrak{X}$. Show $\sum x_n$ is a WUC iff

$$\sup_{\Delta \text{ finite}} \left\| \sum_{n \in \Delta} x_n \right\| < \infty$$

Hint: $\sup_{\Delta} \|\cdot\| \leq$ WUC norm $\leq \begin{matrix} 4\ (\mathbb{C}) \\ 2\ (\mathbb{R}) \end{matrix} \sup_{\Delta}$

② Let $\sum x_n^*$ be a series in $\mathfrak{X}^*$ s.t. $\sum |x_n^*(x)| < \infty$ $\forall x \in \mathfrak{X}$. Prove $\sum x_n^*$ is a WUC.

③ Show that if $\sum |x_n^*(x)| < \infty$ $\forall x \in \mathfrak{X}$ but $\sum x_n^*$ is not convergent, then $\ell_\infty \hookrightarrow \mathfrak{X}^*$

④ Deduce $c_0 \hookrightarrow \mathfrak{X}^* \Rightarrow \ell_\infty \hookrightarrow \mathfrak{X}^*$

⑤ Show that $\ell_\infty \not\hookrightarrow \mathfrak{X}^* \Rightarrow$ all WUCs in $\mathfrak{X}^*$ converge in norm.

(Hint for ③ $\sum |x_n^*(x)| < \infty$ $\forall x^* \in \mathfrak{X} \Rightarrow$ every subseries converges weak*)

Linear Topological Spaces

Definition: A linear topological space is a vector space V with a topology (Hausdorff) τ s.t. if $\mathcal{U}$ is a nbhd base at 0, then $x + \mathcal{U}$ is a nbhd base at x, and s.t. multiplication by scalars is continuous. An LTS is called a locally convex space if it has a nbhd basis at 0 consisting of convex nbhds.

Examples: LTS, $\sim$ LCS $\quad L_p[0,1] \quad 0 < p < 1$

$$\|f-g\|_p = \int |f-g|^p \, d\mu$$

(metric, not a norm) This space is not locally convex since the function $t \to t^p$ is not convex when $0 < p < 1$.

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Examples : (2) $L_0(\mu)$ all measurable functions

$$\rho(f,g) = \int \frac{|f-g|}{1+|f-g|}\,d\mu$$

LTS, $\sim$ LCS

(3) Any Banach space is an LCS

(4) Any Banach space under its weak topology is LCS

(5) Any dual B-space in its weak* topology is an LCS

(6) Let $\mathfrak{X}$ be any B-space and let $\Gamma \subset \mathfrak{X}^*$ be any separating family. Define a nbhd-basis $\mathcal{U}$ at 0 by

$$\mathcal{U}_F^{\varepsilon} := \{ x \in \mathfrak{X} : |x^*(x)| < \varepsilon \quad \forall x^* \in \underset{\text{finite}}{F} \subset \Gamma \}$$

The resulting topology is called the $\sigma(\mathfrak{X}, \Gamma)$ topology.

Notation : $x, y \in \mathfrak{X}$. Then $[x,y] := \{ tx + (1-t)y : 0 \le t \le 1 \}$

$(x,y) := \{ tx + (1-t)y : 0 < t < 1 \}$

<u>DEFINITION</u>: $C \subset \mathfrak{X}$ is convex iff $x, y \in C \Rightarrow [x,y] \subset C$.

<u>Facts</u> ① $C \subset \mathcal{X}$ convex, $\lambda_i \geq 0$ $\sum_{i=1}^{n} \lambda_i = 1$

$$x_i \in C \implies \sum_{i=1}^{n} \lambda_i x_i \in C$$

Proof. (induction on n) True for $n=1$. General case

$$\sum_{i=1}^{n+1} \lambda_i x_i = \sum_{i=1}^{n} \lambda_i x_i + \lambda_{n+1} x_n = \sum_{k=1}^{n} \lambda_k \left(\sum_{i=1}^{n} \frac{\lambda_i}{\sum_{k=1}^{n} \lambda_i} x_i \right) + \lambda_{n+1} x_{n+1}$$

$$= \text{convex combination of } \sum_{i=1}^{n} \frac{\lambda_i}{\sum \lambda_i} x_i \in C \text{ and } x_{n+1} \in C$$

by induction hypothesis

<u>DEFINITION</u>: Let A be a subset of a vector space. Then

$co(A) :=$ smallest convex set containing A
$= \bigcap$ all convex sets containing A

② Let $A \subseteq \mathcal{X}$ Then

$$co(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in A, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N} \right\}$$

<u>DEFINITION</u>: Let A be a subset of an LTS $\mathcal{X}$. Then

$\overline{co}(A) :=$ smallest closed convex set containing A

$= \overline{co(A)}$

Warning: In general $\overline{co}(A) \neq \{\sum_{n=1}^{\infty} \lambda_n x_n : x_n \in A, \lambda_n \geq 0, \sum_{n=1}^{\infty} \lambda_n = 1\} = B$

Example: Let A = set of unit vector basis of ℓ_2. Let $A = (e_n)$. Let

$$x_m := \sum_{n=1}^{m} \frac{1}{m} e_n \in co(A)$$

$$\|x_m\| = \left(\sum_{n=1}^{m} \frac{1}{m^2}\right)^{1/2} = \left(\frac{m}{m^2}\right)^{1/2} = \frac{1}{\sqrt{m}} \to 0$$

But $0 \notin B$

(3) $C \subseteq$ LTS $\mathfrak{X}$ convex $\Rightarrow \overline{C}$ is convex

THEOREM: Let C be a convex set in an LTS $\mathfrak{X}$. Suppose $C^\circ = \text{int}\, C \neq 0$. If $x \in C^\circ$ and $y \in \overline{C}$, then $(x,y) \subset C^\circ$

Proof. Choose a $\overset{\text{sym}}{\text{a}}$ nbhd U of 0 s.t. $x + U \subset C^\circ$. Let $0<t<1$ and observe $t/(t-1)\, U$ is also a nbhd of 0. Since $y \in \overline{C}$, then for each $0<t<1$, $\exists\, y_t \in C$ s.t. $y_t \in y + t/(t-1)\, U$, i.e.

$$(t-1)(y_t - y) \in t\,U$$

But also $0<t<1 \Rightarrow$

$$\underbrace{t(x+U) + (1-t)\, y_t}_{\text{open set}} \subset tC + (1-t)\, y_t \subset C$$

Hence $tx + tU + (1-t)y_t \subset C^o$, and so

$$\begin{array}{c} tx + (t-1)(y_t - y) + (1-t)y_t \in tx + tU + (1-t)y_t \subset C^o \\ \| \\ tx + (1-t)y \end{array}$$

$\blacksquare$

COROLLARY (of proof) C convex in an LTS $\mathfrak{X}$, $C^o \neq \emptyset$. Then C^o is convex.

COROLLARY: Under same hypothesis $\overline{C} = \overline{C^o}$

DEFINITION: Let C be a subset of an LTS $\mathfrak{X}$. A point $x \in C$ is a core point of C if for each $y \in \mathfrak{X}$ $\exists\, \delta > 0$ s.t.

$$|t| < \delta \Rightarrow x + ty \in C$$

A point $x \in \mathfrak{X}$ is called a bounding point of a convex set C if it is neither a core point of C nor a core point of $\mathfrak{X} \setminus C$.

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THEOREM: Let C be a convex set in an LTS $\mathfrak{X}$. Suppose $C^\circ \neq \emptyset$. Then

① x is a core point of $C \iff x \in C^\circ$

② x is a bounding point for $C \iff x \in \overline{C} \setminus C^\circ$ (i.e. x boundary point)

Proof. ① Suppose $x \in C^\circ$. Then $\exists$ a nbhd V of 0 s.t

$$x + V \subset C$$

Let $y \in \mathfrak{X}$. Since the vector space conditions are continuous, choose a $\delta > 0$ s.t. $|t| < \delta \Rightarrow x + ty \in x + V$. Therefore x is a core point of C.

Conversely, now suppose x is a core and choose $z \in C^\circ$. Then $\exists \delta_z > 0$ s.t.

$$|t| < \delta_z \implies x + t(x-z) \in C$$

Let $r = x + t(x-z)$, so $r \in C$. Then

$$x = \frac{1}{1+t} r + \frac{t}{1+t} z \in C^\circ$$

since $z \in C^\circ$ and $r \in C$, and therefore $x \in (r, z) \subseteq C^\circ$

② Suppose $x \in \mathfrak{X}$ is a bounding point for C. Then x is not a core point of C nor is it a core point of $\mathfrak{X} \setminus C$, and so $x \notin C^\circ$ and $x \notin \operatorname{int} \mathfrak{X} \setminus C$, i.e. $x \in \overline{C}$. Hence $x \in \overline{C} \setminus C^\circ$.

Conversely, if x is a boundary point of C, then $x \notin C^\circ$ and $x \notin (\mathfrak{X}\setminus C)^\circ$. Hence x is not a core point of C, nor is x a core point of $\mathfrak{X}\setminus C$, so x is a bounding point.

$\blacksquare$

<u>THEOREM</u>: Let C be a convex set in an LTS s.t. $0 \in C^\circ$. Define the Minkowski gauge functional ρ of C by

$$\rho(x) = \inf\left\{t>0 : \frac{x}{t} \in C\right\}$$

Then

1. $\rho(x) \geq 0 \quad \forall x \in \mathfrak{X}$
2. $\rho(x) < \infty \quad \forall x \in \mathfrak{X}$
3. $\rho(\alpha x) = \alpha \rho(x) \quad \forall \alpha \geq 0 \ \forall x \in \mathfrak{X}$
4. $\rho(\alpha x) = |\alpha| \rho(x)$ if $C = -C$ (i.e. C symmetric)
5. $\rho(x+y) \leq \rho(x) + \rho(y) \quad \forall x, y$
6. $C^\circ = \{x \in \mathfrak{X} : \rho(x) < 1\}$
7. $\overline{C} = \{x \in \mathfrak{X} : \rho(x) \leq 1\}$
8. $\partial C = \{x \in \mathfrak{X} : \rho(x) = 1\}$

Example: Let $\mathfrak{X}$ be a B-space and C the closed unit ball of $\mathfrak{X}$. Let $x \in \mathfrak{X}$. Then

$$\begin{aligned}\rho(x) &= \inf\left\{t>0 : \left\|\frac{x}{t}\right\| \leq 1\right\} \\ &= \|x\|\end{aligned}$$

Proof ① clear

② $\exists$ nbhd V of 0 s.t. $V \subseteq C$. If $x \in \mathfrak{X}$, then $\exists\, t > 0$ s.t. $tx \in V$. Hence $\rho(x) \le 1/t$.

③ If $\alpha > 0$

$$\rho(\alpha x) = \inf\left\{ s > 0 : \frac{\alpha x}{s} \in C \right\} = \inf\left\{ \alpha t > 0 : \frac{\alpha x}{\alpha t} \in C \right\}$$

$$= \alpha \inf\left\{ t > 0 : \frac{x}{t} \in C \right\} = \alpha \rho(x)$$

④ clear

⑤ Let $x, y \in \mathfrak{X}$. Take $c > \rho(x) + \rho(y)$. Write $c = a + b$, where $a > \rho(x)$ and $b > \rho(y)$. Now

$$\frac{x+y}{c} = \frac{x+y}{a+b} = \frac{a}{a+b}\left(\frac{x}{a}\right) + \frac{b}{a+b}\left(\frac{y}{b}\right)$$

Since $a > \rho(x)$, $\exists\, \lambda < a$ such that $\frac{x}{\lambda} \in C$. Hence $\frac{x}{a} \in \left(0, \frac{x}{\lambda}\right) \subset C$
Similarly, $y/b \in C$, so

$$\frac{x+y}{c} \in C \Rightarrow \rho(x+y) \le c$$

Since $c > \rho(x) + \rho(y)$ was arbitrary, we must have $\rho(x+y) \le \rho(x) + \rho(y)$

⑥ Suppose $x \in C^{\circ}$. Then $\exists\, t > 0$ s.t. $x + tx \in C^{\circ}$. Hence

$$\rho(x) \le \frac{1}{1+t} < 1$$

Conversely, suppose $\rho(x) < 1$. Then $\exists\, \alpha < 1$ s.t. $\frac{x}{\alpha} \in C$. Hence

$$x \in \left(0, \frac{x}{\alpha}\right) \subset C^{\circ}$$

and so $x \in C^{\circ}$.

(7) Similarly $\rho(x) > 1$ characterizes the core points of $\mathfrak{X} \setminus C$

Therefore $\rho(x) = 1$ characterizes the bounding points of C.

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LEMMA: Let $\mathfrak{X}$ be an LTS. Let ℓ be a linear functional on $\mathfrak{X}$. If $\exists$ a nbhd U of the origin s.t. $\ell(U)$ is either bounded from above or bounded from below, then ℓ is continuous.

Proof. Suppose $\sup \ell(U) \leq \alpha$ where $\alpha > 0$. Then $\ell(U) \leq \alpha$ so $\ell(-U) \geq -\alpha$, i.e.

$$\ell(U \cap -U) \subseteq [-\alpha, \alpha]$$

Therefore

$$\ell\left(\tfrac{1}{\alpha}(U \cap -U)\right) \subset [-1,1]$$

$$\Rightarrow \ell\left(\tfrac{\varepsilon}{\alpha}(U \cap -U)\right) \subset [-\varepsilon, \varepsilon]$$

for any $\varepsilon > 0$. Since $\varepsilon/\alpha\,(U \cap -U)$ is a nbhd of 0, ℓ is continuous at 0.

Can reduce the case for bounded below to bounded above.

$\blacksquare$

THEOREM: (Mazur's Theorem - Geometric form of Hahn-Banach)
Let $\mathfrak{X}$ be an LTS. Suppose C is a convex subset of $\mathfrak{X}$ s.t. $C^\circ \neq \phi$. If E is a translate of a subspace of $\mathfrak{X}$ (i.e. E is a flat set) s.t. $E \cap C^\circ = \phi$, then $\exists$ a non-zero continuous linear functional ℓ on $\mathfrak{X}$ and a real α s.t.

$$\ell(E) = \alpha$$
$$\ell(C) \leq \alpha$$
$$\ell(C^\circ) < \alpha$$

Proof. By translation we may assume $0 \in C^\circ$. Write $E = x_0 + \mathcal{Y}$ where $\mathcal{Y}$ is a subspace of $\mathfrak{X}$. Notice that $x_0 \notin \mathcal{Y}$ since $0 \in C^\circ$ and $C^\circ \cap E = \phi$. Define $\tilde{\ell}$ on E by

$$\tilde{\ell}(x_0 + y) = 1$$

for all $y \in \mathcal{Y}$. Now if $t \in \mathbb{R}$, define $\tilde{\ell}(tx_0 + y) = t$ for all $y \in \mathcal{Y}$. This defines $\tilde{\ell}$ on $\langle x_0 + \mathcal{Y} \rangle$

Let ρ be the Minkowski functional for C. Recall $\rho(C^\circ) < 1$ characterizes C°. Therefore $\rho(x) \geq 1$ $\forall x \in E$ since $E \cap C^\circ = \phi$. Therefore

$$\tilde{\ell}(x) = 1 \leq \rho(x) \quad \forall x \in E$$

If $x \in E$ and $t > 0$, then

$$\tilde{\ell}(tx) = t \leq t\rho(x) = \rho(tx) \qquad \forall x \in E$$

If $t < 0$, then

$$\tilde{\ell}(tx) = t < 0 \leq \rho(tx) \quad \forall x \in E$$

But $\langle x_0 + \mathcal{Y} \rangle = \{tx : x \in E, t \in \mathbb{R}\}$, and so we have shown that

$$\tilde{\ell}(x) \leq p(x) \quad \forall x \in \langle x_0 + \mathcal{Y} \rangle$$

By the Hahn-Banach theorem, $\tilde{\ell}$ has a linear extension ℓ to all of $\mathfrak{X}$ s.t.

$$\ell(x) \leq p(x) \quad \forall x \in \mathfrak{X}$$

Observe that $\ell(E) = \tilde{\ell}(E) = 1$. Also

$$\ell(x) \leq p(x) < 1 \quad \forall x \in C^\circ$$

Hence ℓ is continuous by the lemma since it is bounded from above on the open set C°. Moreover, since ℓ is continuous

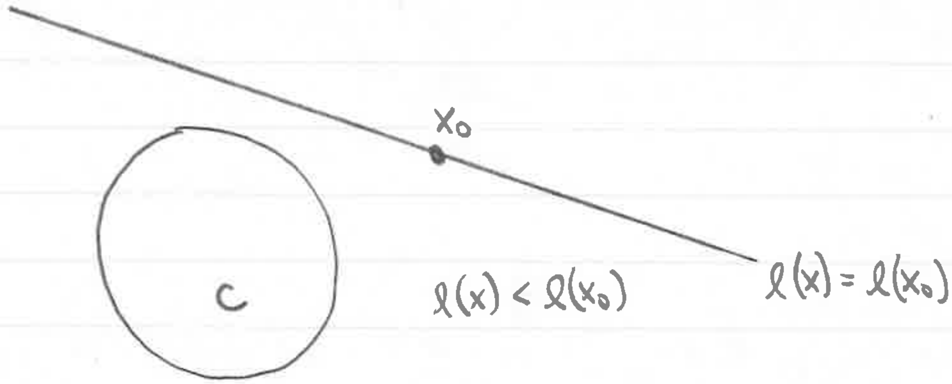
$$\sup \ell(C) \leq \sup \overline{\ell(C^\circ)} \leq 1$$

$\blacksquare$

<u>Corollary</u> (Support Theorem I): If $\mathfrak{X}$ is an LCS and x_0 is not an interior point of a convex C that has non-empty interior, then there is a non-zero continuous linear functional ℓ on $\mathfrak{X}$ s.t.

$$\sup_{z \in C} \ell(z) \leq \ell(x_0)$$

Proof. Take $E = \{x_0\} = x_0 + \{0\}$ and apply last theorem.



