Martingales

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Ω	arbitrary set of points ω
$\mathcal{A}$	σ-field of subsets of Ω
P	probability measure

NOTATION $\quad E f := \int_{\Omega} f \, dP$

Typical sample spaces and σ-fields

$[0,1)$

$\{0,1\} \times \{0,1\} \times \{0,1\}$ (tossing coin 3 times)

all continuous functions $\omega : [0,\infty) \to \mathbb{R}^n$ with $\omega(0) = 0$

($\mathcal{A}$ is the smallest σ-field containing all sets of the form $\{\omega : \omega(t) \in \underset{\uparrow \text{open subset of } \mathbb{R}^n}{\mathcal{O}}\}$

P - Wiener measure)

Take partition $\{B_1, \dots, B_n\}$ of Ω and let $\mathcal{B}$ be all finite unions of partition sets

If $B \subset [0,1)$, we say B is periodic with period $1/n$ if

$$x \in B \Leftrightarrow \underbrace{x + 1/n}_{\text{addition mod } 1} \in B$$

Now take $\mathcal{B} = \{B \in \mathcal{A} : B \text{ is periodic with period } 1/n\}$
(sub-σ-field of $\mathcal{A}$)

<u>DEFINITION</u>: $\mathcal{B}$ any sub-σ-field of $\mathcal{A}$. Suppose f is integrable or non-negative $\mathcal{A}$-measurable and g is a $\mathcal{B}$-measurable function such that

$$(*) \qquad \int_B f\,dP = \int_B g\,dP$$

for all $B \in \mathcal{B}$ (so averages of f and g over B are the same). Then g is called "the" conditional expectation of f given $\mathcal{B}$

THEOREM: (i) g exists

(ii) g is unique in the following sense: if h also satisfies $(*)$ for all $B \in \mathcal{B}$, then $g = h$ a.e. (w.r.t. P and $\mathcal{B}$)

(iii) f integrable $\Rightarrow$ g integrable and $\|g\|_1 \leq \|f\|_1$

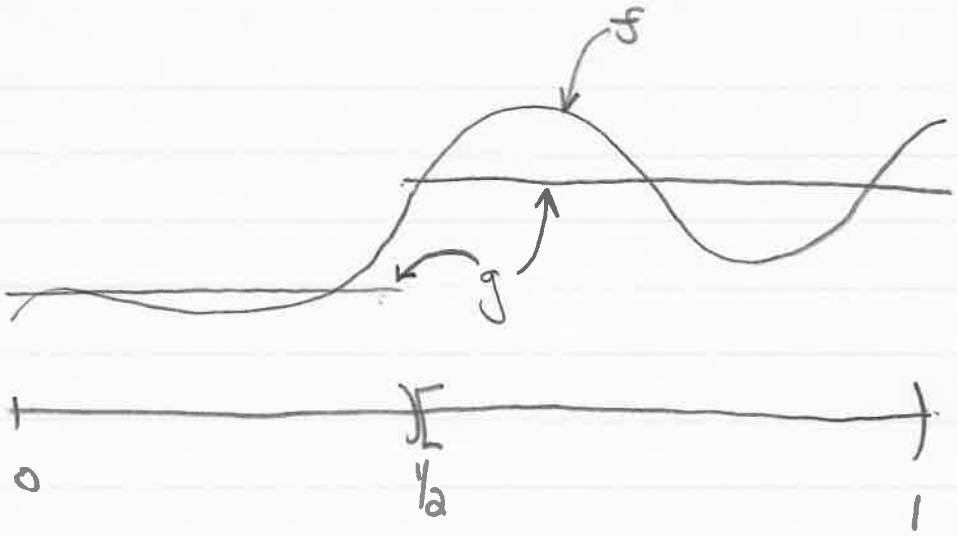
(iv) $f \geq 0$ a.e. $\Rightarrow$ $g \geq 0$ a.e.

EXAMPLES -

1. $\mathcal{B} = \mathcal{A}$, take $g = f$
2. $\mathcal{B} = \{\emptyset, \Omega\}$, take $g := Ef$ (constant)
3. $\mathcal{B}$ generated by a partition $\{B_1, \dots, B_n\}$, take

$$g(\omega) := \frac{\int_{B_k} f}{P(B_k)} \quad \text{if } \omega \in B_k \qquad (= 0 \text{ if } P(B_k) = 0)$$

(so $g = \sum_{k=1}^{n} b_k I_{B_k}$ where b_k defined as above)



The conditional expectation is a "smoother" function than f

Proof of (iii):

$$\|g\|_1 = \int_\Omega |g| = \int_{\underset{\uparrow}{\{g\geq 0\}}} g - \int_{\underset{\uparrow}{\{g<0\}}} g = \int_{\{g\geq 0\}} f - \int_{\{g<0\}} f$$

$\mathcal{B}$- measurable

$$\leq \int_{\{g\geq 0\}\cup\{g<0\}} |f| \leq \int_\Omega |f| = \|f\|_1$$

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Properties of conditional expectation:

Monotonicity $f_1 \leq f_2$ a.e. $\Rightarrow g_1 \leq g_2$ a.e.

$$\begin{tikzcd}
 & \text{MONOTONICITY} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
(f \geq 0 \Rightarrow g \geq 0) & & \text{essential uniqueness}
\end{tikzcd}$$

Proof: Suppose $\{\omega : g_2(\omega) < g_1(\omega)\} =: \{g_2 < g_1\}$ has positive measure. Then, since

$$\{g_2 < g_1\} = \bigcup_{r \in \mathbb{Q}} \{g_2 < r < g_1\}$$

There is at least one $r \in \mathbb{Q}$ s.t. $B := \{g_2 < r < g_1\}$ has positive measure. Now

$$\int_B f_2 - rP(B) = \int_B g_2 - rP(B) = \int_B (g_2 - r) \underset{\substack{\uparrow \\ \text{since } P(B) \neq 0}}{<} 0$$

$$< \int_B (g_1 - r) = \int_B f_1 - rB(B)$$

and so $\int_B f_2 < \int_B f_1$ ↯.

▨

Existence of conditional expectation
(Assume f square integrable)

$$Ef = \int_\Omega f\, dP$$ is the number c which minimizes $E[(f-c)^2] = \|f-c\|_2^2$

(works since probability of space is 1)

$$E[(f-c)^2] = E[(f-Ef)^2] + (Ef-c)^2 \geq E(f-Ef)^2$$

ex. $\mathcal{B}$ generated by $\{B_1, \ldots, B_n\}$

$$g = \sum b_k I_{B_k} \qquad b_k = \frac{\int_{B_k} f}{P(B_k)}$$

Then if $h = \sum c_k I_{B_k}$

$$E(f-h)^2 = \sum_{k=1}^{n} \left[\frac{\int_{B_k} (f-c_k)^2}{P(B_k)}\right] P(B_k) \geq \sum_{k=1}^{n} \left[\frac{\int_{B_k} (f-b_k)^2}{P(B_k)}\right] P(B_k)$$

$$\uparrow$$

from above, working only on B_k

$$= E(f-g)^2$$

Claim: $\exists$ a $\mathcal{B}$-measurable g s.t.

$$\|f-g\|_2 \leq \|f-h\|_2$$

for all $\mathcal{B}$-measurable h.

Proof. Let $\delta^2 = \inf_h \|f-h\|_2^2$ (h $\mathcal{B}$-meas., sq. integrable)

Then $\exists\, g_n$ s.t.

$$\delta^2 \le \|f - g_n\|_2^2 < \delta^2 + 1/4^n \quad \forall n \in \mathbb{N}$$

Now

$$\|g_{n+1} - g_n\|_2^2 = 2\|f - g_{n+1}\|_2^2 + 2\|f - g_n\|_2^2 - 4\|f - \frac{(g_n + g_{n+1})}{2}\|_2^2$$

(*) $$< 2\left(\delta^2 + \frac{1}{4^{n+1}}\right) + 2\left(\delta^2 + \frac{1}{4^n}\right) - 4\delta^2$$

$$= \frac{2}{4^{n+1}} + \frac{2}{4^n} < \frac{1}{4^{n-1}}$$

Let $g(w) = \lim g_n(w)$ if limit exists
$\qquad\;\; = 0$ otherwise

Then g is $\mathcal{B}$-measurable. Now

$$g_n = \underbrace{\sum_{k=1}^{n-1} (g_{k+1} - g_k)}_{\text{converges by } (*)} + g_1$$

Hence $\lim g_n(w)$ exists a.e. and

$$\delta^2 \le \|f - g\|_2^2 = \int (f-g)^2 \le \liminf \int (f - g_n)^2 \le \delta^2$$

We want to show

$$(f-g, h) = \int (f-g)h = 0$$

for any square integrable $\mathcal{B}$-measurable h. But for any $t \in \mathbb{R}$

$$\delta^2 \leq \| f - (g+th) \|_2^2 = \| (f-g) - th \|_2^2$$

$$= \underset{\delta^2}{\underset{\shortparallel}{\| f-g \|_2^2}} - 2t(f-g, h) + t^2 \| h \|_2^2$$

$$\Rightarrow \quad 2t(f-g,h) \leq t^2 \| h \|_2^2$$

$$\Rightarrow \begin{cases} (1) \quad 2(f-g,h) \leq t \| h \|_2^2 \quad \forall t > 0 \\ \qquad \Rightarrow \quad (f-g,h) \leqq 0 \\ (2) \quad 2(f-g,h) \geq t \| h \|_2^2 \quad \forall t < 0 \end{cases}$$

$$\Rightarrow \quad (f-g,h) \geq 0$$

Hence $(f-g,h) = 0$. Now set $h = I_B$, so

$$\int_B f = \int_B g$$

for all B

(Assume now that $f \geq 0$)

Let g_n be the conditional expectation of $\underbrace{f \wedge n}_{\text{bounded, sq. integrable}} = \min\{f, n\}$

Then $g_n \leq g_{n+1}$ a.e. Let $g = \lim g_n$. Then $\forall B \in \mathcal{B}$

$$\begin{tikzcd}
\int_B f \wedge n \arrow[d, "\text{MCT}"] \arrow[r, phantom, "="] & \int_B g_n \arrow[d] \quad \forall n \\
\int_B f \arrow[r, phantom, "="] & \int_B g
\end{tikzcd}$$

For integrable f, use positive and negative parts

NOTATION: Any such g satisfying conditions for conditional expectation of f w.r.t. $\mathcal{B}$ will be denoted by

$$E(f \mid \mathcal{B})$$

PROPERTIES OF CONDITIONAL EXPECTATION

(1) $\mathfrak{f} \leq \mathfrak{f}_2$ a.e. $\Longrightarrow E(\mathfrak{f}_1 | \mathcal{B}) \leq E(\mathfrak{f}_2 | \mathcal{B})$ a.e. $\forall \mathcal{B}$

(2) $\| E(\mathfrak{f} | \mathcal{B}) \|_1 \leq \| \mathfrak{f} \|_1$

(3) $E(\mathfrak{f}_1 + \mathfrak{f}_2 | \mathcal{B}) = E(\mathfrak{f}_1 | \mathcal{B}) + E(\mathfrak{f}_2 | \mathcal{B})$

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$\mathcal{B}$ - periodic sets of period 1/n on $[0,1)$

$$E(f|\mathcal{B}) = g$$

where

$$g(x) := \frac{1}{n} \sum_{k=1}^{n} f(x+\frac{k}{n})$$

PROPERTIES OF CONDITIONAL EXPECTATION (continued)

(4) Let $f \geq 0$ be $\mathcal{A}$-measurable and $h \geq 0$ $\mathcal{B}$-measurable Then

$$E(hf|\mathcal{B}) = h\, E(f|\mathcal{B}) \quad \text{a.e.}$$

If f is integrable and h $\mathcal{B}$-measurable so that fh is integrable, then same conclusion

Proof. Let $g := E(f|\mathcal{B})$. Then hg is $\mathcal{B}$-measurable and non-negative a.e.

To Show:

$$\int_B hf = \int_B gh \quad \forall B \in \mathcal{B}$$

Holds for $h = I_C$, $C \in \mathcal{B}$ since $g = E(f|\mathcal{B})$

$$\int_{B\cap C} f = \int_{B \cap C} g \qquad \forall B \in \mathcal{B}$$

Hence holds for all simple functions h. If $h \geq 0$ is $\mathcal{B}$-measurable, choose simple functions $h_n \uparrow h$ and use monotone convergence theorem ▨

(5) $E[E(f|\mathcal{B})] = Ef$

(6) $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$. Then
(sub-σ-fields)

$$E[E(f|\mathcal{B}) \,|\, \mathcal{C}] = E(f|\mathcal{C})$$

$$E[E(f|\mathcal{C}) \,|\, \mathcal{B}] = E(f|\mathcal{C})$$

(7) Monotone Convergence for conditional expectation. Suppose $0 \leq f_n \uparrow f$. Then $E(f_n|\mathcal{B}) \uparrow E(f|\mathcal{B})$

(Follows from monotone convergence for integrals)

(8) Fatou. Suppose $0 \leq f_n$. Then

$$E(\liminf_{n\to\infty} f_n \,|\, \mathcal{B}) \leq \liminf_{n\to\infty} E(f_n|\mathcal{B})$$

(9) Dominated Convergence. Suppose $f^* := \sup_n |f_n|$ is such that $Ef^* < \infty$. If $f_n \to f$ a.e., then

$$E(f_n|\mathcal{B}) \longrightarrow E(f|\mathcal{B}) \text{ a.e.}$$

Proof.

$$|E(f_n|\mathcal{B}) - E(f|\mathcal{B})| = |E(f_n - f|\mathcal{B})| \le E(\underbrace{|f_n - f|}_{F_n}\,|\mathcal{B})$$

Note $F_n \to 0$ a.e. and $|F_n| \le 2f^*$ integrable. Let $F^* = \sup_n F_n$

$$E[\liminf (F^* - F_n)\,|\mathcal{B}] \le \liminf E[F^* - F_n|\mathcal{B}]$$

$$= \liminf \big(E(F^*|\mathcal{B}) - E(F_n|\mathcal{B})\big)$$

$$= E(F^*|\mathcal{B}) - \limsup E(F_n|\mathcal{B})$$

But $E(\liminf (F^* - F_n)) = E(f^*|\mathcal{B})$, and so we have

$$0 \le \limsup E(F_n|\mathcal{B}) \le 0$$

$$\Longrightarrow E(F_n|\mathcal{B}) \to 0$$

▨

(10) Jensen's Inequality. Suppose φ is convex on some interval. Let f be integrable. Then

$$\varphi(E(f|\mathcal{B})) \leq E(\varphi(f)|\mathcal{B}) \quad \text{a.e.}$$

Example - $\varphi(x) = |x|^p \quad 1 \leq p < \infty$. Then

$$|E(f|\mathcal{B})|^p \leq E(|f|^p | \mathcal{B}) \quad \text{a.e.}$$

Taking expectation of both sides

$$\|E(f|\mathcal{B})\|_p \leq \underset{\substack{\uparrow \\ \text{property 5}}}{\|f\|_p}$$

(also holds for $p=\infty$)

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JENSEN'S INEQUALITY: φ convex on S (convex subset of $\mathbb{R}$)
f integrable. Then
$f(\Omega) \subset S$

$$\varphi\left(E(f|\mathcal{B})\right) \leq E\left(\varphi(f)|\mathcal{B}\right) \text{ a.e.}$$

[[**LEMMA**: There is a sequence $\psi_1, \ldots, \psi_n, \ldots$ of affine functions

$$\psi_n(x) = a_n x + b_n$$

such that $\varphi(x) = \sup_n \psi_n(x)$ $\forall x \in \operatorname{int} S$ and $\varphi(x) \geq \sup_n \psi_n(x)$ $\forall x \in \partial S$]]

Proof. On the set $\{E(f|\mathcal{B}) \in \operatorname{int} S\}$, then

$$\varphi\left(E(f|\mathcal{B})\right) = \sup_n \psi_n\left(E(f|\mathcal{B})\right) = \sup_n E\left(\psi_n(f)|\mathcal{B}\right) \leq E\left(\varphi(f)|\mathcal{B}\right)$$

Consider $B := \{E(f|\mathcal{B}) = a\}$, where a is left-hand boundary point of S
Then $B \subset \{f = a\}$ a.e. since

$$0 = \int_B \left(E(f|\mathcal{B}) - a\right) = \int_B (f - a)$$

and $f - a \geq 0$. Now

$I_B\varphi(a)$ already $\mathcal{B}$-measurable

$$\text{a.e.} \quad I_B\,\varphi\left(E(f|\mathcal{B})\right) = I_B\varphi(a) \overset{\downarrow}{=} E\left(I_B\varphi(a)\,|\,\mathcal{B}\right)$$

$$\overset{f=a \text{ on } B}{\downarrow} = E(I_B\,\varphi(f)\,|\,\mathcal{B})$$

$$= I_B\, E(\varphi(f)\,|\,\mathcal{B})$$

So on B $\quad \varphi(E(f|\mathcal{B})) = E(\varphi(f)\,|\,\mathcal{B})$

▨

⑪ $E(f\,|\,\mathcal{B} \vee \mathcal{D}) = E(f|\mathcal{B})$ a.e.

f $\mathcal{C}$-measurable, $\mathcal{B} \subset \mathcal{C}$ where $\mathcal{C}, \mathcal{D} \subset \mathcal{A}$. Also, $\mathcal{C}$ and $\mathcal{D}$ are independent classes, i.e.

$$P(C \cap D) = P(C)\,P(D) \quad \forall C \in \mathcal{C} \ \forall D \in \mathcal{D}$$

$\mathcal{B} \vee \mathcal{D}$ is the σ-field generated by the union $\mathcal{B} \cup \mathcal{D}$ (so its the smallest σ-field containing both $\mathcal{D}$ and $\mathcal{B}$ - smallest σ-field containing the field of all finite disjoint unions of sets of the form $B \cap D$, $B \in \mathcal{B}$ and $D \in \mathcal{D}$)

To show - Let $g := E(f|\mathcal{B})$. Must show g not only $\mathcal{B}$-measurable but also $\mathcal{B} \vee \mathcal{D}$ measurable, and that

$$(*) \qquad \int_{B\cap D} f = \int_{B \cap D} g \qquad \forall B \in \mathcal{B}, D \in \mathcal{D}$$

LEMMA: Suppose f $\mathcal{C}$-measurable, h $\mathcal{D}$-measurable, where $\mathcal{C}, \mathcal{D}$ are independent. Then

$$E(fh) = Ef \cdot Eh$$

Proof. Holds for indicator functions by independence. Continue in standard way.