HW/ Let $\mathfrak{X}$ be a B-space and $\Phi : \mathfrak{X} \longrightarrow \mathbb{R}$ be a continuous convex function. Show that for each $x_0 \in \mathfrak{X}$ $\exists\, x_0^* \in \mathfrak{X}^*$ s.t.

$$\Phi(x) \geq \Phi(x_0) + x_0^*(x - x_0)$$

Show $\nabla \Phi(x_0)$ exists $\Leftrightarrow$ x_0^* is uniquely determined

$f : \mathfrak{X} \longrightarrow \mathbb{R}$ is Frechet differentiable at x_0 with derivative $\nabla f(x_0)$ iff $\exists\, \ell \in \mathfrak{X}^*$ s.t.

$$\lim_{\|h\| \to 0} \frac{f(x_0+h) - f(x_0) - \ell(h)}{\|h\|} = 0$$

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<u>COROLLARY</u> (Eidelheit Separation Theorem) Let C_1 and C_2 be convex sets in LTS $\mathfrak{X}$. Suppose $C_1^\circ \neq \phi$ and $C_1^\circ \cap C_2 = \phi$. Then there exists a non-zero continuous linear functional ℓ on $\mathfrak{X}$ s.t.

$$\sup \ell(C_1) \leq \inf \ell(C_2)$$

Proof. C_1° is a convex set. Therefore $C_1^\circ - C_2$ is an open convex set and $0 \notin C_1^\circ - C_2$. Apply support theorem to find a continuous linear functional ℓ s.t.

$$\sup \ell(C_1^\circ - C_2) \leq \ell(0) = 0$$

Hence

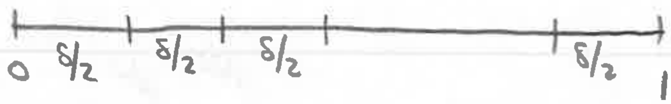
$$\sup \ell(C_1^\circ) \leq \inf \ell(C_2)$$

$$\Rightarrow \sup \ell(C_1) \leq \inf \ell(C_2)$$

$\blacksquare$

10/31 BANACH SPACES

⌈ In L_0, $\{f \in L_0 : \mu\{|f| \geq \varepsilon\}) < \delta\}$ is a nbhd of 0.
Suppose such a nbhd is convex. Any function supported inside one of these intervals is in the nbhd



Suppose there are K of the intervals $I_1, \dots, I_k$. Since this nbhd is convex

$$\sum_{i=1}^{k} \frac{1}{k}(k \chi_{I_i}) = \chi_{[0,1)}$$

is in C. Similarly, if $f \in L_0$ is arbitrary, then $k f \chi_{I_i} \in$ our nbhd.
Thus

$$f = \sum_{i=1}^{k} k f \chi_{I_i} \in C$$

So any convex set with non-empty interior is all of L_0 ⌋

<u>COROLLARY:</u> (Support Theorem II) Let C be a closed convex subset of an LCS $\mathfrak{X}$. If $x \notin C$, then there exists a continuous linear functional ℓ s.t.

$$\ell(x) < \inf \ell(C)$$

Proof. Take a convex nbhd N of x s.t. $N \cap C = \emptyset$. Use Eidelheit separation theorem to find a non-zero continuous linear functional ℓ s.t.

$$\sup \ell(N) \leq \inf \ell(C)$$

Write $N = x + V$, where V is a nbhd of 0. We get

$$\ell(x) + \sup \ell(V) \leq \inf \ell(C)$$

Since 0 is a core point of V and $\ell \neq 0$, there exists $y \in V$ s.t. $\ell(y) > 0$. Therefore

$$\ell(x) + \ell(y) \leq \inf \ell(C)$$

$$\Rightarrow \ell(x) < \inf \ell(C)$$

$\blacksquare$

<u>COROLLARY</u>: In an LCS a convex set is closed iff it is weakly closed.

Proof. Weakly closed $\Rightarrow$ closed always. Conversely, the closure of C is contained in the weak closure of C. If

$$x \in (\text{weak closure of } C) \setminus C$$

then choose a continuous linear functional l s.t.

$$(*) \qquad l(x) < \inf l(C)$$

Choose a net (x_α) in C s.t. $x_\alpha \longrightarrow x$ weakly. Observe

$$l(x_\alpha) \longrightarrow l(x)$$

which contradicts $(*)$.

$\blacksquare$

COROLLARY: Let (x_n) be a sequence in a B-space s.t. $\lim x_n = x$ weakly. Then $\exists$ a sequence of convex combinations of the x_n's s.t. this sequence of convex combinations converges to x in norm.

Proof. Let $C = \overline{co}\{x_n : n \in \mathbb{N}\}$. Then

$$x \in \text{weak closure of } C = \text{norm closure of } C$$

$\blacksquare$

Open Problem: A B-space has the Banach-Saks property if every bounded seq. in $\mathfrak{X}$ has a subsequence whose arithmetic means converge in norm

All L^p's have B-S $\quad 1 < p$

$\mathfrak{X} = $ B-S $\Rightarrow \mathfrak{X}$ is reflexive (Kakutani).

$\exists$ reflexive $\mathfrak{X}$ s.t. $\mathfrak{X}$ fails B-S (Baernstein

Uniformly convex spaces have B-S

Property is not self-dual (Seifert)

What does characterize B-S?

<u>COROLLARY</u>: Let C_1 and C_2 be disjoint closed convex sets in an LCS. Suppose one is compact. Then there exists a non-zero continuous linear functional ℓ on $\mathfrak{X}$ s.t.

$$\sup \ell(C_1) < \inf \ell(C_2)$$

Proof. Consider $C = C_2 - C_1$. This set is convex and closed. Why? Let $x_\alpha - y_\alpha = z_\alpha$ be a convergent net in $C_2 - C_1$ with $x_\alpha \in C_2$ and $y_\alpha \in C_1$ $\forall \alpha$. Suppose C_1 is compact. Then there is a convergent subnet (y_β) of (y_α). But

$$x_\beta = z_\beta - y_\beta$$

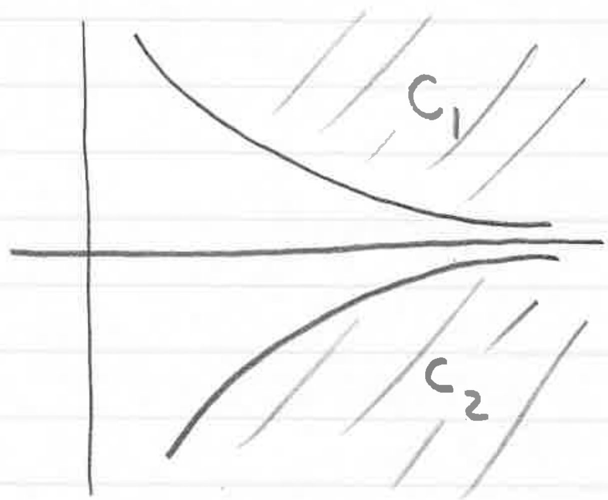
is then convergent to an element of C_2 since C_2 is closed.

Now $0 \notin C$, so there is a continuous linear functional ℓ on $\mathfrak{X}$ s.t.

$$0 = \ell(0) < \inf \ell(C)$$

Hence $\sup \ell(C_1) < \inf \ell(C_2)$. $\boxtimes$

Example to show that compactness is needed



Examples of uses of separation theorems

① Helley's Theorem
② Kuhn-Tucker Theorem (basic theorem of non-linear programming)

<u>DEFINITION</u>: A convex cone in a B-space is a set K such that

$$x, y \in K \Rightarrow \alpha x + \beta y \in K \quad \forall \alpha, \beta \geq 0$$

The dual cone K^* is $\{x^* \in \mathfrak{X}^* : x^*(K) \geq 0\}$

HW/ $K^{**} \cap \mathfrak{X} = K$ if K is a closed convex cone

11/2 BANACH SPACES

<u>**FARKAS'S LEMMA:**</u> Let A be a matrix and b a vector. Then

$$\left(Ax \geq 0 \Rightarrow b^T x \geq 0 \right)$$

iff

$$\exists\, u \geq 0 \text{ s.t. } A^T u = b$$

Proof ($\Leftarrow$) If $A^T u = b$, then if $Ax \geq 0$

$$b^T x = (u^T A)x = u^T (Ax) > 0$$

($\Rightarrow$) Let K be the closed cone generated by the rows of A

$$K = \{ A^T u : u \geq 0 \}$$

If we can show that $b \in K^{**}$, then we'll be done (since $K^{**} = K$)
Observe that if $y \in K^*$, then $x^T y \geq 0$ $\forall x \in K$. Each of the rows a_i of A are in K. Then

$$y \in K^* \Rightarrow y^T a_i \geq 0 \text{ for all rows } a_i \text{ of } A$$

$$\Rightarrow Ay \geq 0$$

Hence $y^T b \geq 0$ $\forall y \in K^*$, so $b \in K^{**}$ ▨

Duality theorem of linear programming

Primal L.P.

$$\max\ c\cdot y \quad \text{s.t. } Ay \le b,\ y \ge 0$$

Dual L.P.

$$\min\ b\cdot x \quad \text{s.t. } A^T x \ge c,\ x \ge 0$$

<u>Primal-dual inequality</u>: If y is feasible for the primal and x is feasible for the dual, then

$$b\cdot x \ge c\cdot y$$

Proof. $c\cdot y \le A^T x\cdot y = x\cdot Ay \le x\cdot b$ ▨

Observe: If $\exists x$ feasible for dual and $\exists\, y$ feasible for primal s.t. $b\cdot x = c\cdot y$, then x solves the dual and y solves the primal (Follows from primal dual inequality)

<u>Main Duality Theorem</u>: If both both the primal and the dual are feasible, then they both have the same optimal value (and both have solutions)

Proof. Consider the inequality

$$(*)\qquad \left[\begin{array}{c|c|c} 0 & -A & b \\ \hline A^T & 0 & -c \\ \hline \multicolumn{3}{c}{I} \end{array}\right] \begin{pmatrix} x \\ \downarrow \\ \hline y \\ \downarrow \\ \hline \rho \end{pmatrix} \geq 0$$

(ρ scaler)

which is equivalent to

$$A y \leq \rho b$$

$$(*)\qquad A^T x \geq \rho c$$

$$x \geq 0,\ y \geq 0,\ \rho \geq 0$$

Let p be a vector s.t. $A^T p \geq 0$ and $p \geq 0$. Let q be a vector s.t. $Aq \leq b$, $q \geq 0$ (Exist by feasibility assumption). Let (x, y, ρ) solve $(*)$. A quick check shows

$$\frac{p+x}{1+\rho}$$

is feasible for the dual, and $\dfrac{q+y}{1+\rho}$ is feasible for the primal

By the primal-dual inequality

$$b \cdot (p+x) \geq c \cdot (q+y)$$

Therefore

$$b \cdot x - c \cdot y + 0 \cdot \rho \geq c \cdot q - b \cdot p = \text{constant}$$

for any solution (x, y, ρ) of $(*)$. Since $(nx, ny, n\rho)$ solves $(*)$ for any $n \in \mathbb{N}$, we must actually have

$$b \cdot x - c \cdot y + 0 \cdot \rho \geq 0$$

By Farkas' lemma, $\exists\, r \geq 0$ s.t.

$$\left[\begin{array}{cc|c} 0 & A & \\ -A^T & 0 & I \\ b^T & -c^T & \end{array}\right] r = \begin{pmatrix} b \\ -c \\ 0 \end{pmatrix}$$

If $r = \begin{pmatrix} s \\ t \\ u \\ v \\ w \end{pmatrix}$ we get

$$b = At + u \qquad -\underline{c} = -A^T s + v$$

$$0 = b \cdot s - c \cdot \underline{t} + w$$

Therefore $At \le b$, $A^T s \geq c$ and $t, s \geq 0$. Hence t is feasible for primal and s is feasible for dual. Also

$$c \cdot t \geq b \cdot s$$

But in general, the primal dual inequality says

$$b \cdot s \geq c \cdot t$$

Hence $b \cdot s = c \cdot t$, so t and s solve the primal and dual problems.

$\blacksquare$

LEMMA: Let V be a vector space. Let $\ell, \ell_1, \dots, \ell_n$ be linear functionals on V. If

$$\ell_1(x) = \ell_2(x) = \dots = \ell_n(x) = 0 \Rightarrow \ell(x) = 0$$

then ℓ is a linear combination of the ℓ_i's.

Proof. Define $T: V \longrightarrow \mathbb{R}^n$ by

$$T(x) = (\ell_1(x), \dots, \ell_n(x))$$

Define linear functional φ on $T(V)$ by $\varphi(Tx) = \ell(x)$. The hypothesis guarantee that φ is well-defined. Note φ is linear on $T(V)$. Take any extension $\tilde{\varphi}$ of φ to all of $\mathbb{R}^n$. Write

$$\tilde{\varphi}(y_1, \dots, y_n) = \sum \alpha_i y_i$$

Observe

$$\ell(x) = \varphi(Tx) = \sum_{i=1}^{n} \alpha_i \ell_i(x) \quad \forall x \in V$$

$\boxtimes$

11/5 BANACH SPACES

HW/ (Ω, Σ, μ) finite measure space. Let $f: \Omega \to \mathfrak{X}$ be a functional s.t. $\exists$ weakly compact convex set $W \subset \mathfrak{X}$ s.t. $f(\Omega) \subset W$. Prove that for each $E \in \Sigma$ $\exists x_E \in \mathfrak{X}$ s.t.

$$x^*(x_E) = \int_E x^* f \, d\mu$$

$\forall x^* \in \mathfrak{X}^*$.

Hint: ① show x_E exists in $\mathfrak{X}^{**}$

② show $x_E \in \mu(E) \cdot W$ ($W \subset \mathfrak{X}^{**}$ is weak* closed and compact)

THEOREM: Let $\mathfrak{X}$ be a vector space and Γ be a subspace of the algebraic dual of $\mathfrak{X}$. Suppose Γ separates the points of $\mathfrak{X}$. A linear functional ℓ on $\mathfrak{X}$ is $\sigma(\mathfrak{X}, \Gamma)$ continuous iff $\ell \in \Gamma$.

Proof. ($\Leftarrow$) If (x_α) is a net in $\mathfrak{X}$ s.t. $\lim x_\alpha = x \in \mathfrak{X}$ for the $\sigma(\mathfrak{X}, \Gamma)$ topology, then $\lim \ell(x_\alpha) = \ell(x)$, so ℓ is $\sigma(\mathfrak{X}, \Gamma)$ continuous

($\Rightarrow$) Choose a nbhd $\mathcal{U}$ of the origin s.t.

$$\ell(\mathcal{U}) \subset [-1, 1]$$

There exists an open subset N of $\mathcal{U}$ s.t

$$N = \{x \in \mathfrak{X} : |\ell_k(x)| < \varepsilon \ , \ 1 \le k \le n\}$$

for some $\ell_1, \ldots, \ell_n \in \Gamma$ and some $\varepsilon > 0$. If

$$\ell_1(x) = \ell_2(x) = \ldots = \ell_n(x)$$

then $|\ell(x)| \le 1$. Hence $\ell(x) = 0$. By lemma,

$$\ell = \sum \alpha_i \ell_i$$

for appropriately chosen α_i's.