Proof of (*)

$$\int_{B\cap D} f = E(f I_B \cdot I_D) \overset{\substack{\text{lemma}\\ \downarrow}}{=} E(f I_B)\, E(I_D)$$

$$= E(g I_B)\, E(I_D) \underset{\substack{\uparrow\\ \text{lemma}}}{=} E(g I_B \cdot I_D)$$

$$= \int_{B\cap D} g$$

▨

Special case $E(\underset{\uparrow}{f} \mid \underset{\uparrow}{\mathcal{D}}) = Ef$ - Take $\mathcal{B} = \{\Omega, \emptyset\}$

independent, i.e. $\exists$ σ-field $\mathcal{C}$ s.t. f $\mathcal{C}$-measurable and $\mathcal{C}, \mathcal{D}$ independent

Define an operator $T := E(\cdot \mid \mathcal{B})$, i.e.

$$T\underset{\substack{\uparrow \\ \text{equivalence class of functions}}}{f} = E(f \mid \mathcal{B})$$

T is a linear operator from $L^p(\Omega, \mathcal{A}, P)$ onto $L^p(\Omega, \mathcal{B}, P)$, $1 \leq p \leq \infty$
Assume $\mathcal{B} \supset \{A \in \mathcal{A} : P(A) = 0\}$ (Then $L^p(\Omega, \mathcal{B}, P)$ is a subspace of $L^p(\Omega, \mathcal{A}, P)$)

a) $\|T\|_p = 1$

b) $T^2 = T$

c) In L^2 have self-adjointness i.e. $(Tf_1 \mid f_2) = (f_1 \mid Tf_2)$

$$(E(f_1 \mid \mathcal{B}) \mid f_2) = E\left[E(f_1 \mid \mathcal{B}) f_2\right] = E\left[E\left[E(f_1 \mid \mathcal{B}) f_2) \mid \mathcal{B}\right]\right]$$

$$= E\left[E(f_1 \mid \mathcal{B}) E(f_2 \mid \mathcal{B})\right]$$

$$\Longrightarrow (Tf_1 \mid f_2) = (Tf_1 \mid Tf_2) \underset{\substack{\uparrow \\ \text{symmetry}}}{=} (f_1 \mid Tf_2)$$

d) $T1 = 1$

e) $Tf \geq 0$, $f \geq 0$

linear, idempotent, self adjoint

<u>THEOREM</u> (Bahadur 1955) If T is an orthogonal projection on $L^2(\Omega, \mathcal{A}, P)$, $T1 = 1$, and $Tf \geq 0$ if $f \geq 0$, then

$$T = E(\cdot \mid \mathcal{B})$$

where $\mathcal{B}$ is the smallest σ-field with respect to which each fixed point h is measurable ($Th = h$)

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Probability space $(\Omega, \mathcal{A}, P)$

T partially ordered set ⟦ transitive $r \leq s,\ s \leq t \Rightarrow r \leq t$

anti-symmetric $s \leq t,\ t \leq s \Rightarrow s = t$ ⟧

$(\mathcal{A}_t)_{t \in T}$ non-decreasing family of sub-σ-fields of $\mathcal{A}$

⟦ if $s \leq t$ then $\mathcal{A}_s \subseteq \mathcal{A}_t$ ⟧

$f = (f_t)_{t \in T}$ family of integrable functions on Ω

<u>DEFINITION</u>: $f = (f_t)_{t \in T}$ is a MARTINGALE if

(i) f_t is $\mathcal{A}_t$-measurable and integrable

(ii) $E(f_t \mid \mathcal{A}_s) = f_s$ whenever $s \leq t$

(averaging property - if $s \leq t$

$$\frac{1}{P(A_s)} \int_{A_s} f_t = \frac{1}{P(A_s)} \int_{A_s} f_s \quad \forall A_s \in \mathcal{A}_s \Big)$$

($E(f_t \mid \mathcal{A}_s) \geq f_s$ submartingale

$E(f_t \mid \mathcal{A}_s) \leq f_s$ supermartingale)

EXAMPLES

1. F integrable. Define $f_t := E[F \mid \mathcal{A}_t]$

$$s \leq t \Rightarrow E(f_t \mid \mathcal{A}_s) = E\Big(E[F \mid \mathcal{A}_t] \mid \mathcal{A}_s \Big) \underset{\substack{\uparrow \\ \text{since } \mathcal{A}_s \subset \mathcal{A}_t}}{=} E(F \mid \mathcal{A}_s) = f_s$$

e.g. $T :=$ all sub-σ-fields of $\mathcal{A}$; $\mathcal{B} \leq \mathcal{C} \Leftrightarrow \mathcal{B} \subset \mathcal{C}$

$\mathcal{A}_{\mathcal{B}} := \mathcal{B} \quad \forall \mathcal{B} \in T$

$$\mathfrak{f}_{\mathcal{B}} := E(F|\mathcal{B})$$

DEFINITION: Reversed martingale - The $(\mathcal{A}_t)_{t \in T}$ is non-increasing and the averaging property holds $\forall\ s \geq t$

⟦Note - If $\mathfrak{f}$ is a submartingale and $s \leq t$, then $E\mathfrak{f}_s \leq E\mathfrak{f}_t$
(expectation increasing)
If $\mathfrak{f}$ is a supermartingale and $s \leq t$, then $E\mathfrak{f}_s \geq E\mathfrak{f}_t$
(expectation decreasing)
For martingales expectations are preserved ⟧

2. Ω = unit interval $[0,1)$ with Lebesgue measure. F integrable

$$g_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} F\left(x + \frac{k}{n}\right)$$

$$= E(F|\mathcal{B}_n)$$

↑
σ-field of measurable periodic sets of period $1/n$

$(\mathcal{B}_n)_{n \in \mathbb{N}}$ is <u>not</u> monotone. Define

$$\mathcal{A}_n := \mathcal{B}_{2^n}$$

$$\mathfrak{f}_n := g_{2^n}$$

Then $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \dots$, so $\mathfrak{f} = (\mathfrak{f}_n : n \in \mathbb{N})$ is a reversed martingale

If F is continuous (or Riemann integrable), then

$$g_n(x) \longrightarrow \int_0^1 F \quad \text{everywhere}$$

If F is Lebesgue integrable, but not Riemann integrable, then $g_1, g_2, \ldots$ does not converge a.e. However, even in this case, the sequence $f_1, f_2, f_3, \ldots$ does converge a.e. to $\int_0^1 F$ (B Jessen 1934 Acta Math)

3. $T := \mathbb{N}$. $\mathcal{A}_n$ σ-field generated by a finite partition Π_n of Ω where Π_{n+1} is a refinement of Π_n and

$$P(A) > 0 \qquad \forall A \in \Pi_n$$

Then $\mathcal{A}_{n+1} \supset \mathcal{A}_n$. Let φ be a finitely additive real-valued set function on $\mathcal{A}$.

$$f_n := \sum_{A \in \Pi_n} \frac{\varphi(A)}{P(A)} I_A$$

CLAIM: $f = (f_1, f_2, \ldots)$ is a martingale

Clearly each f_n is $\mathcal{A}_n$-measurable and integrable (simple function)

To show

$$\int_B f_n = \int_B f_m$$

for $m \leq n$, $\forall B \in \Pi_m$

Now $B = \overset{\text{finite union}}{\underset{j}{\bigcup}} B_j$, where the B_j's are disjoint sets in Π_n

$$\int_B f_n = \int_{\cup B_j} f_n = \sum_j \int_{B_j} f_n \underset{\substack{\uparrow \\ f_n \text{ constant on } B_j}}{=} \sum_j \frac{\varphi(B_j)}{P(B_j)} \cdot P(B_j)$$

$$= \sum_j \varphi(B_j) \underset{\substack{\uparrow \\ \text{additivity of } \varphi}}{=} \varphi(\cup B_j) = \varphi(B)$$

$$= \frac{\varphi(B)}{P(B)} P(B) = \int_B f_m$$

(If we allow $P(B) = 0$ for some B's $\in \Pi_n$, then we get a supermartingale)

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Examples - cont.

$(\Omega, \mathcal{A}, P)$ σ-fields $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots \subset \mathcal{A}$ (indexed by $\mathbb{N}$)
$f = (f_1, f_2, \dots)$

(Note - we can replace the averaging property by $E(f_n | \mathcal{A}_{n-1}) = f_{n-1}$ a.e. for $n \geq 2$. For example

$$E(f_n | \mathcal{A}_{n-2}) = E[E(f_n|\mathcal{A}_{n-1}) \mid \mathcal{A}_{n-2}]$$
$$= E[f_{n-1} | \mathcal{A}_{n-2}]$$
$$= f_{n-2}$$
)

Often easier to work with martingale difference sequence -

$$d_1 = f_1, \quad d_2 = f_2 - f_1, \dots, d_n = f_n - f_{n-1}, \dots$$

Then $f_n = \sum_{k=1}^{n} d_k$. The conditions for a martingale becomes

(1) d_n $\mathcal{A}_n$-measurable and integrable
(2) $E(d_n | \mathcal{A}_{n-1}) = 0$ a.e., $n \geq 2$

Suppose each f_n is square-integrable. Then each d_n is also square-integrable. In fact $d = (d_1, d_2, \dots)$ is an orthogonal sequence

$$E\, d_i d_j = 0 \quad \text{if } i \neq j$$

For if $j<k$,

$$E\, d_j d_k = E\left[E(d_j d_k \mid \mathcal{A}_{k-1}) \right] \underset{\uparrow}{=} E\left[d_j\, E(d_k \mid \mathcal{A}_{k-1}) \right] = 0$$

d_j $\mathcal{A}_j$-measurable $\subset \mathcal{A}_{k-1}$

d_k integrable

Interpretation -

d_n = dollars gambler might win playing game n is a sequence of games

$f_n = d_1 + \dots + d_n$ = fortune after game n

$E(d_n \mid \mathcal{A}_{n-1})$ = expected winning on nth game given past $= 0$
(fair game after 1ST)

Suppose $d = (d_1, d_2, \dots)$ is an independent seq. of integrable functions. Let $\mathcal{A}_n := \sigma(d_1, \dots, d_n)$

Suppose $Ed_n = 0$ for $n \geq 2$.

↑ i.e. $\sigma(d_n)$ is indep. of $\mathcal{A}_{n-1}$

CLAIM: $\left(f_n = \sum_{k=1}^{n} d_k : n \in \mathbb{N} \right)$ is a martingale

$$E(d_n \mid \mathcal{A}_{n-1}) \underset{\substack{\uparrow \\ \text{independence}}}{=} Ed_n = 0$$

Suppose $m_1, m_2, \ldots$ satisfy

$$E(m_n \mid \mathcal{A}_{n-1}) = 1$$

with $f_n = \prod_{k=1}^{n} m_n$ integrable, $\mathcal{A}_n$-measurable.

$$E(f_n \mid \mathcal{A}_{n-1}) \underset{\substack{\uparrow \\ f_n = f_{n-1} m_n}}{=} f_{n-1} E(m_n \mid \mathcal{A}_{n-1}) = f_{n-1} \quad \text{a.e.}$$

Hence $(f_1, \ldots)$ martingale

Suppose $(f_t : t \in T)$ is a martingale with values in convex $S \subset \mathbb{R}$. Suppose φ is convex on the convex set $S \subset \mathbb{R}$. If each $\varphi(f_t)$ is integrable, then $(\varphi(f_t) : t \in T)$ is a submartingale.

$$E(\varphi(f_t) \mid \mathcal{A}_s) \geq \varphi\big(E(f_t \mid \mathcal{A}_s)\big) = \varphi(f_s)$$

Similarly, if f_t is a submartingale and φ is non-decreasing and convex, then $\varphi(f_t)$ is a submartingale.

PROPOSITION: If f is a martingale and g is a martingale both relative to $(\mathcal{A}_t : t \in T)$, then

(i) $(f_t + g_t : t \in T)$ is a martingale
(ii) $(\max(f_t, g_t) : t \in T)$ is a submartingale

(corresponding results for submartingales)

Suppose $f = (f_1, f_2, \dots)$ is a martingale

$$g_n := \sum_{k=1}^{n} \underset{\uparrow}{v_k}\, d_k$$

a_{k-1} measurable (value he places on k^{th} game)

$g = (g_1, g_2, \dots)$ is the transform of f under $v = (v_1, v_2, \dots)$

Let $e_k = v_k d_k$. $E(e_n | a_{n-1}) = v_n E(d_n | a_{n-1}) = 0$.
If the e_k's are integrable (for example if the v_k's are bounded) then g is a martingale.

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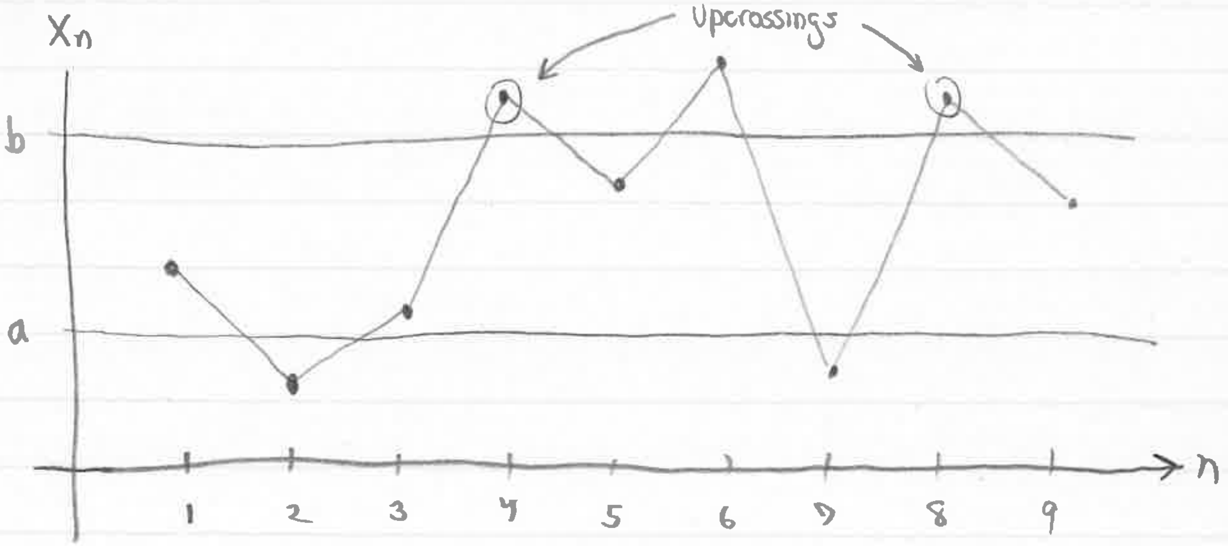
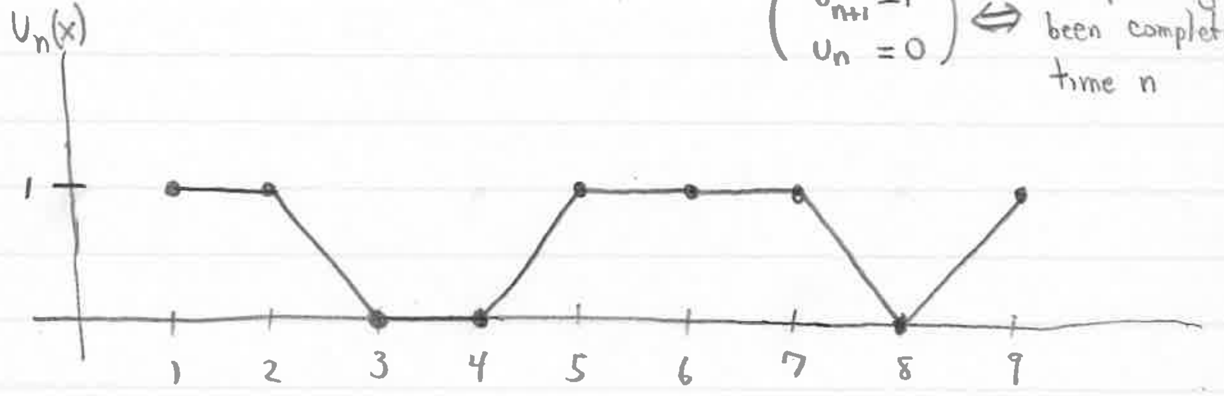
Let $x = (x_1, x_2, \cdots)$ be a sequence of real numbers. For $a<b$ define

$$U_1(x) := 1$$

$$U_{n+1}(x) := \begin{cases} 1 & \text{if } x_n \geq b \\ U_n(x) & \text{if } a < x_n < b \\ 0 & \text{if } x_n \leq a \end{cases}$$



$\begin{pmatrix} U_{n+1} = 1 \\ U_n = 0 \end{pmatrix} \Leftrightarrow$ an upcrossing has been completed at time n



Let

$$U_n^{ab}(x) := \sum_{k=2}^{n} \left[u_{k+1}(x) - u_k(x) \right]^+$$

and define

$$U^{ab}(x) = \lim_{n \to \infty} U_n^{ab}$$

[This gives the number of upcrossings of the interval $[a,b]$ by the seq x]

If $U^{ab}(x) = \infty$, then $\liminf x_n < a < b < \limsup x_n$. Hence if $U^{ab}(x) < \infty$ for each choice of $a<b$, then $\lim x_n$ exists.