$\boxtimes$

<u>COROLLARY</u>: A member x^{**} of $\mathfrak{X}^{**}$ lies in $\mathfrak{X}$ iff x^{**} is weak* continuous on $\mathfrak{X}^*$

<u>COROLLARY</u>: If $\mathfrak{X}$ is separable and $x^{**} \in \mathfrak{X}^{**}$, then $x^{**} \in \mathfrak{X}^{**}$ iff x^{**} is weak* sequentially continuous on $\mathfrak{X}^*$.

This corollary is easily understood once it is known that a linear functional on a dual space is weak*-continuous iff it is weak* continuous on the unit ball with weak* topology. (Krein-Smulian)

Proof. By Krein-Smulian it is enough to prove that if ℓ is weak*-seq. cont. on $B_{\mathfrak{X}^*}$, then ℓ is weak* cont on $B_{\mathfrak{X}^*}$. But $B_{\mathfrak{X}^*}$ in its weak*-topology is a compact metric space. In such spaces seq. cont $\Rightarrow$ cont.

$\boxtimes$

Fact: (Amir–Lindenstrauss) If $\mathfrak{X}$ is WCG Banach space, then $x^{**} \in \mathfrak{X}^{**}$ lies in $\mathfrak{X}$ iff x^{**} is weak*-seq. cont on $\mathfrak{X}^*$

Corollary: Let $T : \mathfrak{X} \to \mathcal{Y}$ be linear. Then T is norm-to-norm continuous iff T is weak-to-weak continuous.

Proof. Suppose T is norm-to-norm continuous. Let (x_α) be a net in $\mathfrak{X}$ s.t. $\lim x_\alpha = x$ weakly. If $y^* \in \mathcal{Y}^*$, then $y^* T \in \mathfrak{X}^*$ so

$$\lim y^* T x_\alpha = y^* T x$$

Hence $Tx_\alpha \to Tx$ weakly.

Conversely, suppose $T : \mathfrak{X} \to \mathcal{Y}$ is weak-to-weak continuous. Obviously $y^* T$ is weakly continuous on $\mathfrak{X}$ $\forall y^* \in \mathcal{Y}^*$, so $y^* T \in \mathfrak{X}^*$. Let

$$x_n \to x$$
$$Tx_n \to y$$

in norm. Notice $x_n \to x$ weakly, so $y^* T x_n \to y^* T x$. But also $y^* T x_n \to y^* y$, so

$$y^* T x = y^* y$$

for all $y^* \in \mathcal{Y}^*$. Hence $Tx = y$, so T is continuous $\boxtimes$

11/7 BANACH SPACES

(Proof continued) Fix $\mu \in C(W, \text{weak})^*$ and observe that the functional

$$x^* \xmapsto{\Phi} \int_W x^* f \, d\mu$$

is linear in x^*. Also, it is continuous, since

$$\left| \int x^* f \, d\mu \right| \leq \int_W |x^* f| \, d|\mu| \leq \int_W \|x^*\| K \, d|\mu|$$

(where $K = \sup_{x \in W} \|x\|$)

$$= K \|x^*\| \, |\mu|(W)$$

Hence this functional lies in $\mathfrak{X}^{**}$.

Claim: $\exists \, x_\mu \in \mathfrak{X}$ s.t. $x^*(x_\mu) = \int_W x^* f \, d\mu$. It suffices to show that Φ is weak* seq. continuous (Separability of $\mathfrak{X}$ used here to insure that we can consider sequences). To this end, suppose (x_n^*) is a seq. in $\mathfrak{X}^*$ s.t.

$$\lim x_n^* = x^* \quad \text{weak}^*$$

Then

$$\Phi(x_n^*) = \int_W x_n^* f \, d\mu \xrightarrow{\text{bounded convergence theorem}} \int_W x^* f \, d\mu = \Phi(x^*)$$

Hence Φ is weak* seq. continuous. Since $\mathfrak{X}$ is separable, Φ is weak* continuous, i.e. $\Phi = x_\mu \in \mathfrak{X}$

Define $T : C(W, \text{weak})^* \to \mathfrak{X}$ by

$$T(\mu) = x_\mu$$

Evidently T is linear. Claim: T is weak*-to-weakly continuous. To see this, suppose (μ_α) is a net in $C(W)^*$ s.t.

$$\lim \mu_\alpha = \mu \quad \text{weak*}$$

Then if $x^* \in \mathfrak{X}^*$ we have

$$x^* T\mu_\alpha = \int x^* f \, d\mu_\alpha \longrightarrow \int x^* f \, d\mu = x^* T\mu$$

since $x^* f \in C(W)$. Hence $T\mu_\alpha \to T\mu$ weakly.

Now evidently

$$M = T\left(\overline{B}_{C(W)^*}\right)$$

is convex and weakly compact (since $\overline{B_{C(W)^*}}$ is weak* compact)

To complete the proof, observe that $W \subseteq M$. To see why, take $x \in W$ and let δ_x be the point mass measure at x. Then if $x^* \in \mathfrak{X}^*$

$$x^*(T(\delta_x)) = \int x^* f \, d\delta_x = x^* f(x)$$

and so $x = f(x) = T(\delta_x) \in M$. Therefore $\overline{co}(W) \subseteq M$, so

$\overline{co}(W)$ is weakly compact.

$\blacksquare$

<u>THEOREM</u> (Mazur) The closed convex hull of a norm compact subset of $\mathfrak{X}$ is also norm compact.

Proof. Define $T : C(W, \text{norm})^* \longrightarrow \mathfrak{X}$ just as before. Use the total boundedness of W to find for each n a measurable function $f_n : W \longrightarrow \mathfrak{X}$ taking only finitely many values in such a way that

$$\| f_n(x) - f(x) \|_{\mathfrak{X}} \leq 1/n$$

$\forall x \in W$. Write

$$f_n = \sum_{i=1}^{p_n} x_{in} \chi_{A_{in}}$$

and define $T_n : C(W)^* \longrightarrow \mathfrak{X}$ by

$$T_n(\mu) := \sum_{i=1}^{p_n} x_{in} \mu(A_{in})$$

Obviously $\overline{T_n(B_{C(W)^*})}$ is compact $\forall n$ since T_n is finite rank operator.

If $x^* \in \mathfrak{X}^*$ and $\mu \in \overline{B_{C(W)^*}}$, then

$$| x^* T(\mu) - x^* T_n(\mu) | = \left| \int_W (x^* f - x^* f_n) \, d\mu \right|$$

$$\leq \int_W \| x^* \| \, \| f(x) - f_n(x) \| \, d|\mu|(x)$$

$$\leq \|x^*\| \cdot 1/n \; |\mu|(W) \leq \|x^*\| \, 1/n$$

Therefore

$$\sup_{\|x^*\| \leq 1} |x^* T\mu - x^* T_n \mu| \longrightarrow 0 \text{ unif in } \|x^*\| \leq 1$$

i.e.

$$\lim_n \|T(\mu) - T_n(\mu)\| \longrightarrow 0 \text{ unif in } \|x^*\| \leq 1$$

Hence everything in $\overline{T(B_{C(\omega)^*})} = M$ is within $\varepsilon/2$ of something in $T_{n_0}(B_{C(\omega)^*})$ for appropriately chosen n_0. It follows that M is totally bounded, so M is compact

Proceed as before.

$\boxtimes$

11/9 BANACH SPACES

DEFINITION: A point x_0 of a subset C of a vector space is called an extreme point of C if

$$x, y \in C\ ,\ 0<t<1\ ,\ tx+(1-t)y = x_0 \Rightarrow x=y=x_0$$

DEFINITION: Let $\mathfrak{X}$ be an LCS. Let C be a convex subset of $\mathfrak{X}$. A subset F of C is called a face of C if $F \neq \phi$, F is convex and F contains all line segments in C whose interior intersects F, i.e. $x, y \in C$, $(x,y) \cap F \neq \phi \Rightarrow [x,y] \subset F$.

COROLLARY: ① A face of a face of C is a face of C.

② If $\{F_\alpha\}$ is a family of faces of C, then $\bigcap F_\alpha$ is also a face of C provided it is non-empty.

③ $x \in C$ is an extreme point of C iff $\{x\}$ is a face

Proof. Easy

LEMMA: Let C be a non-empty compact convex subset of an LSC $\mathfrak{X}$. If C is not a singleton, then C has a proper closed face.

Proof. Take two distinct points x_0, y_0 in C. Take a linear functional $x^* \in \mathfrak{X}^*$ s.t.

$$x^*(x_0) < x^*(y_0)$$

Since C is compact, we know x^* achieves its maximum value α on C. Set

$$F = \{x \in C : x^*(x) = \alpha\}$$

Suppose $y, z \in C$ and $\exists\ 0 < t < 1$ s.t. $ty + (1-t)z \in F$. Then

$$t x^*(y) + (1-t) x^*(z) = \alpha$$

But $x^*(y) \leq \alpha$ and $x^*(z) \leq \alpha$, so we must actually have

$$x^*(y) = \alpha = x^*(z)$$

Therefore $y, z \in F$. It follows that F is a face. F is obviously closed and $x_0 \notin F$.

$\blacksquare$

THEOREM (Krein-Milman) Let C be a compact convex set of an LCS $\mathfrak{X}$. Then C has at least one extreme point. Consequently, $C = \overline{co}(\text{ext}\, C)$

Proof. Let $\mathcal{F}$ be the family of all closed faces of C. Order $\mathcal{F}$ by reverse inclusion, i.e.

$$F_1 \leq F_2 \iff F_2 \subset F_1$$

Let (F_α) be a linearly order subset of $\mathcal{F}$. ZORNICATE! This produces a maximal element F_0 of $\mathcal{F}$. By last lemma, F_0 is a singleton $\{x_0\}$, and so x_0 is an extreme point.

To finish, we have to show

$$C = \overline{co}(\text{ext } C)$$

To this end, suppose $x \in C \setminus \overline{co}(\text{ext } C)$. Appeal to separation theorem to find $x^* \in \mathfrak{X}^*$ s.t.

$$(*) \qquad x^*(x) > \sup x^*(\overline{co}(\text{ext } C))$$

Since C is compact, x^* achieves a maximum value β on C. Put

$$F := \{ x \in C : x^*(x) = \beta \}$$

Then F is a closed convex subset of C. Hence F has an extreme point y_0 of F. Claim: $y_0 \in \text{ext } C$. If $x, y \in C$ and $0 < t < 1$, and

$$tx + (1-t)y = y_0$$

then $t x^*(x) + (1-t) x^*(y) = x^*(y_0) = \beta$. Hence $x^*(x) = x^*(y) = \beta$, so $x, y \in F$. But $y_0 \in \text{ext } F$, so $x = y = y_0$. Therefore y_0 is an extreme point of C. But this is impossible by $(*)$ $\boxtimes$

HW/ (Milman) Let A be a weakly compact subset of a B-space. Prove

$$\mathrm{ext}\left(\overline{\mathrm{co}}(A)\right) \subseteq A$$

(Hint: Prove $\max x^*(A) = \max\left(x^*\left(\overline{\mathrm{co}}(A)\right)\right)$

COROLLARY. The closed unit ball of a dual B-space has extreme points

COROLLARY: Neither c_0 nor $L_1[0,1]$ are isometric to dual spaces.

Proof. Neither space has any extreme points in the ball. Take (α_n) in B_{c_0}. Choose α_{n_0} s.t.

$$|\alpha_{n_0}| < 1/2$$

Put

$$\beta_1 = (\alpha_n) + \tfrac{1}{4} e_{n_0}$$

$$\beta_2 = (\alpha_n) - \tfrac{1}{4} e_{n_0}$$

Then $\|\beta_1\| \le 1$, $\|\beta_2\| \le 1$ and $\frac{1}{2}\beta_1 + \frac{1}{2}\beta_2 = (\alpha_n)$. Hence (α_n) is not extreme

Banach Spaces (11/12)

Goal: Theorem (Bishop Phelps, 1960)

If K is a closed convex subset of a B-Space $\mathfrak{X}$, then the collection of fcnals which attain their sup on K is norm dense in $\mathfrak{X}^*$. In particular the collection of fcnals which attain their norm on $\overline{B}_{\mathfrak{X}}$ is norm dense, i.e. $\mathfrak{X}$ is "subreflexive"

Note that for every $x \in S_{\mathfrak{X}} = \{x \in \mathfrak{X} : \|x\| = 1\}$ $\exists$ $f \in \mathfrak{X}^*$ $\ni$ $f(x) = \|f\| = 1$.

However not all $f \in \mathfrak{X}^*$ necessarily attain their norm.

Example

$$g = (1-1, 1-1/2, 1-1/3, 1-1/4, \dots) \in \ell_\infty$$

then $\|g\| = 1$ but if $\exists$ $(\alpha_n) \in \ell_1$ $\ni$

$$\|\alpha\| = 1 \text{ and } g(\alpha) = \|g\| = 1$$

We have $\sum |\alpha_n| = 1$ $\quad \sum \alpha_n (1 - 1/n) = 1$

patently incompatable.

Defn.

Let K be closed subset of $\mathfrak{X}$

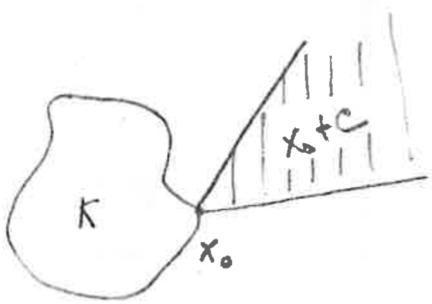
① $f \in \mathfrak{X}^*$ is called a support fcnal of K if $f \neq 0$ and

$$\sup f(K) = f(x_0)$$

for some $x_0 \in K$. f is said to support K at x_0

② $x_0 \in K$ is called a support point if $\exists$ $f \in \mathfrak{X}^*$ which supports K at x_0

③ $x_0 \in K$ is called a conical support point of K wrt C if $(x_0 + C) \cap K = \{x_0\}$



Cone $\begin{cases} C \text{ is convex and closed} \\ \alpha C \subset C \quad \forall\, \alpha \geq 0 \\ x, -x \in C \Rightarrow x = 0 \end{cases}$

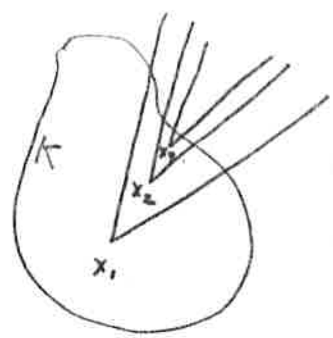
If $f \in S_{\mathfrak{X}^*}$ and $\gamma > 0$ then

$$C(f, \gamma) := \{ x \in \mathfrak{X} : \gamma \|x\| \leq f(x) \} \text{ is a cone.}$$

Lemma 1

Let K be a closed set in $\mathfrak{X}$. Let $f \in S_{\mathfrak{X}^*}$ and $0 < \gamma < 1$. Let $\epsilon > 0$ and suppose $\sup f(K) < \infty$. Let $x \in K$. Then $\exists\, x_0 \in K \ni x_0$ is a conical support pt. wrt $C(f, \gamma)$. Moreover if $\sup f(K) \leq f(x) + \epsilon$ we can choose $x_0 \ni \|x - x_0\| \leq \epsilon/\gamma$.

proof



Let $x_1 = x$ and let $K_1 = (x_1 + C) \cap K$. Suppose $x_1, \ldots, x_n$ and closed sets $K_1, \ldots, K_n$ have be chosen.

Since $\sup f(K_n) < \infty$ $\exists\, x_{n+1} \in K_n \ni$

$$\sup f(K_n) \leq f(x_{n+1}) + \frac{1}{n+1}$$

Set $K_{n+1} = (x_{n+1} + C) \cap K_n$. Clearly $K_{n+1} \subset K_n$ and K_{n+1} is closed.