Let $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2, \dots)$ be a submartingale. Let

$$V_n := u_n(\mathfrak{f})$$

(V_n is $\mathfrak{a}_{n-1}$ measurable since depends only on $\mathfrak{f}_1, \dots, \mathfrak{f}_{n-1}$)

(i.e. $V_n(\omega) = u_n(\mathfrak{f}(\omega))$, where $\mathfrak{f}(\omega) = (\mathfrak{f}_1(\omega), \mathfrak{f}_2(\omega), \dots)$ Then

$$(b-a)\, U_n^{ab}(\mathfrak{f}) = \sum_{k=2}^{n} (b-a)(V_{k+1} - V_k)^+ \quad \forall n \in \mathbb{N}$$

[Note that $(b-a)[u_{n+1}(x) - u_n(x)]^+ \leq (x_n - a)(u_{n+1}(x) - u_n(x))$. If $x_n \geq b$, then $u_{n+1}(x) = 1$ and $u_{n+1}(x) - u_n(x) \geq 0$. If $a < x_n < b$, then $u_{n+1}(x) = u_n(x)$, so ok. If $x_n < a$, then $u_{n+1}(x) = 0$ and so $[u_{n+1}(x) - u_n(x)]^+ = 0$ while right-hand side is product of two ≤ 0]

Hence

$$(b-a)\, U_n^{ab}(\mathfrak{f}) \leq \sum_{k=2}^{n} (\mathfrak{f}_k - a)(V_{k+1} - V_k)$$

Taking expectations

$$E\left[(\mathfrak{f}_k - a)(V_{k+1} - V_k) \right]$$

$$= E\left[V_{k+1}(\mathfrak{f}_k - a) \right] - E\left[V_k (\mathfrak{f}_k - a) \right]$$

$$= E\left[V_{k+1}(\mathfrak{f}_k - a) \right] - E\left[E\{ V_k (\mathfrak{f}_k - a) \,|\, a_{k-1} \} \right]$$

$$= E\left[V_{k+1}(\mathfrak{f}_k - a) \right] - E\left[V_k \, E(\mathfrak{f}_k - a \,|\, a_{k-1}) \right]$$

$$\leq E\left[V_{k+1}(\mathfrak{f}_k - a) \right] - E\left[V_k (\mathfrak{f}_{k-1} - a) \right]$$

(telescoping sum)

and so

$$(b-a)\, E\, U_n^{ab}(\mathfrak{f}) \leq E\left[V_{n+1}(\mathfrak{f}_n - a) \right] - E\left[\underbrace{V_2 (\mathfrak{f}_1 - a)}_{\text{always non-negative}} \right]$$

$$\leq E\left[V_{n+1}(\mathfrak{f}_n - a) \right]$$

$$\leq E\, (\mathfrak{f}_n - a)^+$$

$$\leq E\, |\mathfrak{f}_n| + |a|$$

Let us define $\|f\|_1 := \sup_n \|f_n\|_1$. Then

$$(b-a)\, E\, U_n^{ab}(f) \le \|f\|_1 + |a| \quad (\text{may be } \infty)$$

By the monotone convergence theorem

$$E\, U^{ab}(f) \le \frac{\|f\|_1 + |a|}{b-a}$$

(Upcrossing Inequality Doob (Martingale case), Snell (Submartingale Case))

$\|f\|_1 < \infty$

THEOREM (Doob): If f is an L^1-bounded submartingale (a martingale or supermartingale), then f converges a.e.

Proof. (I) $f_\infty := \lim f_n$ exists a.e.
(II) f_∞ integrable (hence finite a.e.)

(Note (I) $\Longrightarrow$ (II) since by Fatou $E|f_\infty| \le \liminf E|f_n| \le \|f\|_1 < \infty$)

To show that $P(\liminf f_n < \limsup f_n) = 0$, note that

$$P(\liminf f_n < \limsup f_n) \le \sum_{\substack{a<b \\ a,b\in\mathbb{Q}}} P(\liminf f_n < a < b < \limsup f_n)$$

But for $a<b$, $a,b \in \mathbb{Q}$, $P(\liminf f_n < a < b < \limsup f_n) \le P(U^{ab}(f) = \infty)$
But $E\, U^{ab}(f) < \infty$ by the upcrossing inequality, so $P(U^{ab}(f) = \infty) = 0$

▨

Any non-negative martingale or supermartingale is L^1- bounded

$$\|\xi\|_1 = \|\xi_1\|_1$$

since $E\,\xi_n = E|\xi_n| = \|\xi_n\|_1$ and the expectations are preserved or non-increasing

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THEOREM: If $f=(f_1,f_2,\dots)$ is a reversed martingale, then f converges a.e.

Proof. For a reversed martingale, $\|f\|_1 \leq \|f_1\|_1 < \infty$.

$$(*) \quad E\,\underbrace{V^{ab}(\overset{\text{(martingale)}}{f_n,f_{n-1},\dots,f_1,f_1,\dots})}_{V^{ab}(f)\,=\,\text{limit of}\,\uparrow\,\text{as } n\to\infty} \leq \frac{\|f\|_1+a}{b-a}$$

$V^{ab}(f)=+\infty$ iff $\liminf f_n < a < b < \limsup f_n$. But $V^{ab}(f)$ must be finite since it has finite expectation by (*).

▨

DEFINITION: A family of integrable functions $(f_t : t\in T)$ is uniformly integrable if

$$\left(\sup_{t\in T}\int_{|f|>b}|f_t|\right)\longrightarrow 0 \quad \text{as } b\to\infty$$

Example - $f=(f_1,f_2,\dots)$ Double-or-nothing (HW #3)

If $0<b<2^n$

$$\int_{|f_n|>b}|f_n| = \int_{|f_n|>b} 2^n$$

$$= 2^n P(|f_n| > b) = 2^n \left(\frac{1}{2^n}\right) = 1$$

and so

$$\sup_n \int_{|f_n|>b} |f_n| \not\to 0 \quad \text{as } b \to \infty$$

No uniform integrability

Example: $f_t := E(\underset{\underset{\text{integrable}}{\uparrow}}{F} \mid \underset{\underset{\text{sub-}\sigma\text{-field}}{\uparrow}}{\mathcal{A}_t}) \qquad t \in T$

Hence each f_t is integrable. Now for any $a > 0$

$$\int_{|f_t|>b} |f_t| \leq \int_{|f_t|>b} E(|F| \mid \mathcal{A}_t) = \int_{\substack{|f_t|>b \\ |F|\leq a}} |F| + \int_{\substack{|f_t|>b \\ |F|>a}} |F|$$

$$\leq a P(|f_t| > b) + \int_{|F|>a} |F|$$

$$\leq \frac{a}{b} \|f_t\|_1 + \int_{|F|>a} |F|$$

$$\leq \frac{a}{b} \|F\|_1 + \int_{|F|>a} |F|$$

$$\sup_{t\in T} \int_{|f_t|>b} |f_t| \leq \frac{a}{b}\|F\|_1 + \int_{|F|>a} |F|$$

$$\Rightarrow \limsup_{b\to\infty} \sup_{t\in T} \int_{|f_t|>b} |f_t| \leq \int_{|F|>a} |F| \qquad (\forall a)$$

$$\Rightarrow \limsup_{b\to\infty} \sup_{t\in T} \int_{|f_t|>b} |f_t| = 0 \qquad (\text{let } a\to\infty)$$

UNIFORMLY INTEGRABLE

THEOREM: $f=(f_1, f_2, \ldots)$ is unif. integrable. Then

$$E(\liminf_{n\to\infty} f_n) \leq \liminf_{n\to\infty} Ef_n \leq \limsup_{n\to\infty} Ef_n \leq E(\limsup_{n\to\infty} f_n)$$

In particular, if $f_n \to f$ a.e., then f is integrable and $f_n\to f$ in L^1

Proof. Let $b>0$ and set

$$g_n := \begin{cases} f_n & f_n \geq -b \\ 0 & f_n < -b \end{cases}$$

$\left(\text{Choose } \varepsilon>0 \text{ \& Choose } b \text{ s.t. } \int_{|f_n|>b} |f_n| < \varepsilon \ \forall n\right)$

Then $f_n \leq g_n$ and so

$$E(\liminf f_n) \leq E(\liminf g_n) \overset{\text{Fatou (}g_n\text{ bdd below)}}{\leq} \liminf Eg_n$$

$$E g_n = \int_{f_n \geq -b} f_n = E f_n - \int_{f_n < -b} f_n$$

$$\leq E f_n + \varepsilon$$

so $\liminf E g_n \leq \liminf E f_n + \varepsilon \Rightarrow E(\liminf f_n) \leq \liminf E f_n$

▨

CONTINUITY THEOREM FOR CONDITIONAL EXPECTATIONS:

Let $(\mathcal{A}_n : n \in \mathbb{N})$ be a seq. of sub-σ-fields. F integrable

(1) $E(F|\mathcal{A}_n) \to E(F| \underset{\nwarrow\, \mathcal{A}_\infty}{\bigvee_{n=1}^{\infty} \mathcal{A}_n})$ a.e. and in L^1-norm

if $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$

(2) $E(F|\mathcal{A}_n) \to E(F| \bigcap_{n=1}^{\infty} \mathcal{A}_n)$ a.e. and in L^1-norm

if $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \dots$

Proof. Let $f_n = E(F|\mathcal{A}_n)$. Then $f = (f_1, f_2, \dots)$ is a martingale in case (1) and a reversed martingale in case (2). f is uniformly integrable and so L^1-bounded (see second example above). Hence by the convergence theorem $\exists f_\infty$ s.t. $f_n \to f_\infty$ a.e. ($\Rightarrow f_n \to f_\infty$ in L^1-norm by unif. integrability and the preceding theorem)

CASE 1: f_n is $\mathcal{A}_\infty$-measurable $\forall n \Rightarrow f_\infty$ is $\mathcal{A}_\infty$-measurable. To show - $f_\infty = E(F|\mathcal{A}_\infty)$, i.e.

$$\int_A F = \int_A f_\infty \quad \forall A \in \mathcal{A}_\infty$$

But

$$(*) \qquad \int_A F = \int_A f_\infty \qquad \forall\ A \in \mathcal{A}_n$$

because $\forall\ A \in \mathcal{A}_n$ and $k \in \mathbb{N}$

$$\int_A F = \int_A f_{n+k} \longrightarrow \int_A f_\infty \quad \text{as } k \to \infty$$

$(A \in \mathcal{A}_n \subset \mathcal{A}_{n+k})$

$$\uparrow$$

$$\left| \int_A f_{n+k} - \int_A f_\infty \right| \leq \int_A |f_{n+k} - f_\infty| \leq \|f_{n+k} - f_\infty\|_1 \to 0$$

Hence $(*)$ holds on $\cup \mathcal{A}_n$ and so it holds on $\mathcal{A}_\infty = \sigma(\cup \mathcal{A}_n)$

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$$\mathcal{A}_1 \supset \mathcal{A}_2 \supset \dots \longrightarrow \mathcal{A}_\infty = \bigcap_{n=1}^{\infty} \mathcal{A}_n$$

(reversed martingale) $f_n = E(F|\mathcal{A}_n) \to f_\infty$ a.e. and in L^1 (by u.i.)

To show - $f_\infty = E(F|\mathcal{A}_\infty)$

f_∞ integrable by Fatou.

$$f_\infty = \lim_{k \to \infty} \underbrace{f_{n+k}}_{\mathcal{A}_{n+k} \subset \mathcal{A}_n \text{ measurable } \forall n}$$

$\Rightarrow f_\infty$ $\mathcal{A}_\infty$-measurable

If $A \in \mathcal{A}_\infty$, then $A \in \mathcal{A}_n$ $\forall n$ so

$$\int_A F = \int_A f_n = \int_A f_{n+k} \longrightarrow \int_A f_\infty$$

$$\Rightarrow \int_A F = \int_A f_\infty$$

THEOREM: If $f = (f_1, f_2, \ldots)$ is a martingale, then TFAE

(1) f is uniformly integrable
(2) There is an integrable function F s.t. $f_n = E(F|\mathcal{a}_n)$ a.e. $\forall n$
(3) f converges in L^1
(4) f is L^1-bounded (hence converges a.e. to f_∞) and $\|f_\infty\|_1 = \|f\|_1$
(5) $f_\infty = \lim f_n$ a.e. exists and $(f_1, f_2, \ldots, f_\infty)$ is a martingale relative to $(\mathcal{a}_1, \mathcal{a}_2, \ldots, \mathcal{a}_\infty)$ (so that $f_n = E(f_\infty|\mathcal{a}_n)$ a.e.)

Proof. (1) $\Rightarrow$ (2) Let $F = \lim f_n$ (integrable). Want to show

$$(*) \qquad \int_A F = \int_A f_n \quad \forall A \in \mathcal{a}_n$$

But for every k,

$$\int_A f_n = \int_A f_{n+k} \quad (\text{Since } E(f_{n+k}|\mathcal{a}_n) = f_n)$$

and by u.i. $\int_A f_{n+k} \to \int_A F$, so $(*)$ holds.

(2) $\Rightarrow$ (3) Continuity theorem

(3) $\Rightarrow$ (4) $\big|\, \|f\|_\infty - \|f_n\|_1 \big| \leq \|f_n - f_\infty\|_1 \to 0$.

$(|f_n| : n \in \mathbb{N})$ is a submartingale and so expectation increasing

$$\|f_1\|_1 \leq \|f_2\|_1 \leq \ldots \leq \|f_n\|_1 \leq \ldots$$

Hence $\|f\|_1 = \sup \|f_n\|_1 = \lim \|f_n\|_1 = \|f_\infty\|_1$

(5) $\Rightarrow$ (2) immediate

(1) $\Rightarrow$ (5)

(4) $\Rightarrow$ (1) : Need the following

LEMMA : Suppose $h_1, h_2, \dots, h_\infty$ are non-negative integrable with $h_n \to h_\infty$ a.e. If $Eh_n \to Eh_\infty$, then $\|h_n - h_\infty\|_1 \to 0$

Proof.

$$\|h_\infty - h_n\|_1 = \int |h_\infty - h_n| = 2\underbrace{\int (h_\infty - h_n)^+}_{\le h_\infty \text{ int}} - \underset{\substack{\downarrow \\ 0}}{\int (h_\infty - h_n)}$$

and $h_\infty - h_n \to 0$

$\Rightarrow E(h_\infty - h_n) \to 0$ (Lebesgue Dom.)

To show (1) from (4) let $h_n = |f_n|$ and $h_\infty = |f_\infty|$. Then by the lemma,

$$\| \, |f_n| - |f_\infty| \, \|_1 \to 0$$

and so (f_n) is uniformly integrable

▨

EXAMPLE – $(\Omega, \mathcal{a}, P)$

Π_n finite partition of Ω (elements of Π_n are measurable)
Let $\mathcal{a}_n$ be the sub-σ-field generated by Π_n. We assume Π_{n+1} is a refinement of Π_n, so $\mathcal{a}_n \subset \mathcal{a}_{n+1}$.
φ set function on $\mathcal{a}$, finitely additive. Let

$$f_n := \sum_{\substack{A \in \Pi_n \\ P(A) > 0}} \frac{\varphi(A)}{P(A)} I_A$$

(supermartingale)
Now assume $\varphi \geq 0$, countably additive, $\varphi(\Omega) < \infty$, and φ is absolutely cont. w.r.t. P ($\varphi << P$). Then $f = (f_1, f_2, \dots)$ is a uniformly integrable martingale

(martingale follows from $\varphi << P$ for then $\varphi(A) = 0$ when $P(A) = 0$)

($\varphi << P$ iff $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $A \in \mathcal{a}$, $P(A) < \delta \Rightarrow \varphi(A) < \varepsilon$)

Now f is non-negative, hence L^1-bounded, hence converges a.e. to f_∞, say. Claim: f u.i.
Let $\varepsilon > 0$. Want to show $\exists$ b s.t.

$$\int_{f_n > b} f_n < \varepsilon \quad \forall n$$

Claim: $\varphi(f_n > b) = \int_{f_n > b} f_n$. If this were true, then it would suffice to find b s.t. $P(\xi_n > b) < \delta$ (where δ chosen from abs. continuity). But

$$P(\xi_n > b) \leq \frac{E\xi_n}{b} = \frac{\varphi(\Omega)}{b} < \delta$$

↑

choose $b > \frac{\varphi(\Omega)}{\delta}$

Now if $B \in \pi_n$, $\int_B \xi_n = \frac{\varphi(B)}{P(B)} \cdot P(B) = \varphi(B)$ if $P(B) \neq 0$. Of course, if $P(B) = 0$, then $\int_B \xi_n = 0 = P(B) = \varphi(B)$. Hence

$$\int_B \xi_n = \varphi(B) \quad \forall B \in a_n$$

Significance of ξ_∞ -

$$\varphi(A) = \int_A \xi_\infty \quad \forall A \in a_\infty$$

↑

Radon-Nikodym derivative of φ w.r.t. P on a_∞

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examples -

① Let F be integrable on $[0,1)$. Then

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{2^n-1} F\left(x+\frac{k}{2^n}\right) = E(F) \text{ a.e.}$$

For what we are taking the limit of is $E(F|\underset{\substack{\uparrow \\ \frac{1}{2^n}\text{-periodic measurable sets}}}{a_n})$

$a_n \supset a_{n+1}\ \forall n$. By the continuity theorem

$$E(F|a_n) \to E(F|\underset{\substack{\nwarrow \\ A_\infty}}{\bigcap_{n=1}^{\infty} A_n})$$

a.e. and in L^1-norm. But

$$A \in a_\infty \Rightarrow P(A)=0 \text{ or } P(A)=1$$

and so $E(F|a_\infty) = E(F)$. We can also see this in the following way: certainly true for G with a finite number of discontinuities and bounded (then Riemann integrable). Choose such a G with $\|F-G\|_1 < \varepsilon$. Then

$$\|EF - E(F|a_\infty)\|_1 \leq \overset{\text{equal}}{\|EF-EG\|_1 + \|E(G|a_\infty)} - E(F|a_\infty)\|_1$$

$$\leq \varepsilon + \|G-F\|_1 < 2\varepsilon$$

To show such a G exists, let

$$\mathcal{F}_n := E(F|\mathcal{D}_n)$$

$\uparrow$ σ-field generated by the dyadic partition

$\mathcal{D}_n \uparrow \mathcal{B}$ Borel σ-field, so $\mathcal{F}_n = E(F|\mathcal{D}_n) \to E(F|\mathcal{A}) = F$ a.e.
If $\varepsilon > 0$, $\exists\, n_0$ s.t. $\|\mathcal{F}_{n_0} - F\| < \varepsilon$. Take $G = \mathcal{F}_{n_0}$ suitably modified (over a small set) so that it is continuous

② (KOLMOGOROV'S 0-1 LAW)

$X_1, X_2, \dots$ seq. of independent random variables

$$\mathcal{A}_n := \sigma(X_{n+1}, X_{n+2}, \dots)$$

Then $\mathcal{A}_n \supset \mathcal{A}_{n+1}$.