Let $y \in K_{n+1}$. Then $y - x_{n+1} \in C \Rightarrow \gamma \|y - x_{n+1}\|$

$\leq f(y - x_{n+1}) = f(y) - f(x_{n+1})$

$$\leq \sup f(K_{n+1}) - f(x_{n+1})$$

$$\leq \sup f(K_n) - f(x_{n+1})$$

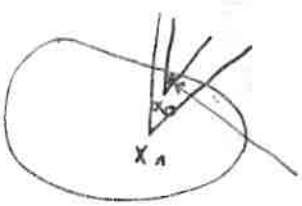
$$\leq \frac{1}{n+1}$$

$$\therefore \quad \| y - x_{n+1} \| \leq \frac{1}{r(n+1)}$$

$$\Rightarrow \operatorname{diam}(K_{n+1}) < \frac{2}{r(n+1)} \to 0.$$

$$\therefore \bigcap_{n=1}^{\infty} K_n = \{x_0\}$$

claim: $(x_0 + C) \cap K \subset K_n$



$(x_0+C) \cap K$

$x_0 \in K_1 = (x_1 + C) \cap K$ so

$$x_0 \in x_1 + C \Rightarrow x_0 + C \in x_1 \in C + C \subset x_1 + C$$

$$(x_0 + C) \cap K \subset (x_1 + C) \cap K = K_1$$

Assume $(x_0 + C) \cap K \subset K_n$ then $x_0 \in K_{n+1} \Rightarrow x_0 \in x_{n+1} + C$

$$\Rightarrow x_0 + C \in x_{n+1} + C \Rightarrow (x_0 + C) \cap K \in (x_{n+1} + C) \cap K_n$$

$$= K_{n+1} \quad \boxtimes$$

$x_0 \in (x_0 + C) \cap K \subset \bigcap K_n = \{x_0\}$. Thus x_0 is a conical support point.

Now suppose $\sup f(K) \leq f(x) + \epsilon$

$$x_0 \in K_1 \Rightarrow x_0 \in x + C$$

$$\Rightarrow x_0 - x \in C$$

$$\Rightarrow \gamma \| x_0 - x \| \leq f(x_0 - x) = f(x_0) - f(x)$$

$$\leq \sup f(K) - f(x) \leq \epsilon$$

$\boxtimes$

<u>Lemma 2</u>

Suppose $f, g \in \mathfrak{X}^*$, $\|f\| = \|g\| = 1$ and suppose

(*) $\quad |f(x)| \leq \gamma$ whenever $g(x) = 0$ and $\|x\| = 1$

Then either $\|f - g\| \leq 2\gamma$ or $\|f + g\| \leq 2\gamma$

<u>Proof</u>

Let $h = f|_{\ker g} : \ker g \to \mathbb{R}$

By (*) $\|h\| \le \gamma$. Let h^* be a H-B extension to $\mathfrak{X}$ w/ $\|h^*\| \le \gamma$.

$$\ker g \subset \ker (f - h^*)$$

So $$f - h^* = \alpha g$$

$$|1 - |\alpha|| = |\,\|f\| - \|\alpha g\|\,|$$

$$\le \|f - \alpha g\|$$

$$= \|h^*\| \le \gamma$$

Case 1: $\alpha \ge 0$

$$\|f - g\| = \|f - \alpha g - (1-\alpha) g\|$$

$$\le \|f - \alpha g\| + |1 - \gamma|\,\|g\|$$

$$= \|h^*\| + |1 - \alpha|$$

$$\le \gamma + \gamma = 2\gamma$$

Case 2: $\alpha < 0$

$$\|f + g\| = \|f - \alpha g + (1 + \alpha g)\|$$

$$\leq \| f - \alpha g \| + |1+\alpha| \, \|g\|$$

$$< \gamma + \gamma = 2\gamma \qquad \boxtimes$$

<u>Theorem</u>

If K is closed, bdd, convex set $f \in S_{\mathfrak{X}^*}$, $\epsilon > 0$, and $0 < \gamma < 1/2$, let $x \in K$ with

$$\sup f(K) \leq f(x) + \epsilon$$

Then $\exists\, x_0 \in K$ and $g \in \mathfrak{X}^* \ni$

(1) $\|g\| = 1$

(2) $\sup g(K) = g(x_0)$

(3) $\|f - g\| \leq 2\gamma$

(4) $\|x - x_0\| \leq \epsilon/\gamma$

Banach Spaces (11/4) Larry Riddle

Proof of Bishop Phelps

By lemma 1, $\exists$ a conical support pt. x_0 wrt $C(f, \gamma)$ $\ni$

$$\| x - x_0 \| \leq \epsilon/\gamma \qquad (\Longrightarrow \text{[redacted] and } (4))$$

claim: Interior of $x_0 + C$ is not empty. Suffices to show $C^\circ \neq \phi$.

$$C = \{ x \in \mathfrak{X} : \gamma \|x\| \leq f(x) \}$$

We know $\|f\| = 1$ and $\gamma < 1$. Choose y_0 w/ $\|y_0\| = 1$ $\ni$

$$f(y_0) > \gamma$$

Then $y_0 \in C$. $M := \{ x \in \mathfrak{X} : \gamma < f(x) \}$ is open

and $y_0 \in M$. Thus $\exists$ nhd W of y_0 $\ni$ $W \subset M$

$$\phi \neq B_{\mathfrak{X}} \cap W \subset W \subset M$$

$\uparrow$ $y_0 \in \overline{B}_{\mathfrak{X}}$

$$\Longrightarrow \phi \neq B_{\mathfrak{X}} \cap W \subset M \cap B_{\mathfrak{X}} \subset C$$

and $B_{\mathfrak{X}} \cap W$ is open $\Longrightarrow$ $C^\circ \neq \phi$

$(x_0+C)^\circ \cap K = \emptyset$ since $x_0 \notin (x_0+C)^\circ \cap K$

$\subset (x_0+C) \cap K = \{x_0\}$.

By Edelheit $\exists\, g \in \mathcal{X}^* \ni \|g\| = 1$ and

$$\sup g(K) \leq \inf[g(x_0+C)]$$

$$g(x_0) \underset{x_0 \in K}{\leq} \sup g(K) \leq \inf g(x_0+C) \underset{x_0 \in x_0+C}{\leq} g(x_0)$$

$\Longrightarrow$ $g(x_0) = \sup g(K)$ $\quad (\Longrightarrow (2))$

Claim $(\ker g) \cap C^\circ = \emptyset$, for if $y \in (\ker g) \cap C^\circ$ then

$$x_0 + y \underset{y \in C^\circ}{\in} x_0 + C^\circ = (x_0+C)^\circ$$

$$g(x_0) \leq \inf g(x_0+C) \underset{OMT}{<} g(x_0+y) \underset{y \in \ker g}{=} g(x_0) \qquad \text{Contra}$$

Suppose $\|x\| = 1$ and $g(x) = 0$. Then $x \notin C^\circ$ so

$$\gamma = \gamma\|x\| \geq f(x)$$

Same holds for $-x$ so $|f(x)| \leq \gamma$

by lemma 2, either $\|f+g\| \le 2\gamma$ or $\|f-g\| \le 2\gamma$

$0<\gamma<1/2 \Rightarrow 0<2\gamma<1$. Since $\|f\|=1$, $\exists\, y_0$

w/ $\|y_0\|=1$ and $f(y_0) > 2\gamma > \gamma \Rightarrow y_0 \in C$

$\therefore\ g(x_0) \le g(\underbrace{x_0+y_0}_{\text{in } x_0+C}) = g(x_0)+g(y_0)$

$\Rightarrow g(y_0) \ge 0$

$\|f+g\| \ge (f+g)(y_0) = f(y_0)+g(y_0) \ge f(y_0) > 2\gamma$

$\Rightarrow \|f-g\| \le 2\gamma$ by lemma 2. $\Rightarrow$ (3)

$\boxtimes$

<u>Coro</u>

The collection of fcnals which attain their sup on a closed convex bdd set in $\mathfrak{X}$ is norm dense in $\mathfrak{X}^*$. In part. the fcnals which attain their norm on $B_{\mathfrak{X}}$ is norm dense.

<u>proof</u>

Let $f \in \mathfrak{X}^*$ and $\|f\|=1$. Since K is bdd $\exists\, x \in K$

w/ $\sup f(K) \le f(x)+1$

Given $\delta < 1$ apply theorem w/ $\gamma = \delta/2$. $\exists\ g \in \mathfrak{X}^*$

$\ni \|g\| = 1$, $\sup g(K) = g(x_0)$ for some $x_0 \in K$

and $\| f - g \| \leq \delta$ $\boxtimes$

Examples

(1) In c_0, the collection of fcnals in $\ell_1 = c_0^*$ which attain their norm on B_{c_0} is

$$\{ (\alpha_n) \in \ell_1 : \alpha_n \neq 0 \text{ for only finitely many } n\text{'s} \}$$

(2) In $L_1(\mu)$ the collection is

$$\{ g \in L_\infty(\mu) : \mu \{ x : |g(x)| = \|g\|_\infty \} > 0 \}$$

This is dense since every simple fcn is in it and simple fcns are dense.

Remark: James has shown that a B-Space is reflexive $\iff$ every fcnal in $\mathfrak{X}^*$ attains its norm on $B_{\mathfrak{X}}$

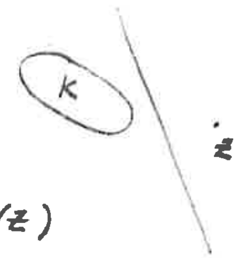
Coro

K closed, convex, bdd. $\Rightarrow$ the support points of K are dense in ∂K.

proof

Let $x \in \partial K$ and $\delta > 0$. Choose $z \in \mathfrak{X} \setminus K \ni$

$$\| x - z \| \leq \delta/4$$



Choose $f \in \mathfrak{X}^* \ni \|f\| = 1$ and $\sup f(K) < f(z)$

then $|f(z) - f(x)| \leq \|z - x\| \leq \delta/4$

so $\sup f(K) \leq f(z) \leq f(x) + \delta/4$

Apply Bishop Phelps w/ $\epsilon = \delta/4$ and $\gamma = 1/4$. By (2)

x_0 is a support pt. and by (4) $\|x - x_0\| \leq \delta$. $\boxed{\diagup}$

Coro (Bollobás)

K as before, $f \in \mathfrak{X}^*$, $\|f\| = 1$ and $x \in K$ w/

$$\sup f(K) \leq f(x) < \epsilon^2/2 \qquad 0 < \epsilon < 1$$

then $\exists$ $g \in \mathcal{X}^*$, $\|g\| = 1$ and $x_0 \in K \ni$

(i) $\sup g(K) = g(x_0)$

(ii) $\|x - x_0\| \leq \epsilon$

(iii) $\|g - f\| \leq \epsilon$

Proof

Apply Bishop Phelps w/ $\epsilon = \epsilon^2/2$ and $\gamma = \epsilon/2$. $\boxtimes$

Banach Spaces (11/16) Larry Riddle

Applications of Bishop - Phelps

(1) Numerical Ranges

Let $\mathfrak{X}$ be a B-Space and $T \in B(\mathfrak{X})$. The numerical range of T, $V(T)$ is

$$V(T) := \{ f(Tx) : \|x\| = 1, \|f\| = 1, f(x) = 1 \}$$

Theorem

$$\overline{V(T^*)} = \overline{V(T)}$$

note: $V(T^*) := \{ x^{**}(T^* f) : \|f\| = 1, \|x^{**}\| = 1, x^{**}(f) = 1 \}$

proof

$\lambda \in V(T) \implies \lambda = f(Tx)$ where $\|x\| = \|f\| = f(x) = 1$

Consider $\hat{x} = Jx \in \mathfrak{X}^{**}$.

$$\|\hat{x}\| = \|x\| = 1 \quad \text{and} \quad \hat{x}(f) = f(x) = 1$$

$$\lambda = f(Tx) = T^* f(x) = \hat{x}(T^* f) \in V(T^*)$$

$$\therefore \quad V(T) \subset V(T^*)$$

Claim: $V(T^*) \subset \overline{V(T)}$

let $\mu \in V(T^*)$ and $\epsilon > 0$. $\exists\ x^{**} \in \mathfrak{X}^{**}$, $f \in \mathfrak{X}^* \ni$

$\|x^{**}\| = 1$, $\|f\| = 1$, $x^{**}(f) = 1$

$\mu = x^{**}(T^* f)$

Goldstine $\Rightarrow$ $S_{\mathfrak{X}}$ is w^* dense in $S_{\mathfrak{X}^{**}}$ $\Rightarrow$

$\exists\ x \in S_{\mathfrak{X}} \ni$

(i) $|\hat{x}(f) - x^{**}(f)| \leq \epsilon^2/2$

(ii) $|\hat{x}(T^* f) - x^{**}(T^* f)| \leq \epsilon$

(i) $\Rightarrow$ $\sup f(B_{\mathfrak{X}}) = \|f\| = 1 = x^{**}(f) \underset{\uparrow (i)}{\leq} \hat{x}(f) + \epsilon^2/2$

by Bolobas's coro, $\exists\ g \in S_{\mathfrak{X}^*}$, $x_0 \in S_{\mathfrak{X}} \ni$

(iii) $g(x_0) = 1$

(iv) $\|f - g\| \leq \epsilon$

(v) $\|x - x_0\| \leq \epsilon$

$\hat{x}_0(T^* g) = T^* g(x_0) = g(T x_0) \in V(T)$

$$|\hat{x}_0(T^*g)-\mu| = |\hat{x}_0(T^*g) - x^{**}(T^*f)|$$

$$\leq |g(Tx_0) - f(Tx_0)| + |f(Tx_0) - f(Tx)| + |f(Tx) - x^{**}(T^*f)|$$

$$\leq \|g-f\| \|T\| \|x_0\| + \|f\| \|T\| \|x_0 - x\| + \underset{\uparrow (ii)}{\epsilon}$$

$$\leq \epsilon \|T\| + \|T\| \epsilon + \epsilon$$

$$= (2\|T\|+1)\epsilon$$

$$\therefore \mu \in \overline{V(T)} \quad \boxtimes$$

(2) <u>Very Smooth B-Spaces</u>

H.B. $\Rightarrow \forall x \in S_{\mathfrak{X}} \ \exists f_x \in S_{\mathfrak{X}^*} \ni f_x(x) = 1$. We call $\mathfrak{X}$ <u>smooth</u> if f_x is unique. We call $\mathfrak{X}$ <u>very smooth</u> if $\forall x \in S_{\mathfrak{X}} \ \exists! \, x^{***} \in S_{\mathfrak{X}^{***}} \ni$

$$x^{***}(x) = 1 \qquad (\mathfrak{X} \subset \mathfrak{X}^{**})$$

Equivalently $\mathfrak{X}$ is very smooth $\Leftrightarrow$ $\mathfrak{X}$ is smooth and the map $x \longmapsto {}^\wedge$ of $S_{\mathfrak{X}}$ into $S_{\mathfrak{X}^*}$ is norm – w cont.

Theorem

$\mathfrak{X}^*$ very smooth $\Rightarrow$ $\mathfrak{X}$ reflexive.

proof

Denote by $f \mapsto F_f$ the norm-to-wk cont support map of $S_{\mathfrak{X}^*}$ into $S_{\mathfrak{X}^{**}}$.

Let $f \in S_{\mathfrak{X}^*}$. By James theorem it suffices to show that f obtains its norm on $S_{\mathfrak{X}}$.

Bishop Phelps $\Rightarrow$ $\exists$ a seq $(f_n) \subset S_{\mathfrak{X}^*}$ which do attain their norm on $S_{\mathfrak{X}}$ and for which $f_n \xrightarrow[\|\cdot\|]{} f$

Let $(x_n) \subset S_{\mathfrak{X}}$ $\ni$ $f_n(x_n) = 1$. Then $\hat{x}_n(f) = 1$ and $\hat{x}_n \in S_{\mathfrak{X}^{**}}$, so $\hat{x}_n = F_{f_n}$ by smoothness of $\mathfrak{X}^*$

But $f_n \xrightarrow[\|\cdot\|]{} f$ so $F_{f_n} \xrightarrow{w} F_f$

$\mathfrak{X}^*$ very smooth

$\therefore$ $\hat{x}_n \xrightarrow{w} F_f$

$\therefore \; F_f \in w(\mathfrak{X}) = \mathfrak{X}^{-} \; (\in \mathfrak{X}^{**}) = \mathfrak{X}$

i.e. $F_f \in \mathfrak{X}$

so $F_f = \hat{x}$ for some $x \in S_{\mathfrak{X}}$

$\therefore \quad 1 = F_f(f) = \hat{x}(f) = f(x)$

$\therefore$ f attains its norm at x

So $\mathfrak{X}$ is reflexive $\boxtimes$

11/19 BANACH SPACES

Preparation for Krein-Milman Theorem

DEFINITION: A B-space $\mathfrak{X}$ is called strictly convex "rotund" if $x, y \in \overline{B_{\mathfrak{X}}}$ implies $(x,y) \subseteq B^{\circ}_{\mathfrak{X}}$

COROLLARY: In a strictly convex space each point on the surface of the unit ball is an extreme point of the unit ball.

Example: l_p $1<p<\infty$ is rotund.

Take (α_n) and (β_n) in the closed unit ball of l_p. Suppose $\exists\ 0<t<1$ s.t

$$\| t(\alpha_n) + (1-t)\beta_n \|_p = 1$$

i.e.

$$\sum_{n=1}^{\infty} (t\alpha_n + (1-t)\beta_n)^p = 1$$

But if $\varphi(y) = |y|^p$, then for $x \neq y$

$$\varphi(tx + (1-t)y) < t\varphi(x) + (1-t)\varphi(y)$$

If $\exists n_0$ s.t. $\alpha_{n_0} \neq \beta_{n_0}$ then

$$1 = \sum_{n=1}^{\infty} |t\alpha_n + (1-t)\beta_n|^p < \sum_{n=1}^{\infty} t|\alpha_n|^p + (1-t)|\beta_n|^p$$

$$= t\,\|(\alpha_n)\|^p + (1-t)\,\|(\beta_n)\|^p \leq 1$$

Hence $\alpha_n = \beta_n$ $\forall n$, so $\alpha = \beta$. Hence $((\alpha_n), (\beta_n)) \subset B^{\circ}_{\ell_p}$

Extreme points in the ball of $C(K)^*$

Obvious extreme points: $\pm$ The point masses. Take $x \in K$ and let δ_x = point mass measure. Suppose

$$\delta_x = t\mu + (1-t)\lambda$$

where $\mu, \lambda \in \overline{B_{C(K)^*}}$. Then

$$|\delta_x|(E) \leq t|\mu|(E) + (1-t)|\lambda|(E)$$

If $E = \{x\}$, then

$$1 \leq t|\mu|(E) + (1-t)|\lambda|(E)$$

Hence $|\mu|(E) = |\lambda|(E) = 1$. Therefore $\mu = \pm\delta_x$, $\lambda = \pm\delta_x$. Since

$$\delta_x = t\mu + (1-t)\lambda$$

it follows directly that $\mu = \lambda = \delta_x$. Hence $\delta_x \in \text{ext}(B_{C(K)^*})$

Fact: If $\mu \in B_{C(K)^*}$ is not $\pm$ a point mass, then $\mu \notin \text{ext}(C(K)^*)$

WLOG $|\mu|(K) = 1$

Suppose μ is not a point mass. Then $\exists$ a Borel set A s.t.

$$|\mu|(A) \neq 0 \quad \text{and} \quad |\mu|(K \setminus A) \neq 0$$

Define

$$\mu_1(E) = \frac{\mu(E \cap A)}{|\mu|(A)}$$

$$\mu_2(E) = \frac{\mu(E \setminus A)}{|\mu|(K \setminus A)}$$

Then $|\mu_1|(K) = \dfrac{|\mu|(A)}{|\mu|(A)} = 1$ and $|\mu_2|(K) = \dfrac{|\mu|(K \setminus A)}{|\mu|(K \setminus A)} = 1$. But

$$\mu = |\mu|(A)\mu_1 + |\mu|(K \setminus A)\mu_2$$

Hence $\mu \in (\mu_1, \mu_2)$ since $\mu_1 \neq \mu_2$

Consequences and related facts:

① If μ is non-atomic, then $\overline{B_{L_1(\mu)}}$ has no extreme points (same proof)

② $\overline{B_{C(K)^*}} = \text{weak}^* \text{-} \overline{co}$ ($\pm$ point masses) , i.e. if $\mu \in \overline{B_{C(K)^*}}$

then $\exists$ a net λ_α of convex combinations of $\pm$ point masses s.t.

$$\lim_\alpha \int_K f \, d\lambda_\alpha \longrightarrow \int_K f \, d\mu \quad \forall f \in C(K)$$

In particular, if $K = [0,1]$, then we can get by with a sequence

<u>Corollary of Krein-Milman:</u> If W is a weakly compact convex set in B-space, then $W = \text{norm-}\overline{co}(\text{ext } W)$

Proof: $W = \text{weak-}\overline{co}(\text{ext } W) = \text{norm-}\overline{co}(\text{ext } W)$

Fact: $\overline{B_{\ell_1}} = \overline{co}(\text{ext } \overline{B_{\ell_1}})$ since $\text{ext } \overline{B_{\ell_1}} = \pm$ unit vectors

and

$$(\alpha_n) = \sum \alpha_n e_n \; , \; \|(\alpha_n)\| = \sum |\alpha_n|$$

11/21 BANACH SPACES

Let (Ω, Σ, μ) be a finite measure space. If (f_n) is a sequence in $L_\infty(\mu)$ s.t. $\lim f_n = f$ weakly. Then $\lim f_n = f$ a.e.

Proof. (Bourgain) **WLOG** $f_n \to 0$ weakly. It is possible to find a null set N s.t.

$$\left\| \sum_{i=1}^{\infty} \alpha_i f_i \right\|_\infty = \sup_{t \in \Omega \setminus N} \left| \sum_{i=1}^{\infty} \alpha_i f_i(t) \right|$$

for all finitely non-zero sequences of rationals. Hence

$$(*) \qquad \left\| \sum_{i=1}^{\infty} \alpha_i f_i \right\|_\infty = \sup_{t \in \Omega \setminus N} \left| \sum_{i=1}^{\infty} \alpha_i f_i(t) \right|$$

for all finitely non-zero sequences of reals.

Now suppose $\exists\, \varepsilon > 0$ and a $t_0 \in \Omega \setminus N$ and subsequence (f_{n_i}) of (f_n) s.t.

$$f_{n_i}(t_0) > \varepsilon$$

By one of our theorems, there exist a sequence φ_m of convex combinations of the tail of (f_{n_i}) s.t.

$$\lim_m \varphi_m = 0 \quad (\text{norm})$$

Now by $(*)$

$$\|\varphi_m\| = \sup_{t\in\Omega\setminus N} |\varphi_m(t)| \geq |\varphi_m(t_0)|$$

$$= \left|\sum_{i=1}^{k} \alpha_i f_{n_i}(t_0)\right| \quad \left(\text{where } \alpha_i \geq 0,\ \sum_{i=1}^{k}\alpha_i = 1\right)$$

$$\geq \sum_{i=1}^{k} \alpha_i \varepsilon = \varepsilon$$

Similar argument work in case $f_{n_i}(t_0) < -\varepsilon \ \forall (n_i)$. Hence

$$\lim f_m(t) = 0 \quad \forall t \in \Omega\setminus N$$

Therefore $f_m \to 0$ a.e.

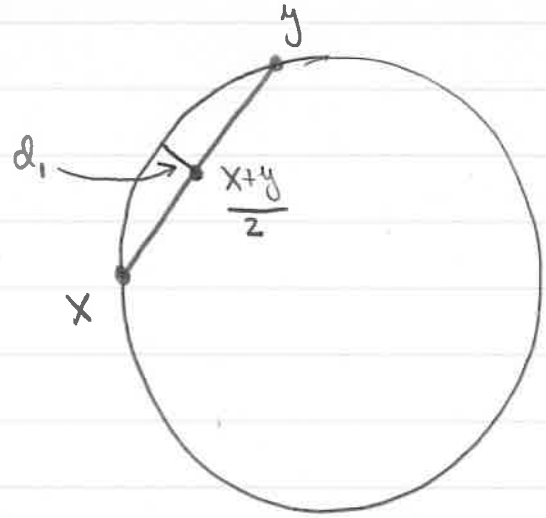
▨

(Same result if $(f_n) \in L_\infty(\mu, \mathfrak{X}^*)$ converges w^* to f)

One of the earliest geometric classes of spaces was introduced by Clarkson in 1936 TAMS. His purpose was to initiate the study of Radon-Nikodym property.

<u>DEFINITION</u>: A B-space $\mathfrak{X}$ is uniformly convex (uniformly rotund) if for each $\varepsilon>0$ $\exists\, \delta>0$ s.t.

$$\|x\|, \|y\| \leq 1,\ \|x+y\| \geq 2-\delta \implies \|x-y\| < \varepsilon$$



d_1 small $\Rightarrow$ chord small

Corollary: Uniform convex $\Rightarrow$ strict convex

Examples: ① $L_2(\mu)$ is uniformly convex

Recall $f, g \in L_2(\mu) \Rightarrow \|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$

If $\|f+g\|$ is near 2, then

$$(\text{nearly } 2)^2 + \|f-g\|^2 = 2(\text{nearly } 1) + 2(\text{nearly } 1)$$

$$\Rightarrow \|f-g\| \text{ nearly } 0$$

② $1 < p < \infty \Rightarrow L_p(\mu)$ is uniformly convex. Follows from

Lemma: (Clarkson's inequalities)

a) $1 < p \le 2 \Rightarrow 2^{p-1}(\|f\|_p^p + \|g\|_p^p) \le \|f+g\|_p^p + \|f-g\|_p^p$
$$\le 2(\|f\|_p^p + \|g\|_p^p)$$

b) $2 < p < \infty \Rightarrow 2(\|f\|_p^p + \|g\|_p^p) \le \|f+g\|_p^p + \|f-g\|_p^p \le 2^{p-1}(\|f\|_p^p + \|g\|_p^p)$

Proof of lemma: a) Observe that it suffices to prove

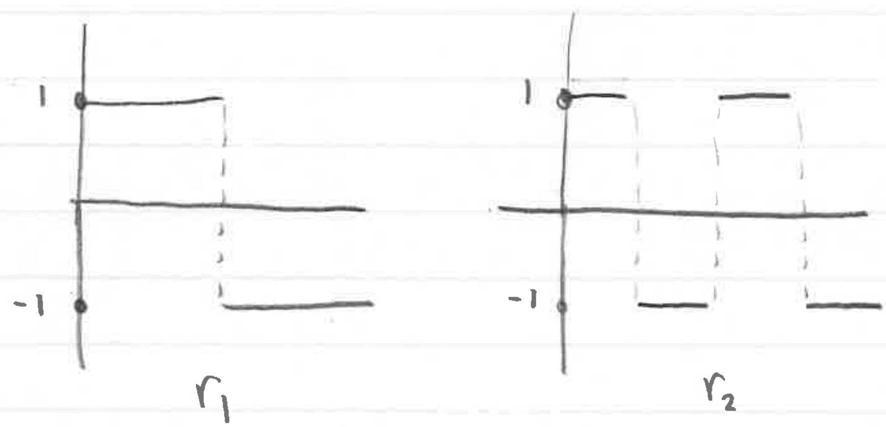
$$(*) \qquad \|f+g\|_p^p + \|f-g\|_p^p \le 2\left(\|f\|_p^p + \|g\|_p^p\right)$$

Why? Replace f by $f+g$ and g by $f-g$ to see that if $(*)$ holds, then the other inequality holds.

Observe that

$$\frac{1}{2}\left(\|f+g\|_p^p + \|f-g\|_p^p\right) \le \int_0^1 \|r_1(t)f + r_2(t)g\|_p^p \, dt$$

(where r_1 = 1ST Radamacher function and r_2 = 2ND Radamacher



)

$$= \int_0^1 \left(\int_\Omega |r_1(t)f(\omega) + r_2(t)g(\omega)|^p \, d\mu(\omega) \right) dt$$

$$= \int_\Omega \int_0^1 |r_1(t)f(\omega) + r_2(t)g(\omega)|^p \, dt \, d\mu$$

$$= \int_\Omega \int_0^1 \left(|r_1(t)f(\omega) + r_2(t)g(\omega)|^2\right)^{p/2} dt \, d\mu$$

$$\leq \int_\Omega \left[\int_0^1 | r_1(t) f(\omega) + r_2(t) g(\omega) |^2 \, dt \right]^{p/2} d\mu$$

(Why? If $0 < \alpha < 1$, and $0 \leq \varphi^\alpha \in L^{1/\alpha}[0,1]$, then

$$\int_0^1 \varphi^\alpha(t)\, dt \leq \left(\int_0^1 (\varphi^\alpha)^{1/\alpha} dt \right)^\alpha \left(\int_0^1 1^{1/\beta} dt \right)$$

Hölder where $\alpha + \beta = 1$)

$$= \int_\Omega \left[\int_0^1 r_1^2(t) f^2(\omega) + 2 r_1(t) r_2(t) f(\omega) g(\omega) + r_2^2(t) g^2(\omega)\, dt \right]^{p/2} d\mu$$

$$= \int_\Omega \left(f^2(\omega) + g^2(\omega) \right)^{p/2} d\mu \qquad (r_i \text{ orthonormal in } L^2)$$

$$\leq \int_\Omega (|f|^2)^{p/2} + (|g|^2)^{p/2} \, d\mu \qquad \left((1+t)^r \leq 1 + t^r,\ 0 \leq r \leq 1 \right)$$

$$= \|f\|_p^p + \|g\|_p^p$$

b) is proved similarly

11/26 BANACH SPACES

HW/ Application of Krein-Milman

LIAPUNOV'S THEOREM: Let Σ be a σ-field of subsets of Ω. Let $\mu_1, \dots, \mu_n$ be finite non-atomic signed measures on Σ. Define

$$F(E) = (\mu_1(E), \dots, \mu_n(E))$$

Prove $F(\Sigma)$ is a compact convex subset of $\mathbb{R}^n$.

Proof due to Lindenstrauss 1964

Steps ① Put $\mu = \sum_{i=1}^{n} |\mu_i|$ Define $T: L_\infty(\mu) \to \mathbb{R}^n$ by

$$T(f) = \left(\int_\Omega f \, d\mu_1, \dots, \int_\Omega f \, d\mu_n \right)$$

Use Radon-Nikodym theorem to prove T is weak*-cont on $L_\infty(\mu)$

② Let $K = \{ g \in L_\infty(\mu) : 0 \leq g \leq 1 \}$ Prove $T(K)$ is compact and convex

③ Prove $F(\Sigma) \subseteq T(K)$

④ Pick $x \in T(K)$ and write $K_x = \{ g \in K : T(g) = x \}$ and show that K_x is compact and convex and hence has extreme points

⑤ Show if $\mu(E) > 0$, then $\chi_E L_\infty(\mu)$ is infinite dimensional (non-atomicity used)

⑥ Show that if $g \in K_X$ and g is not a characteristic function, then $g \notin \text{ext}(K_X)$

(Hint: If g is not characteristic then $\exists\, E \in \Sigma$ with $r \leq g \leq 1-r$ on E for some $r > 0$. Find a $g_0 \in L_\infty(\mu)$ with $-r \leq g_0 \leq r$ s.t. $T(g_0) = 0$
Conclude g was not extreme

⑦ Conclude $T(K) = F(\Sigma)$

⑧ Stop

Why uniformly convex spaces are good

① $\mathfrak{X}$ unif. convex, (x_n) seq. in $\mathfrak{X}$ s.t. $\|x_n\| \to 1$ and $\|x_n + x_m\| \to 2$ then $\lim x_n$ exists

Proof. If $\|x_n\| \le 1$ $\forall n$, then follows directly from definition. In general

$$\left\| \frac{x_n}{\|x_n\|} \right\| \to 1 \quad \text{and} \quad \left\| \frac{x_n}{\|x_n\|} + \frac{x_m}{\|x_m\|} \right\| \to 2$$

Hence $\lim_n \frac{x_n}{\|x_n\|}$ exists. Since $\lim \|x_n\| = 1$, we see that $\lim x_n$ exists.

② (Milman - Pettis 1939) Uniformly convex spaces are reflexive

Proof: Let $x^{**} \in \mathfrak{X}^{**}$ have norm 1. It suffices to show that $x^{**} \in \mathfrak{X}$.

Use Goldstine's theorem to find a net (x_α) in $B_{\mathfrak{X}}$ s.t.

$$\lim x_\alpha = x^{**}$$

in the weak* topology of $\mathfrak{X}^{**}$. Notice $x_\alpha + x_\beta \longrightarrow 2x^{**}$ w^* and

$$\| x_\alpha + x_\beta \| \le \|x_\alpha\| + \|x_\beta\| = 2$$

Claim: $\lim_{\alpha,\beta} \|x_\alpha + x_\beta\| = 2$

Suppose $\exists\, \delta > 0$ s.t. on a subnet $\|x_\alpha + x_\beta\| \le 2 - \delta$
Take $x^* \in \mathfrak{X}^*$ s.t. $\|x^*\| = 1$ and

$$x^{**}(x^*) > 1 - \delta/4$$

Then $x^*(x_\alpha + x_\beta) \longrightarrow 2x^{**}(x^*) > 2(1 - \delta/4) = 2 - \delta/2$.
On the other hand,

$$|x^*(x_\alpha + x_\beta)| \le \|x_\alpha + x_\beta\| \le 2 - \delta.$$

Hence $2 - \delta/2 < 2 - \delta$ $\lightning$. Therefore $\lim \|x_\alpha + x_\beta\| = 2$

Hence $\|x_\alpha\| \le 1$ $\forall \alpha$ and $\|x_\alpha + x_\beta\| \longrightarrow 2$. By definition of unif. convexity, $\lim x_\alpha$ exists in norm. But $x^{**} = w^*\text{-limit } x_\alpha$ so $x^{**} = \lim x_\alpha$ in norm. Hence $x^{**} \in \mathfrak{X}$.