$$\mathcal{A}_\infty := \bigcap_{n=1}^{\infty} A_a \quad \text{tail field}$$

Let $\mathcal{B}_n = \sigma(X_1, \dots, X_n)$. Then $\mathcal{B}_n$ and $\mathcal{A}_n$ are independent.

$$\mathcal{B}_\infty := \bigvee_{n=1}^{\infty} \mathcal{B}_n \supset \mathcal{A}_\infty$$

Let $A \in \mathcal{A}_\infty$

$$P(A) = E I_A = E(\underbrace{I_A \mid \mathcal{B}_n}_{\text{independent}})$$

Now $E(I_A \mid \mathcal{B}_n) \longrightarrow E(I_A \mid \mathcal{B}_\infty)$ a.e. $= I_A$ a.e., and so

$$P(A) = I_A \quad \text{a.e.}$$

Hence I_A is constant a.e., so $P(A) = 0$ or 1

③ (STRONG LAW OF LARGE NUMBERS) $X_1, X_2, \ldots$ i.i.d
If EX_1 exists, then

$$\frac{S_n}{n} \longrightarrow EX_1 \quad \text{a.e.}$$

(Assume X_1 integrable)

$$\text{CLAIM:} \quad \frac{S_n}{n} = E(X_1 \mid \underset{\substack{\uparrow \\ \text{means } \sigma\text{-field generated by } S_n}}{\tilde{S}_n}) = E(X_1 \mid \overbrace{S_n, S_{n+1}, \ldots}) \xrightarrow{\text{continuity th}^{\text{m}}} \underset{\substack{\uparrow \\ \text{constant a.e.} \\ \text{by K. 0-1 law}}}{Y}$$

$$Y = E(X_1 \mid \text{intersection})$$

$$c = EY = E\left(\frac{S_n}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} EX_i \underset{\substack{\uparrow \\ \text{i.i.d}}}{=} \frac{1}{n}\sum_{i=1}^{n} EX_1 = EX_1$$

$$E(X_1 \mid S_n, S_{n+1}, S_{n+2}, \ldots) = E(X_1 \mid S_n, X_{n+1}, X_{n+2}, \ldots)$$

$$= E(X_1 \mid S_n) \qquad \left[E[\mathfrak{f} \mid \mathcal{B} \vee \mathcal{D}] = E(\mathfrak{f} \mid \mathcal{B}) \ \forall\ \mathfrak{f}\ \mathcal{C}\text{-measurable}, \ \mathcal{B} \subset \mathcal{C}, \text{ and } \mathcal{C}, \mathcal{D} \text{ independent} \right]$$

Let $g = E(X_1|S_n)$. g is the (essentially) unique $\sigma(S_n)$-measurable function satisfying

$$\int_\Omega X_1\, \varphi(S_n) = \int_\Omega g\, \varphi(S_n)$$

$$\| \qquad\qquad\qquad\qquad \varphi \text{ bounded measurable}$$

$$\int_\Omega X_2\, \varphi(S_n)$$

$$\| \\ \vdots \\ \|$$

$$\int_\Omega X_n\, \varphi(S_n)$$

Hence $g = E(X_j|S_n)$ for $1 \leq j \leq n$. Hence

$$ng = E\left(\sum_{j=1}^{n} X_j \,\middle|\, S_n\right) = E(S_n|S_n) = S_n$$

$$\Longrightarrow g = S_n/n$$

//

$$(\Omega, \mathcal{A}, P) \xrightarrow{\xi} (\mathbb{R}^n, \underset{\underset{\text{Borel sets}}{\uparrow}}{\mathcal{B}}, \underset{\nwarrow\ \text{distribution of } \xi}{\mu})$$

Define μ by $\mu(B) := P(\xi^{-1}(B))$. Then μ is a measure.

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DEFINITION: $f = (f_1, f_2, \ldots)$ any sequence of functions

$$f_n^*(\omega) := \sup_{1 \leq k \leq n} |f_k(\omega)|$$
$$f^*(\omega) := \sup_k |f_k(\omega)|$$

WEAK L^1-INEQUALITY: $f = (f_1, f_2, \ldots)$ non-negative submartingale

Then

$$P(f_n^* > \lambda) \leq \frac{1}{\lambda} \int_{\{f_n^* > \lambda\}} f_n \qquad \forall \lambda > 0$$

$$\lambda P(f^* > \lambda) \leq \|f\|_1 = \sup_n \|f_n\|_1$$

(So f L^1-bounded $\Rightarrow$ $f^* < \infty$ a.e.)

COROLLARY: If f is a martingale, then for $1 \leq p < \infty$

$$\lambda^p P(f^* > \lambda) \leq \|f\|_p^p \qquad \forall \lambda > 0$$

($|f_n|^p$ is a submartingale)

Proof of the Weak L^1-Inequality - Let

$\forall k \in \mathbb{N}$ $\quad A_k := \{\omega : f_k(\omega) > \lambda \text{ and } f_j(\omega) \leq \lambda \;\forall\, j < k\} \in \mathcal{A}_k$

(set where $f_k > \lambda$ for 1st time).

$$\lambda P\left(f_n^* > \lambda\right) = \lambda \sum_{k=1}^{n} P(A_k)$$

$$\leq \sum_{k=1}^{n} \int_{A_k} f_k$$

$$\leq \sum_{k=1}^{n} \int_{A_k} f_n \qquad \left(\underline{\underline{\text{submartingale property}}}\right)$$

$$= \int_{\cup A_k} f_n = \int_{\{f_n^* > \lambda\}} f_n$$

▨

$\underline{\underline{L^p\text{ - INEQUALITIES } (1 < p < \infty)}}$: $f = (f_1, f_2, \ldots)$ non-negative submartingale or f is a martingale. Then

$$\| f^* \|_p \leq \underset{\substack{\uparrow \\ \text{no smaller constant works}}}{q} \| f \|_p$$

$\left(1/p + 1/q = 1 \right)$

Proof. Enough to prove $\| f_n^* \|_p \leq q \, \| f_n \|_p$ (by MCT) for a non-negative submartingale. We use the following

LEMMA: If $f, g \geq 0$, and

$$\lambda P(g>\lambda) \leq \int_{(g>\lambda)} f \qquad \forall \lambda > 0$$

then

$$\|g\|_p \leq q \, \|f\|_p$$

Proof of lemma -

$$\|g\|_p^p = E g^p = \int_\Omega \overset{(*)}{\int_0^g} p\lambda^{p-1} \, d\lambda \, dP$$

Fubini

$$= \int_0^\infty p\lambda^{p-1} \int_\Omega I(g>\lambda) \, dP \, d\lambda$$

$$= \int_0^\infty p\lambda^{p-1} P(g>\lambda) \, d\lambda$$

$$\varphi(\lambda, \eta) = \begin{cases} 1 & 0<\lambda<\eta \\ 0 & \text{otherwise} \end{cases}$$

. φ measurable on $\mathbb{R}^2$.

(*) $$\int_\Omega \int_0^\infty \varphi(\lambda, g(\omega)) \, p\lambda^{p-1} \, d\lambda \, dP(\omega)$$

$$\leq \int_0^\infty p\lambda^{p-1} \frac{1}{\lambda} \int_\Omega I(g>\lambda) f \, dP \, d\lambda$$

Fubini

$$= \int_\Omega f \underbrace{\int_0^g p\lambda^{p-2} \, d\lambda}_{\frac{p}{p-1}\lambda^{p-1}\Big|_0^g} dP$$

$$= q E\left(f g^{p-1}\right)$$

$$\leq q \left(E f^p\right)^{1/p} \left(E g^{(p-1)q}\right)^{1/q}$$

$$= q \|f\|_p \|g\|_p^{p/q}$$

Now divide through by $\|g\|_p^{p/q}$ to get

$$\|g\|_p \leq q \|f\|_p$$

If $0<\|g\|_p<\infty$. If $\|g\|_p=0$, inequality is trivial. If $\|g\|_p=\infty$ repeat argument with $g\wedge n$. It will turn out that $\|g\|_p=\infty$ is impossible, since we get $\|g\wedge n\|_p \leq q\|f\|_p$ $\forall n$ [illegible].