$\blacksquare$

③ $\left(\sum_{n=1}^{\infty} \ell_1^n\right)_{\ell_2}$ is reflexive but not uniformly convexifiable

④ In a uniformly convex space, a closed convex set has a unique element of smallest norm. In fact, minimizing sequences converge.

Proof. Let C be a closed convex subset of a unif. convex space $\mathfrak{X}$. Let (x_n) be a seq. in C s.t.

$$\lim \|x_n\| = \inf\{\|x\| : x \in C\} = d$$

We shall show that $\lim x_n$ exists.

Case 1: $d = 0$ In this case $0 \in C$ and $\lim x_n = 0$.

Case 2: WLOG $d = 1$. Then $\|x_n\| \to 1$. Also,

$$\|x_n + x_m\| = 2 \left\| \frac{x_n + x_m}{2} \right\| \geq 2d = 2$$

$\frac{x_n+x_m}{2} \in C$

But $\|x_n + x_m\| \leq \|x_n\| + \|x_m\| \leq (1+\varepsilon) + (1+\varepsilon)$ for large n. Hence $\|x_n + x_m\| \to 2$. By unif. convexity, $\lim x_n$ exists in norm. If $x_0 = \lim x_n$, then $x_0 \in C$ and $\|x_0\| = d$.

If $x_1 \in C$ also has $\|x_1\| = d$, then apply above result to sequence $(x_0, x_1, x_0, x_1, x_0, x_1, \dots)$ to get contradiction.

$\boxtimes$

FACT: If C is a weakly compact subset of a B-space, then C has a member of smallest norm.

Proof. Take $(x_n) \in C$ s.t. $\lim \|x_n\| = \inf\{\|x\| : x \in C\}$. Let (y_n) be a weakly convergent subsequence, say $w\text{-}\lim y_n = y \in C$. Then

$$d \leq \|y\| \leq \underline{\lim} \|y_n\| \to d$$

so $\|y\| = d$

11/28 BANACH SPACES

DEFINITION: Let C be a closed bounded convex set.

1) We say $x_0 \in C$ is an exposed point if $\exists\, x^* \in \mathfrak{X}^*$ s.t.

$$x^*(x_0) > x^*(x) \quad \forall x \in C \setminus \{x_0\}$$

2) We say $x_0 \in C$ is strongly extreme if for each $\varepsilon > 0$ $\exists\, x^* \in \mathfrak{X}^*$ and a number K s.t. $x^*(x_0) \geq K$ and

$$\operatorname{diam}\left(C \cap [x^* \geq K]\right) < \varepsilon$$

(also called denting point)

3) We say $x_0 \in C$ is strongly exposed if $\exists\, x^* \in \mathfrak{X}^*$ s.t.

$$x^*(x_0) > x^*(x) \quad \forall x \in C \setminus \{x_0\}$$

and s.t. if (x_n) is a sequence in C with $\lim x^*(x_n) = x^*(x_0)$, then $\lim \|x - x_n\| = 0$.

Examples: ① $(1,0,0,\dots) \in \ell_1$ is strongly exposed in $\overline{B_{\ell_1}}$ by $(1,0,0,\dots) \in \ell_\infty$

Why? Notice that $\underset{\ell_\infty}{(1,0,\dots)}\,\underset{\ell_1}{(\alpha_1,\alpha_2,\dots)} = \alpha_1$. If $\sum |\alpha_n| \leq 1$ and α_1 is closed to 1, then $(\alpha_1,\alpha_2,\dots)$ is closed to $(1,0,\dots)$

② Every point on the surface of the unit ball of a uniformly convex space is strongly exposed

Let $\mathfrak{X}$ be unif. convex and take $x_0 \in \mathfrak{X}$ with $\|x_0\| = 1$. Choose $x^* \in \mathfrak{X}^*$ s.t. $\|x^*\| = 1$ and $x^*(x_n) = 1$. Suppose (x_n) is a seq. in the unit ball of $\mathfrak{X}$ s.t

$$x^*(x_n) \longrightarrow x^*(x) = 1$$

Then $x^*(x_n + x_m) \longrightarrow 2$. Hence $\|x_n\| \longrightarrow 1 = \|x_0\|$ and $\|x_n + x_m\| \longrightarrow 2$. Therefore $\lim x_n = y$ exists in norm. To see that $y = x_0$, replace (x_n) by

$$(x_1, x_0, x_2, x_3, x_0, x_4, x_0, \dots)$$

and use same argument to conclude the above sequence converges.

FACT: TFAE

① $\mathfrak{X}$ is an Asplund space; i.e. every continuous real valued convex function on $B_{\mathfrak{X}}$ is Fréchet differentiable on a dense G_δ

② $\mathfrak{X}^*$ has RNP; i.e. every absolutely continuous function $f : [0,1] \longrightarrow \mathfrak{X}^*$ is differentiable a.e.

③ Every separable subspace of $\mathfrak{X}$ has a separable dual

④ $\mathfrak{X}^*$ has Krein Milman property; i.e. every closed bound convex subset of $\mathfrak{X}^*$ is the norm-$\overline{co}$ (ext C)

⑤ Every closed bounded convex subset of $\mathfrak{X}^*$ is the norm closed convex hull of its strongly exposed points

⑥ Every closed bound convex subset of $\mathfrak{X}^*$ has a strongly extreme point.

⑦ $I : (\mathfrak{X}^*, w^*) \longrightarrow (\mathfrak{X}^*, \text{weak})$ is universally Lusin measurable

OPEN: Is ② $\Leftrightarrow$ ④ for non-dual spaces, i.e. general B-spaces?

Known: $\mathfrak{X}$ has RNP iff $\mathfrak{X}$ has strong KMP, i.e. every closed bounded subset A of $\mathfrak{X}$ has an extreme point of the norm-$\overline{co}$ (A)

Known: If $\mathfrak{X}$ fails RNP, $\exists$ an equivalent norm on $\mathfrak{X}$ s.t. if $B_{\mathfrak{X}}$ is unit ball in the new norm then $\exists$ a closed bdd subset $A \subset B_{\mathfrak{X}}^{0}$ s.t.

$$\overline{co}(A) = \overline{B_{\mathfrak{X}}}$$

OPERATOR THEORY

$B(\mathcal{X},\mathcal{Y})$ – uniform operator top. given by operator norm

– strong operator top. $T_\alpha \longrightarrow T$ iff $\lim T_\alpha x = Tx$ $\forall x$

– weak operator top $T_\alpha \longrightarrow T$ iff $\lim y^* T_\alpha x = y^* Tx$
$\forall y^* \in \mathcal{Y}^*$ $\forall x \in \mathcal{X}$

DEFINITION: Let $T \in B(\mathcal{X},\mathcal{Y})$. The operator $T^*: \mathcal{Y}^* \to \mathcal{X}^*$ is defined by

$$T^* y^* = y^* \circ T$$

Fact: ① T^* is linear

② The operator $T \longrightarrow T^*$ is a linear isometry of $B(\mathcal{X},\mathcal{Y})$ into $B(\mathcal{Y}^*,\mathcal{X}^*)$

Proof: $\|T^*\| = \sup_{\|y^*\|\le 1} \|T^* y^*\| = \sup_{\|y^*\|\le 1} \|y^* T\|$

$$= \sup_{\|y^*\|\le 1} \sup_{\|x\|\le 1} |y^* Tx| = \sup_{\|x\|\le 1} \sup_{\|y^*\|\le 1} |y^*(Tx)|$$

HB

$$= \sup_{\|x\|\le 1} \|Tx\| = \|T\|$$

11/30 BANACH SPACES

DEFINITION: $T \in B(\mathfrak{X}, \mathcal{Y})$ is a $\left\{\begin{matrix}\text{compact} \\ \text{weakly compact}\end{matrix}\right\}$ operator if $\overline{T(B)}$ is a $\left\{\begin{matrix}\text{compact} \\ \text{weakly compact}\end{matrix}\right\}$ subset of $\mathcal{Y}$

DEFINITION: $T \in B(\mathfrak{X}, \mathcal{Y})$ is completely continuous (Dunford-Pettis) if T takes weakly compact sets into norm compact sets

DEFINITION: A Banach space $\mathfrak{X}$ is said to have the Dunford-Pettis property if $\forall$ B-spaces $\mathcal{Y}$ every weakly compact operator in $B(\mathfrak{X}, \mathcal{Y})$ is completely continuous.

COROLLARY: All compact operators are completely continuous.

Fact: T completely continuous, W weakly compact $\Rightarrow T(W)$ is norm compact.

Proof Take a sequence Tx_n in $T(W)$ (here $x_n \in W$). Use weak compactness of W and Eberlein - Smulian to find a subsequence (z_j) of (x_n) s.t.

$$\lim z_j = x_0 \text{ weakly}$$

$x_0 \in W$. Since T is weak-to-weak continuous. Hence

$$\lim T(z_j) = T(x_0) \text{ weakly}$$

$$\Rightarrow \lim T(z_j) = T(x_0) \text{ in norm}$$

since $\overline{T(W)}$ is norm compact

$\blacksquare$

Fact: No infinite dimensional reflexive B-space has D-P

Proof. Let $\mathfrak{X}$ be reflexive, and $I : \mathfrak{X} \to \mathfrak{X}$ the identity operator. Then I is weakly compact. If I is completely continuous then

$$B_{\mathfrak{X}} = \underbrace{I(B_{\mathfrak{X}})}_{\text{weakly compact}} = \text{norm compact} \; \lightning$$

$\blacksquare$

Fact: No infinite dim reflexive subspace of a D-P space is the range of a bounded projection; i.e. no reflexive subspace of a D-P space is complemented.

Fact: (Lindenstrauss - Tzafriri) If $\mathfrak{X}$ is a B-space s.t. each of its subspaces is the range of a continuous projection, then $\mathfrak{X}$ is isomorphic to $l_2(\Gamma)$ for some set Γ.

Proof of penultimate fact: Let $\mathfrak{X}$ have D-P. Let R be a reflexive subspace of $\mathfrak{X}$ s.t. $\exists$ projection $P: \mathfrak{X} \to R$ with $P(\mathfrak{X}) = R$. Evidently $P(B_{\mathfrak{X}})$ is weakly compact. But

$$\underbrace{P(B_{\mathfrak{X}}) = P(\underbrace{P(B_{\mathfrak{X}})}_{\text{weakly compact}})}_{} = \underbrace{\text{rel. norm compact}}_{\text{since } P \text{ is completely continuous}}$$

But P maps $\mathfrak{X}$ onto R, so by the Open Mapping Theorem $P(B_{\mathfrak{X}})$ is open. Hence R has a relatively norm compact open set, so R is finite dimensional. ▨

(Back to adjoints)

<u>PROPOSITION</u>: Let $T \in B(\mathfrak{X}, \mathcal{Y})$. Then $T^* : \mathcal{Y}^* \to \mathfrak{X}^*$ is weak*-to-weak* continuous.

Proof Let (y^*_α) be a net in $\mathcal{Y}^*$ s.t. $\lim y^*_\alpha = y^*$ weak*. Then

$$T^* y^*_\alpha (x) = y^*_\alpha (Tx) \to y^*(Tx) = T^* y^* (x)$$

for all $x \in \mathfrak{X}$, so $\lim T^* y^*_\alpha = T^* y^*$ weak*. ▨

HW / Let $T \in B(\mathcal{Y}^*, \mathcal{X}^*)$. Then $\exists\, S \in B(\mathcal{X}, \mathcal{Y})$ s.t. $S^* = T$ iff T is w^*-w^* continuous.

<u>THEOREM</u>: (Gantmacher) $T \in B(\mathcal{X}, \mathcal{Y})$ is weakly compact iff $T^{**}(B_{\mathcal{X}^{**}}) \subset \mathcal{Y}$

Proof. Let $T : \mathcal{X} \to \mathcal{Y}$ be weakly compact. Then $T^{**} : \mathcal{X}^{**} \to \mathcal{Y}^{**}$

Then

$$T^{**}(B_{\mathcal{X}^{**}}) \subseteq \overline{T^{**}(B_{\mathcal{X}})}^{w^*}$$

(since $B_{\mathcal{X}}$ is w^* dense in $\mathcal{X}^{**}$ and T^{**} is w^*-w^* continuous)

$$= \overline{T(B_{\mathcal{X}})}^{w^*} \quad (\text{in } \mathcal{Y}^{**})$$

since $T^{**}|_{\mathcal{X}} = T$

$$= \overline{T(B_{\mathcal{X}})}^{w} \quad (\text{in } \mathcal{Y}^{**})$$

since $T(B_{\mathcal{X}})$ is rel. w. compact

$$= \overline{T(B_{\mathcal{X}})}^{w} \quad (\text{in } \mathcal{Y})$$

$$\subset \mathcal{Y}$$

Conversely, suppose $T^{**}(B_{\mathcal{X}^{**}}) \subset \mathcal{Y}$. But $T^{**}(\overline{B_{\mathcal{X}^{**}}})$ is w^* compact in $\mathcal{Y}^{**}$ by Alaoglu, so $T^{**}(\overline{B_{\mathcal{X}^{**}}})$ is weakly compact in $\mathcal{Y}$. Then $T(B_{\mathcal{X}}) \subset T^{**}(\overline{B_{\mathcal{X}^{**}}})$ is rel. weakly compact. $\blacksquare$

12/3 BANACH SPACES

COROLLARY: $\mathfrak{X}$ or $\mathcal{Y}$ reflexive $\Rightarrow$ every member of $B(\mathfrak{X},\mathcal{Y})$ is weakly compact.

Proof. $\mathcal{Y}$ reflexive $\Rightarrow$ every bounded subset of $\mathcal{Y}$ is relatively weakly compact

$\mathfrak{X}$ reflexive $\Rightarrow$ $\mathfrak{X}^{**} = \mathfrak{X}$ $\Rightarrow$ $T^{**}(\mathfrak{X}^{**}) = T^{**}(\mathfrak{X}) = T(x) \subset \mathcal{Y}$.

$\blacksquare$

COROLLARY: The subspace $WC(\mathfrak{X},\mathcal{Y})$ of $B(\mathfrak{X},\mathcal{Y})$ consisting of the weakly compact operators is closed

Proof. Let (T_n) be a Cauchy sequence in $WC(\mathfrak{X},\mathcal{Y})$. Then $\lim T_n = T$ exists in norm of $B(\mathfrak{X},\mathcal{Y})$. Then

$$\lim \| T_n^{**} - T^{**} \| = 0$$

In particular,

$$\lim \| T_n^{**}(x^{**}) - T^{**}(x^{**}) \| = 0$$

for all $x^{**} \in \mathfrak{X}^{**}$. But $T_n^{**}(x^{**}) \in \mathcal{Y}$ $\forall n$, and so $T^{**}(x^{**}) \in \mathcal{Y}$ for all $x^{**} \in \mathfrak{X}^{**}$. Therefore T is weakly compact.

$\blacksquare$

Lemma: $T \in WC(\mathfrak{X}, \mathcal{Y})$ if and only if $T^*: \mathcal{Y}^* \to \mathfrak{X}^*$ is weak*-to-weakly continuous.

Proof. ($\Leftarrow$) For every $x^{**} \in \mathfrak{X}^{**}$, $T^{**}(x^{**})$ is continuous in the weak* topology of $\mathcal{Y}^*$, so $T^{**}(x^{**}) \in \mathcal{Y}$. Hence T is weakly compact.

($\Rightarrow$) Let y^*_α be a net in $\mathcal{Y}^*$ s.t. $\lim y^*_\alpha = y^*$ w^*. Fix $x^{**} \in \mathfrak{X}^{**}$ and consider

$$\begin{array}{ccc} \lim x^{**} T^*(y^*_\alpha) & \overset{?}{=} & x^{**} T^*(y^*) \\ \| & & \| \\ \lim T^{**} x^{**}(y^*_\alpha) & \overset{?}{=} & T^{**}(x^{**}) y^* \\ & \uparrow & \\ & \text{yes since } T^{**}(x^{**}) \in \mathcal{Y} & \end{array}$$

$\blacksquare$

Theorem (Gantmacher's Theorem) $T \in WC(\mathfrak{X}, \mathcal{Y})$ if and only if $T^* \in WC(\mathcal{Y}^*, \mathfrak{X}^*)$

Proof. ($\Rightarrow$) If T is weakly compact then T^* is weak* to weakly continuous. Hence $T^*(B_{\mathcal{Y}^*})$ is relatively weakly compact

($\Leftarrow$) Suppose T^* is weakly compact. Then T^{**} is also weakly compact by the first part, so $T^{**}(B_{\mathfrak{X}^{**}})$ is rel. weakly compact in $\mathcal{Y}^{**}$. Hence

$$T(B_{\mathfrak{X}}) = T^{**}(B_{\mathfrak{X}}) \subset T^{**}(B_{\mathfrak{X}^{**}})$$

is rel. weakly compact in $\mathcal{Y}^{**}$, so $T(B_{\mathcal{X}})$ is rel. weakly compact in $\mathcal{Y}$

$\blacksquare$

THEOREM (Davis-Figiel-Johnson-Pelczynski) $T \in B(\mathcal{X}, \mathcal{Y})$ is weakly compact iff $\exists$ reflexive R and continuous operators U and V s.t.

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "T"] \arrow[dr, "U"'] & & \mathcal{Y} \\ & R \arrow[ur, "V"'] & \end{tikzcd}$$

THEOREM (Schauder's Theorem) $T \in B(\mathcal{X}, \mathcal{Y})$ is compact iff T^* is compact.

Proof ($\Rightarrow$) Let (y_n^*) be a seq. in $B_{\mathcal{Y}^*}$. Notice

$$|y_n^*(z) - y_n^*(y)| \le \|z - y\| \quad \forall z, y \in \mathcal{Y}$$

Hence the sequence (y_n^*) of functions is equi-continuous on bounded subsets of $\mathcal{Y}$. The set (y_n^*) is equicontinuous on the compact set $\overline{T(B_{\mathcal{X}})}$ By Arzela-Ascoli the sequence (y_n^*) has a subsequence $(y_{n_j}^*)$ which is uniformly Cauchy on $\overline{T(B_{\mathcal{X}})}$, i.e.

$$\lim_{i,j} |(y_{n_j}^* - y_{n_i}^*)Tx| = 0$$

unif. in $\|x\| \leq 1$, i.e.

$$\lim_{i,j} \left| \left(T^*(y^*_{n_j}) - T^*(y^*_{n_i}) \right) x \right| = 0$$

unif. in $\|x\| \leq 1$, so

$$\lim_{i,j} \| T^*(y^*_{n_j}) - T^*(y^*_{n_i}) \| = 0$$

Therefore T^* maps bounded sequences into sequences with convergent subsequences, i.e. T^* is compact.

($\Leftarrow$) T^* compact $\Rightarrow$ T^{**} compact $\Rightarrow$ T compact.

$\boxtimes$

<u>THEOREM</u>: An operator $T \in B(\mathcal{X}, \mathcal{Y})$ is compact if $\exists$ a sequence (T_n) of finite rank operators in $B(\mathcal{X}, \mathcal{Y})$ s.t.

$$\lim \| T_n - T \| = 0$$

in uniform operator topology.

Proof. Want to show $T(B_{\mathcal{X}})$ is totally bounded. Let $\varepsilon > 0$. Choose n_0 s.t. $\| T - T_{n_0} \| \leq \varepsilon/2$. This means

$$\| Tx - T_{n_0} x \| \leq \varepsilon/2$$

for all $x \in B_{\mathfrak{X}}$. Since $T_{n_0}(B_{\mathfrak{X}})$ is a bounded subset of a finite dimensional space, it can be covered by a finite number of $\varepsilon/2$-balls. Hence the set $T(B_{\mathfrak{X}})$ can be covered by finitely many ε-balls.

$\square$

COROLLARY (of proof) The space $K(\mathfrak{X}, \mathcal{Y})$ of compact members of $B(\mathfrak{X}, \mathcal{Y})$ is a closed subspace of $B(\mathfrak{X}, \mathcal{Y})$

COROLLARY: If $\mathcal{Y}$ has a basis, then every member of $K(\mathfrak{X}, \mathcal{Y})$ is the operator top. limit of finite rank operators.

12/5 BANACH SPACES

THEOREM: If $\mathcal{Y}$ has a basis, then the finite rank operators are dense in $K(\mathcal{X},\mathcal{Y})$

Proof. Let (y_n) be a basis for $\mathcal{Y}$, i.e.

$$y = \sum_{n=1}^{\infty} \underbrace{y_n^*(y)}_{\text{coefficients}} y_n$$

For each m s.t.

$$P_m(y) = \sum_{n=1}^{m} y_n^*(y)\, y_n$$

We know $\sup_m \|P_m\| < \infty$ (Uniform boundedness principal). Hence (P_m) is equicontinuous on $\mathcal{Y}$ that tends pointwise to the identity.
Let $\mathcal{X}$ be arbitrary B-space and $T: \mathcal{X} \to \mathcal{Y}$ compact. Then $T(B_{\mathcal{X}})$ is relatively compact. Thus

$$P_n(Tx) \longrightarrow Tx \quad \text{unif. in } \|x\| \le 1$$

$$\Rightarrow \lim_n \| \underbrace{P_n T}_{\text{finite rank}} - T \|_{B(\mathcal{X},\mathcal{Y})} = 0$$

$\boxtimes$

<u>Approximation Problem</u>: For any B-spaces $\mathfrak{X}, \mathcal{Y}$ are the finite rank operators dense in $K(\mathfrak{X},\mathcal{Y})$?

<u>Basis Problem</u>: Do all separable B-spaces have basis?

Enflo showed answer to both is <u>no</u>

Let $f \in L_1(\mu)$ (finite measure space (Ω, Σ, μ)). Let $\mathcal{B}$ be a sub-σ-field of Σ. Define λ on $\mathcal{B}$ by

$$\lambda(E) = \int_E f \, d\mu$$

for $E \in \mathcal{B}$. Then λ is $\mu|_{\mathcal{B}}$-continuous. By Radon-Nikodym, $\exists$ $\mathcal{B}$-measurable g s.t.

$$\lambda(E) = \int_E g \, d\mu$$

for all $E \in \mathcal{B}$, and the integral makes sense for all $E \in \Sigma$, so $g \in L_1(\mu)$. We say g is the conditional expectation of f given $\mathcal{B}$ and write

$$E(f|\mathcal{B}) = g$$

Properties:

① $E(\cdot|\mathcal{B})$ is a projection on $L_1(\mu)$ and is linear
② $E(\cdot|\mathcal{B})$ preserves order
③ $E(\cdot|\mathcal{B})$ is a contraction on $L_1(\mu)$ that also

has property that $\|E(f|\mathcal{B})\|_\infty \le \|f\|_\infty$

Let $E(f|\mathcal{B}) = g$

$$\|g\|_1 = \Big|\int_{(\cdot)} g \, d\mu\Big|(\Omega) = \sup_{\pi_{\mathcal{B}}} \sum_{E \in \pi_{\mathcal{B}}} \Big|\int_E g \, d\mu\Big|$$

Variation

partitions from $\mathcal{B}$

$$= \sup_{\pi_{\mathcal{B}}} \sum \Big|\int_E f \, d\mu\Big| \le \sup_{\pi_\Sigma} \sum_{E \in \pi_\Sigma} \Big|\int_E f \, d\mu\Big| = \|f\|_1$$

partitions from Σ

To prove $\|E(f|\mathcal{B})\|_\infty \le \|f\|_\infty$, notice that

$$-\|f\|_\infty \chi_\Omega \le f \le \|f\|_\infty \chi_\Omega$$

$$\Rightarrow E(-\|f\|_\infty \chi_\Omega|\mathcal{B}) \le E(f|\mathcal{B}) \le E(\|f\|_\infty \chi_\Omega)$$

$$\Rightarrow -\|f\|_\infty \chi_\Omega \le E(f|\mathcal{B}) \le \|f\|_\infty \chi_\Omega$$

$$\Rightarrow \|E(f|\mathcal{B})\|_\infty \le \|f\|_\infty$$

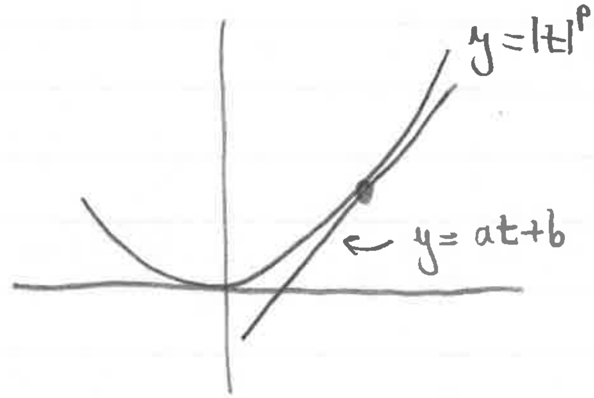
Riesz Convexity theorem $\Rightarrow E(\cdot|\mathcal{B}) : L_p(\mu) \longrightarrow L_p(\mu)$ is a contraction. However, we will prove this via Jensen's Inequality

<u>THEOREM</u> (Jensen's Inequality) If $f \in L_p(\mu)$, then

$$\| E(f|B) \|_p \leq \|f\|_p$$

for $1 \leq p < \infty$

Proof.



Observe $|t|^p = \sup\{at+b : at+b \text{ is a support line for } y=|x|^p\}$
Take such a support line $ax+b$. If $f \in L_p(\mu)$, then

$$a \cdot E(f|B) + b = E(af+b|B)$$

$$\leq E(|f|^p | B)$$

Now take sup on left over all a,b s.t. $ax+b$ supports $|x|^p$ to see

$$|E(f|B)|^p \leq E(|f|^p|B)$$

$$\Rightarrow \|E(f|B)\|_p^p = \int_\Omega |E(f|B)|^p \, d\mu \leq \int_\Omega E(|f|^p|B) \, d\mu$$

$$= \int_\Omega |f|^p \, d\mu = \|f\|_p^p \qquad \boxtimes$$

THEOREM: Let (Ω, Σ, μ) be a finite measure space. Let $\mathfrak{X}$ be any B-space. Then finite rank operators are dense in $K(L_p(\mu), \mathfrak{X})$, $1 \le p < \infty$

Proof. We shall first prove that if $\mathcal{Y}$ is a B-space, then finite rank operators are dense in $K(\mathcal{Y}, L_p(\mu))$ for $1 \le p \le \infty$.
Let $T : \mathcal{Y} \longrightarrow L_p(\mu)$ be compact. For each partition π, define

$$E_\pi(f) = \sum_{A \in \pi} \frac{\int_A f \, d\mu}{\mu(A)} \chi_A \qquad \left[\tfrac{0}{0} = 0\right]$$

Each $E_\pi = E(\cdot \mid \sigma(\pi))$. Hence $\|E_\pi\| \le 1$. Also $E_\pi(f) \to f$ since it holds for all simple functions. Therefore $E_\pi \to I$ uniformly on compact sets, so

$$E_\pi T x \longrightarrow T x \quad \text{unif on } \|x\| \le 1$$

$$\Longrightarrow \underbrace{E_\pi T}_{\text{finite rank}} \longrightarrow T \quad \text{in operator norm}$$

$\square$

12/7 BANACH SPACES

Now suppose $1 \le p < \infty$. We have just shown that if $T : \mathfrak{X} \to L_p(\mu)$ is compact then

$$\lim_{\pi} \| E_\pi T - T \| = 0$$

If $S : L_p(\mu) \longrightarrow \mathfrak{X}$ is compact, then $S^* : \mathfrak{X}^* \longrightarrow L_q(\mu)$ is compact. Hence

$$\lim_{\pi} \| E_\pi S^* - S^* \| = 0$$

Since $E_\pi^* = E_\pi$:

$$\int_\Omega E_\pi(f) g \, d\mu = \int_\Omega \sum_{E \in \pi} \frac{\int_E f \, d\mu}{\mu(E)} \chi_E \, g \, d\mu$$

$$= \sum_{E \in \pi} \frac{\int_E f \, d\mu \int_E g \, d\mu}{\mu(E)} = \int_\Omega f \, E_\pi(g) \, d\mu$$

Accordingly

$$\| S E_\pi - S \| = \| (S E_\pi - S)^* \| = \| E_\pi S^* - S^* \| \to 0$$

$\blacksquare$

Martingales and B-spaces

LEMMA: Let (T_n) be a bounded sequence in $B(\mathfrak{X})$. Suppose

$$\lim T_n x = x$$

for all x in a dense subset of $\mathfrak{X}$. Then $\lim T_n x = x \quad \forall x \in \mathfrak{X}$.

Proof. Take $x \in \mathfrak{X}$ and $\varepsilon > 0$. Observe that

$$\| T_n x - x \| \leq \| T_n x - T_n y \| + \| T_n y - y \| + \| x - y \|$$

$$\leq \beta \| x - y \| + \| T_n y - y \| + \| x - y \|$$

Choose y s.t. $T_n y \to y$ and $\| x - y \| < \varepsilon$. Then $\| T_n x - x \| < \gamma \varepsilon$ for sufficiently large n.

$\blacksquare$

THEOREM: Let (Ω, Σ, μ) be a finite measure space. Let $1 \leq p < \infty$. Let (B_n) be an increasing seq. of sub-σ-fields of Σ. Suppose $\sigma(\cup B_n) = \Sigma$. Then

can be removed

$$\lim \| E(f | B_n) - f \|_p = 0$$

for all $f \in L_p(\mu)$

Proof. $\| E(\cdot | B_n) \| \leq 1$, so $(E(\cdot | B_n))$ is bounded

Also $\lim E(f|B_n) = f$ for all f which is B_{n_0}-measurable for some n_0. Since $\sigma(\bigcup B_n) = \Sigma$ it follows that such f's are dense in $L_p(\mu)$

[[Caratheodory Extension theorem: $E \in \Sigma = \sigma(\mathcal{F}) \Rightarrow \exists (E_n)$ in $\mathcal{F}$ s.t. $\lim \mu(E \Delta E_n) = 0$]]

$\boxtimes$

<u>DEFINITION</u>: Let (Ω, Σ, μ) be a finite measure space. Let (B_n) be an increasing sequence of sub-σ-fields of Σ. A sequence (f_n) in $L_1(\mu)$ denoted by (f_n, B_n) is called a martingale if

① f_n is B_n-measurable

② $\int_E f_m \, d\mu = \int_E f_n \, d\mu \quad \forall E \in B_n \quad \forall m \geq n$

i.e.

$$E(f_m | B_n) = f_n$$

$\forall m \geq n$.

(Mean Martingale Convergence thm)

<u>THEOREM</u>: Let (Ω, Σ, μ) be a finite measure space. If (f_n, B_n) is an $L_1(\mu)$-bounded unif. integrable martingale, then $\lim f_n$ exists in L_1-norm

Proof (#1) Take $E \in \bigcup B_n$ and observe that

$$\lim \int_E f_n \, d\mu =: \lambda(E)$$

exists trivially. Let π be a partition of Ω into $\cup B_n$ sets.

$$\sum_{E\in\pi} |\lambda(E)| = \lim_n \sum_{E\in\pi} \left| \int_E f_n \, d\mu \right|$$

$$\leq \overline{\lim_n} \sum_{E\in\pi} \int_E |f_n| \, d\mu$$

$$= \overline{\lim_n} \int |f_n| \, d\mu = \sup_n \|f_n\|_1 < \infty$$

Hence λ is a finite signed measure. Also uniform integrability

$$\Rightarrow \lim_{\mu(E)\to 0} \int_E |f_n| \, d\mu = 0 \text{ unif in } n$$

$$\Rightarrow \lambda(E) \to 0 \text{ as } \mu(E) \longrightarrow 0, \quad E \in \cup B_n$$

$$\Rightarrow \lambda << \mu \quad \text{w.r.t. } \cup B_n$$

Therefore λ has a μ-cont extension to a measure λ on $\Sigma_1 = \sigma(\cup B_n)$
By Radon-Nikodym theorem $\exists f \in L_1(\Sigma_1, \mu)$ s.t.

$$\lambda(E) = \int_E f \, d\mu \quad \forall E \in \Sigma$$

Therefore

$$\lim_n \int_E f_n \, d\mu = \int_E f \, d\mu$$

$\forall E \in \bigcup B_n$. In particular, the martingale property gives

$$\int_E f_n \, d\mu = \int_E f \, d\mu \quad \forall E \in B_n$$

$$\Rightarrow \quad f_n = E(f \mid B_n)$$

By last theorem,

$$\lim_n \| E(f \mid B_n) - f \|_{L_1(\Sigma_1, \mu)} = 0$$

$$\Rightarrow \lim_n \| f_n - f \|_{L_1(\Sigma_1)} = 0$$

$$\Rightarrow \lim_n \| f_n - f \|_{L_1(\Sigma)} = 0$$

$\blacksquare$

Proof (#2) (f_n) is in a weakly compact subset of $L_1(\mu)$. Therefore it has a subsequence (f_{n_j}) that converges weakly to some $f \in L_1(\mu)$. Integration over $E \in \bigcup B_n$ is a cont. linear functional on $L_1(\mu)$. Therefore

$$\lim_j \int_E f_{n_j} \, d\mu = \int_E f \, d\mu \quad \forall E \in \bigcup B_n$$

But for such E, we know $\lim_n \int f_n \, d\mu$ exists, and so

$$\lim_n \int_E f_n \, d\mu = \int_E f \, d\mu$$

Now continue proof as before.

$\boxed{}$

<u>**Corollary**</u>: If a martingale in $L_1(\mu)$ is bounded in some $L_p(\mu)$ for some p with $1 < p < \infty$, then it converges in $L_p(\mu)$ as well as in $L_1(\mu)$

Proof. Hölder's inequality $\Rightarrow$ bdd sets in $L_p(\mu)$ are uniformly integrable and bounded in $L_1(\mu)$. Hence $\lim f_n = f$ exists in $L_1(\mu)$ norm. Pass to a subseq (f_{n_j}) s.t. $\lim f_{n_j} = f$ a.e. Use Fatou's lemma to get

$$\int_\Omega |f|^p \, d\mu \le \varliminf_n \int |f_n|^p \, d\mu < \infty$$

Hence $f \in L_p(\mu)$. Just as before $E(f|B_n) = f_n$. Apply the first theorem to see

$$\lim_n E(f|B_n) = \lim f_n = f$$

in L_p

12/10 BANACH SPACES

Let (f_n, B_n) be a martingale in $L_p(\mu)$ $1<p<\infty$. Suppose

$$\int_\Omega f_1 \, d\mu = 0$$

i.e. $\int_\Omega f_n \, d\mu = 0$. WLOG suppose $f_0 = 0$. Set

$$d_k := f_k - f_{k-1}$$

Consider $\sum_{k=1}^n \alpha_k d_k$ where the α_k's are scalars. Observe $\left(\sum_{k=1}^n \alpha_k d_k, B_n\right)$ is a martingale, since for $m>n$

$$
\begin{aligned}
E\left(\sum_{k=1}^m \alpha_k d_k \mid B_n\right) &= E\left(\sum_{k=1}^n \alpha_k d_k \mid B_n\right) + E\left(\sum_{k=n+1}^m \alpha_k d_k \mid B_n\right) \\
&= E\left(\sum_{k=1}^n \alpha_k d_k \mid B_n\right) + \sum_{k=n+1}^m \alpha_k \cancelto{0}{E(d_k \mid B_n)} \\
&= E\left(\sum_{k=1}^n \alpha_k d_k \mid B_n\right) \\
&= \sum_{k=1}^n \alpha_k d_k
\end{aligned}
$$

Hence

$$\left\| \sum_{k=1}^n \alpha_k d_k \right\|_p = \left\| E\left(\sum_{k=1}^m \alpha_k d_k\right) \mid B_n \right\|_p \le \left\| \sum_{k=1}^m \alpha_k d_k \right\|_p$$

and so (d_k) is a monotone basis of its span. Also, since $L_p(\mu)$ $(1<p<\infty)$ martingales converge whenever they are bounded, we see that

(d_k) is a boundedly complete basis of its span.

(Definition: A basis (x_n) of $\mathfrak{X}$ is called boundedly complete if

$$\sup_m \left\| \sum_{n=1}^m \alpha_n x_n \right\|_{\mathfrak{X}} < \infty \Rightarrow \sum \alpha_n x_n \text{ converges} \quad)$$

Examples: ① c_0 unit vector basis not boundedly complete

② ℓ_p unit vector basis $(1<p<\infty)$ is boundedly complete

③ Let (d_k) be a difference sequence of a martingale in $L_p(\mu)$, $1<p<\infty$. Then

$$\sup_n \left\| \sum_{k=1}^n \alpha_k d_k \right\|_p < \infty \Rightarrow \left(\sum_{k=1}^n \alpha_k d_k \right) \text{ is an } L_p\text{-bdd martingale}$$

$$\Rightarrow \sum_{k=1}^n \alpha_k d_k \text{ convergent}$$

④ The sequence of Haar functions is a martingale difference sequence, and therefore the Haar basis is a boundedly complete basis of its closed span in $L_p(\mu)$ $(1<p<\infty)$. Since its span is dense, we see that the Haar system is a boundedly complete basis of all the L_p-spaces.

The truth is that any (non-trivial) martingale difference sequence in $L_p(\mu)$ $(1<p<\infty)$ is also an unconditional basis of its span.

<u>**THEOREM**</u>: A non-weakly compact operator $T : c_0 \longrightarrow \mathfrak{X}$ fixes a copy of c_0; i.e. $\exists\, Y \subset c_0$ s.t. Y is isomorphic to c_0 and $T|Y$ is an isomorphism.

(Actually any non-weakly compact $T : C(K) \longrightarrow \mathfrak{X}$ fixes a copy of c_0. The proof is essentially the same)

History: ① $\mathfrak{X}$ weakly complete $\Rightarrow$ all $T \in B(C(K), \mathfrak{X})$ weakly compact
Pettis 1940

② $c_0 \not\hookrightarrow \mathfrak{X} \Rightarrow$ all $T \in B(C(K), \mathfrak{X})$ weakly compact
Pelcynski 1960

③ non-weakly compact $T : C(K) \longrightarrow \mathfrak{X}$ fixes copy of c_0
Rosenthal 1970

Proof. Take $T : c_0 \longrightarrow \mathfrak{X}$ non-weakly compact. Then $T^* : \mathfrak{X}^* \longrightarrow \ell_1$ is not weakly compact. Hence

$$\{T^*(x^*) : \|x^*\| \le 1\}$$

is not uniformly integrable. Therefore $\exists\, \delta$ $\exists$ disjoint seq (A_n) of finite subsets of $\mathbb{N}$ and a seq. (x_n^*) in the unit ball of $\mathfrak{X}^*$ s.t.

$$|T^*(x_n^*)\chi_{A_n}| \ge \delta$$

for all n, i.e. $|x_n^*(T\chi_{A_n})| \ge \delta$. Apply Rosenthal's lemma. We can assume WLOG that

$$\left| x_n^* T\left(\chi_{\bigcup_{\substack{j\in\Delta \\ j\neq n}} A_j} \right) \right| < \delta/2$$

for all finite subsets Δ of $\mathbb{N}$.

Notice (χ_{A_j}) is a basis for a subspace of c_0 which is isomorphic to c_0

$$\left(\| \sum \alpha_j \chi_{A_j} \|_{c_0} = \sup_j |\alpha_j| \right)$$

Claim: T is an isomorphism on this subspace:

① $\| T(\sum \alpha_j \chi_{A_j}) \| \leq \|T\| \, \| \sum \alpha_j \chi_{A_j} \|_{c_0}$

② $\| T(\sum \alpha_j \chi_{A_j}) \| \geq | x_n^* T(\sum \alpha_j \chi_{A_j}) |$

$$\geq | x_n^* T(\alpha_n \chi_{A_n}) | - | x_n^* T(\sum_{j\neq n} \alpha_j \chi_{A_j}) |$$

$$\geq |\alpha_n| \, | x_n^* T(\chi_{A_n}) | - \| \sum \alpha_j \chi_{A_j} \|_{c_0} \delta/2$$

$$\geq |\alpha_n| \delta - \| \sum \alpha_j \chi_{A_j} \|_{c_0} \delta/2$$

$$\Rightarrow \| T(\sum \alpha_j \chi_{A_j}) \| \geq \| \sum \alpha_j \chi_{A_j} \|_{c_0} \delta - \| \sum \alpha_j \chi_{A_j} \|_{c_0} \delta/2$$

$$= \delta/2 \, \| \sum \alpha_j \chi_{A_j} \|_{c_0}$$

and so T^{-1} is bounded.

12/12 BANACH SPACES

Examples of compact operators

① Let $K(s,t)$ be cont. on $[0,1]\times[0,1]$. Define $T: C[0,1] \to C[0,1]$ by

$$Tf(x) = \int_0^1 K(x,y) f(y)\, dy$$

Then.

$$|Tf(s) - Tf(t)| \le \int_0^1 |K(s,y) - K(t,y)|\,|f(y)|\, dy$$

Let $\varepsilon > 0$. Since K is uniformly continuous $\exists\, \delta$ s.t. $|s-t| < \delta$ implies

$$|K(s,y) - K(t,y)| < \varepsilon \quad \forall y$$

Thus

$$|s-t| < \delta,\ \|f\| \le 1 \implies |Tf(s) - Tf(t)| < \int_0^1 \varepsilon \|f\|\, dy \le \varepsilon$$

and so $T(\overline{B_{C[0,1]}})$ is equicontinuous in $C[0,1]$. Therefore T is compact by Arzela-Ascoli Theorem.

② Let $K(x,y)$ be measurable on $[0,1]\times[0,1]$. Let $1 < p, q < \infty$ (Maybe $1/p + 1/q \ne 1$). Let $r = \min\{p, q/q-1\}$. Define

$$T : L_p \longrightarrow L_q$$

$$Tf(x) = \int_0^1 K(x,y)\, f(y)\, dy$$

(Hille - Tamarkin operator)

Suppose $\int_0^1\int_0^1 |K(x,y)|^{\frac{r}{r-1}}\, dx\,dy < \infty$. Then T is compact

Proof. Take a seq. (K_n) of functions of the form $\sum_{i=1}^{k} \alpha_i \chi_{A_i \times B_i}$ s.t.

Borel sets

$$\int_0^1\int_0^1 |K_n - K|^{\frac{r}{r-1}}\, dx\, dy \to 0$$

Define $T_n : L_p \longrightarrow L_q$ by

$$T_n f(x) = \int_0^1 K_n(x,y)\, f(y)\, dy$$

Notice $T_n f$ is simple over the partition associated with the B_i's for K_n.

Claim: $\|T_n - T\|$.

Since each T_n is finite rank, this will prove T is compact. Take $f \in L_p$ and compute

$$\|T_n f - Tf\|_q^q = \int_0^1 |T_n f - Tf|^q\, dx$$

$$= \int_0^1 \left| \int_0^1 (K_n(x,y) - K(x,y)) f(y)\, dy \right|^q dx$$

$$\leq \int_0^1 \left| \left(\int_0^1 |K_n(x,y) - K(x,y)|^{\frac{r}{r-1}} dy \right)^{q(r-1)/r} \left(\int_0^1 |f|^r dy \right)^{q/r} \right| dx$$

$$\leq \int_0^1 \left| \left(\int_0^1 |K_n(x,y) - K(x,y)|^{\frac{r}{r-1}} dy \right)^{\frac{q(r-1)}{r}} \left(\int_0^1 |f|^p dy \right)^{q/p} \right| dx$$

$\|f\|_r \leq \|f\|_p$
$r \leq p$

$$= \left(\int_0^1 \left| \int_0^1 |K_n(x,y) - K(x,y)|^{\frac{r}{r-1}} dy \right|^{\frac{q(r-1)}{r}} dx \right) \|f\|_p^q$$

Note: This exponent is < 1

$$\int_0^1 f^\alpha d\mu \leq \left(\int f d\mu \right)^\alpha$$

$f \geq 0 \quad 0 \leq \alpha < 1$

$$\leq \left[\int_0^1 \int_0^1 |K_n(x,y) - K(x,y)|^{\frac{r}{r-1}} dy\, dx \right]^{\frac{q(r-1)}{r}} \|f\|_p^q$$

$$= \|K_n - K\|_{\frac{r}{r-1}}^q \|f\|_p^q \longrightarrow 0 \text{ unif in } \|f\|_p \leq 1$$

Hence $\|T_n f - Tf\|_q \to 0$ unif in $\|f\|_p \leq 1$. This proves $\|T_n - T\| \to 0$

$\boxtimes$

Theorems about compact operators

THEOREM: $T \in B(\mathcal{X})$ compact $\Rightarrow$ T has a countable number of eigenvalues with no cluster point except possibly at 0. The dimension of each eigenspace (corresponding to non-zero eigenvalue) of T is finite

Proof. Second statement is obvious since non-zero multiples of identity operator are not compact in infinite dimensional spaces

1st part: Let (λ_α) be the collection of distinct eigenvalues of T. Then if $Tx_\alpha = \lambda_\alpha x_\alpha$ and $x_\alpha \neq 0$, then (x_α) is linearly independent

Given $\varepsilon > 0$, then any seq. (λ_k) of distinct eigenvalues of T s.t. $|\lambda_k| \geq \varepsilon$ is a finite seq. Suppose $\exists$ ε and such a seq (λ_k) with (λ_k) infinite. Select $\|x_k\| = 1$ s.t. $Tx_k = \lambda_k x_k$. Set

$$M_n = \mathrm{sp}\{x_1, \ldots, x_n\}$$

Then $M_n \uparrow$, so $\forall n$ $\exists$ $u_n \in M_n$ s.t. $\|u_n\| = 1$ and $\|u_n - x\| \geq 1/2$ for all $x \in M_{n-1}$ (Riesz's lemma). Notice that $Tu_n \in M_n$ since each M_n is invariant under T. Also

$$\lambda_n I x - Tx \in M_{n-1}$$

for all $x \in M_n$. Then if $x \in M_n$

$$z := (\lambda_n I - T)x + Tu_m \in M_{n-1}$$

provided $1 \le m < n$. Finally

$$1 \le m < n \Rightarrow \|Tu_n - Tu_m\| = \|\lambda_n u_n - \lambda_n u_n - Tu_n + Tu_m\|$$

$$= \|\lambda_n (u_n - z)\| = |\lambda_n| \, \|u_n - \frac{z}{\lambda_n}\|$$

$$\ge \lambda_n \cdot 1/2 \ge \varepsilon/2$$

Hence (Tu_n) has no convergent subsequence. Hence T is not compact

$\blacksquare$

END OF COURSE

Mathematics 447 (Uhl)

Fall 1979 Final Exam

1. Spaces and duals:

(a) Complete the table:

$\mathfrak{X}$	$\mathfrak{X}^*$	$x^*(x)$
$L_1(\mu)$	$L_\infty(\mu)$	$\int fg\,d\mu$
$L_p(\mu)$ $1<p<\infty$		
$C[0,1]$		
$L_0[0,1]$		
$l_p(\mathfrak{X})$		
$(\mathfrak{X} \times R)_{l_2}$		
c_0		
c		

(b) Give an example of a separable space whose dual is not separable.

(c) Prove that if $\mathfrak{X}$ is separable and reflexive, then $\mathfrak{X}^*$ is separable.

2. Convex sets.

(a) Prove that the weak closure of a convex set is the same as the norm closure.

(b) Let C be a closed bounded convex subset of a reflexive B-space. Say why $C = \text{norm-}\overline{co}(\text{ext}\, C)$.

3. Exposed points

Prove that every point on the surface of the unit ball of a strictly convex B-space is exposed.

4 $C[0,1]$

(a) Give necessary and sufficient conditions for a sequence (f_n) to be weakly Cauchy in $C[0,1]$.

(b) Let (f_n) be weakly Cauchy in $C[0,1]$. Prove that $\lim_n f_n$ exists in $L_1[0,1]$ norm.

(c) Is $C[0,1]$ weakly sequentially complete?

5. Operators

(a) Show that every operator in $B(c_0, L_2[0,1])$ is compact.

(b) Give an example of a compact operator.

(c) Suppose $\mathcal{X}$ is a reflexive B-space and $T \in B(\mathcal{X}, \mathcal{Y})$ is onto. Prove that $\mathcal{Y}$ is also reflexive.

(d) Let $T \in B(\mathcal{X}, \mathcal{Y})$ be compact. Show that T^* maps weak*-convergent sequences into norm convergent sequences.

6. c_0 and L_1

(a) Prove that neither c_0 nor L_1 are reflexive.

(b) Prove that neither c_0 nor L_1 are dual spaces.

(c) Is every subspace of a dual space a dual space in its own right?

6 (continued)

(d) Prove that no weakly sequentially complete B-space has a subspace isomorphic to c_0.

7. Series

Let $\sum x_n$ be a series in a B-space such that each of its subseries is convergent. Show that if $\sum y_j$ is any rearrangement of $\sum x_n$, then $\sum y_j$ converges in norm to $\sum x_n$. Hint: Orlicz-Pettis makes this easy.

8. Let A be a subset of a B-space $\mathfrak{X}$ such that A is weak*-compact inside $\mathfrak{X}^{**}$. Prove that A is weakly compact in $\mathfrak{X}$.