▨

$(1 < p < \infty)$

THEOREM: f L^p- bounded non-negative submartingale or martingale. Then f converges a.e. and in L^p

Proof. $\|f\|_1 \leq \|f\|_p < \infty$, so f L^1-bounded $\Rightarrow$ f converges a.e. (to f_∞ say)

Now $f^* \in L^p$ by the L^p-inequality. $|f_n - f_\infty|^p \leq (2f^*)^p$, so

$$E|f_n - f_\infty|^p \underset{\overset{\uparrow}{\text{DCT}}}{\longrightarrow} E|f_\infty - f_\infty|^p = E0 = 0$$

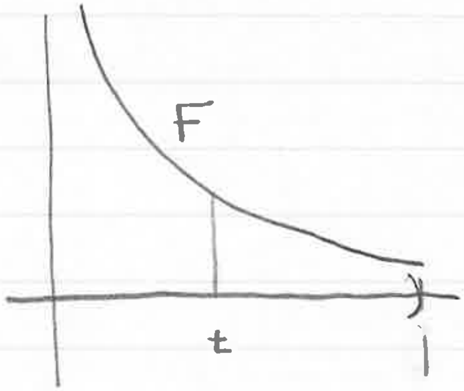
▨

---

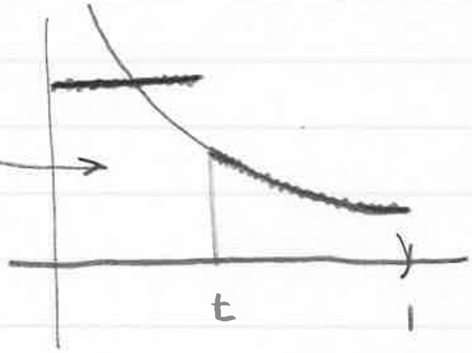
q best possible constant

$\Omega = [0,1)$

$\mathcal{A}_t = \sigma\{A : \text{Borel measurable} ; A \subset [t,1)\}$



$E(F|\mathcal{A}_t)$



ex. $F(x) = x^{-1/r}$

9/25 MARTINGALES

L log L INEQUALITY : $f = (f_1, f_2, \dots)$ non-negative submartingale

$$\|f^*\|_1 \le 2 + 2 \underbrace{\sup_n E\, f_n \log^+ f_n}_{\text{L log L bounded}}$$

$$\log^+ t = \begin{cases} \log t & t \ge 1 \\ 0 & t \le 1 \end{cases}$$

(f L^p-bdd, $p>1$ $\Rightarrow$ f $L\log L$ bdd $\Rightarrow$ f L^1-bdd)

[Application: $X_1, X_2, \dots$ i.i.d.

$$X_1 \in \underset{\substack{\uparrow \\ E|X_1|\log^+|X_1| < \infty}}{L\log L} \quad \underset{\substack{\uparrow \\ \text{Burkholder 1962}}}{\iff} \quad \sup_n \underset{\substack{\uparrow \\ \text{reversed martingale}}}{\left|\frac{X_1 + \dots + X_n}{n}\right|} \in L^1$$

]

Proof of Inequality: Let n be fixed and define

$$h_k := E\big(f_n I(f_n > \lambda) \mid \mathcal{A}_k\big)$$

for $k \ge 1$. Shall show $\|f_n^*\|_1 \le 2 + 2E\, f_n \log^+ f_n$. $h = (h_1, h_2, \dots)$ is a martingale. For $1 \le k \le n$,

$$f_k \underset{\substack{\uparrow \\ \text{submartingale}}}{\le} E(f_n \mid \mathcal{A}_k) = E\big(f_n I(f_n \le \lambda) \mid \mathcal{A}_k\big) + h_k$$

$$\leq \lambda + h_k$$

Hence $P(f_n^* > 2\lambda) \leq P(h_n^* > \lambda) \leq \frac{1}{\lambda}\int_{\Omega} h_n = \frac{1}{\lambda}\int_{\{f_n>\lambda\}} f_n$

$$\|f_n^*\|_1 = \int_0^\infty P(f_n^* > \lambda)\,d\lambda = 2\int_0^\infty P(f_n^* > 2\lambda)\,d\lambda$$

$$\leq 2\int_0^1 1\,d\lambda + 2\int_1^\infty \frac{1}{\lambda}\int_{\{f_n>\lambda\}} f_n\,dP\,d\lambda$$

$$= 2 + 2\int_{\{f_n>1\}} f_n \int_1^{f_n} \frac{1}{\lambda}\,d\lambda\,dP$$

$$= 2 + 2\int_{\{f_n>1\}} f_n \log f_n \leq 2 + 2E(f_n \log f_n)$$

▨

Stopping Time : Given Ω and $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots \subset \mathcal{A}$

$\tau : \Omega \longrightarrow \mathbb{N} \cup \{\infty\}$ defines a stopping time if $\{\tau \leq n\} \in \mathcal{A}_n$ for each $n \in \mathbb{N}$ $(\Rightarrow \tau$ is $\mathcal{A}_\infty$-measurable$)$

example - $f = (f_1, f_2, \ldots)$ f_n is $\mathcal{A}_n$-measurable.

$$\tau(\omega) := \inf \{n : f_n(\omega) > \lambda\} \quad \forall \omega \in \Omega$$

$(\inf \phi = \infty)$.

$$\{\tau \leq n\} = \bigcup_{k=1}^{n} \{f_k > \lambda\} \in \mathcal{a}_n \quad \forall n \in \mathbb{N}$$

CLAIM: If τ_1 and τ_2 are stopping times, then so are

$$\tau_1 \vee \tau_2 \quad (\max)$$
$$\tau_1 \wedge \tau_2 \quad (\min)$$

Since

$$\{\tau_1 \vee \tau_2 \leq n\} = \{\tau_1 \leq n\} \cap \{\tau_2 \leq n\}$$

$$\{\tau_1 \wedge \tau_2 > n\} = \{\tau_1 > n\} \cap \{\tau_2 > n\}$$

Consider $f = (f_1, f_2, \ldots)$ as in the above example. Define

$$f_\tau : \omega \longmapsto f_{\tau(\omega)}(\omega)$$

[Two cases : (1) $\tau < \infty$ a.e. (2) there is an extra arbitrary $\mathcal{a}$-measurable function f_∞]

CLAIM: f_τ is $\mathcal{a}$-measurable

$$\{f_\tau \in \mathcal{B}\} = \bigcup_{n=1}^{\infty} \{\underbrace{f_n \in \mathcal{B}}_{\mathcal{A}_n\text{-meas}}, \underbrace{\tau = n}_{\mathcal{A}_n\text{-meas}}\} \cup \{\underbrace{f_\infty \in \mathcal{B}}_{\mathcal{A}\text{-meas}}, \underbrace{\tau = \infty}_{\mathcal{A}\text{-meas}}\}$$

($f_{\tau \wedge n}$ is actually $\mathcal{A}_n$-measurable)

DEFINITION: f stopped at τ is the seq. f^τ where

$$f^\tau_n := f_{\tau\wedge n} = \begin{cases} f_\tau & \tau < n \\ f_n & \text{if } \tau \geq n \end{cases}$$

$$= \sum_{k=1}^{n} I(\tau \geq k)\, d_k$$

$d_k = f_{k+1} - f_k$

(submartingale)

LEMMA: If f is a martingale and τ is a stopping time, then f^τ is a martingale (submartingale). Moreover

$$E f_1 \underset{(\leq)}{=} E f_{\tau\wedge n} \underset{(\leq)}{=} E f_n \quad \forall n$$

If f is a L^1-bounded martingale or a non-negative sub or supermartingale then

$$\|f^\tau\|_1 \leq \|f\|_1$$

Proof. $$f_n^\tau = \sum_{k=1}^{n} \underbrace{\overset{1-I(\tau\le k-1)}{I(\tau\ge k)}}_{a_{k-1}\text{- measurable}} d_k$$

$$\Rightarrow \quad E\left[I(\tau\ge k)\,d_k \,\middle|\, a_{k-1}\right] = I(\tau\ge k)\,E(d_k | a_{k-1}) = 0$$

Also

$$E(f_n - f_{\tau\wedge n}) = E\left[\sum_{k=2}^{n} I(\tau<k)\,E(d_k | a_{k-1})\right] \ge 0$$

$$\|f_\tau\|_1 \underset{\substack{\uparrow\\ \text{Fatou}}}{\le} \liminf_{n\to\infty} \|f_{\tau\wedge n}\|_1 \le \|f\|_1$$

<u>Application</u>

(1) $\lambda P(f^* > \lambda) \le \lambda P(|f_\tau| > \lambda) \le \|f_\tau\|_1 \le \|f\|_1$

$(\tau = \inf\{n : |f_n| > \lambda\})$

(2) Suppose f non-neg. supermartingale

$$E f_\tau \le E f_1 = f_1 \quad (\text{assume constant})$$

Suppose $\tau = \inf\{n : \tau > f_1\}$. CLAIM - $P\{\tau = \infty\} > 0$

(for otherwise $\xi_1 < \xi_\tau$ a.e. $\Rightarrow E\xi_1 < E\xi_\tau \leqslant E\xi_1$ $\lightning$)

9/27 MARTINGALES

Let $Mf := f^*$, $(M^+f)(\omega) := \sup_n f_n(\omega)$

$$(M^-f)(\omega) := -\inf_n f_n(\omega)$$

THEOREM: Suppose f is a submartingale with $E(M^+ \overset{\substack{\text{differences}\\ \downarrow}}{d}) < \infty$ Then f converges a.e. on $\{M^+f < \infty\}$ (f need not be L^1-bdd)

Proof. Let $\lambda > 0$

$$\tau(\omega) := \inf\{n \in \mathbb{N} : f_n(\omega) > \lambda\}$$

and let $g = f^\tau$. g is an L^1-bounded submartingale. So g converges a.e. But $f = g$ on the set where $\tau(\omega) = \infty$ since

$$f^\tau_n = f_{\tau \wedge n} = f_n$$

$\{\tau(\omega) = \infty\} = \{M^+f \leq \lambda\} \uparrow \{M^+f < \infty\}$ as $\lambda \to \infty$

Hence f converges a.e. on $\{M^+f < \infty\}$

To show g is L^1-bdd — Shall show that M^+g is integrable.

$$\longrightarrow g_n^+ = \begin{cases} f_\tau^+ & \text{if } \tau \leq n \\ f_n^+ \leq \lambda & \text{if } \tau > n \end{cases}$$

submartingale

(submart. $\vee$ submart. = submart.)

(*)

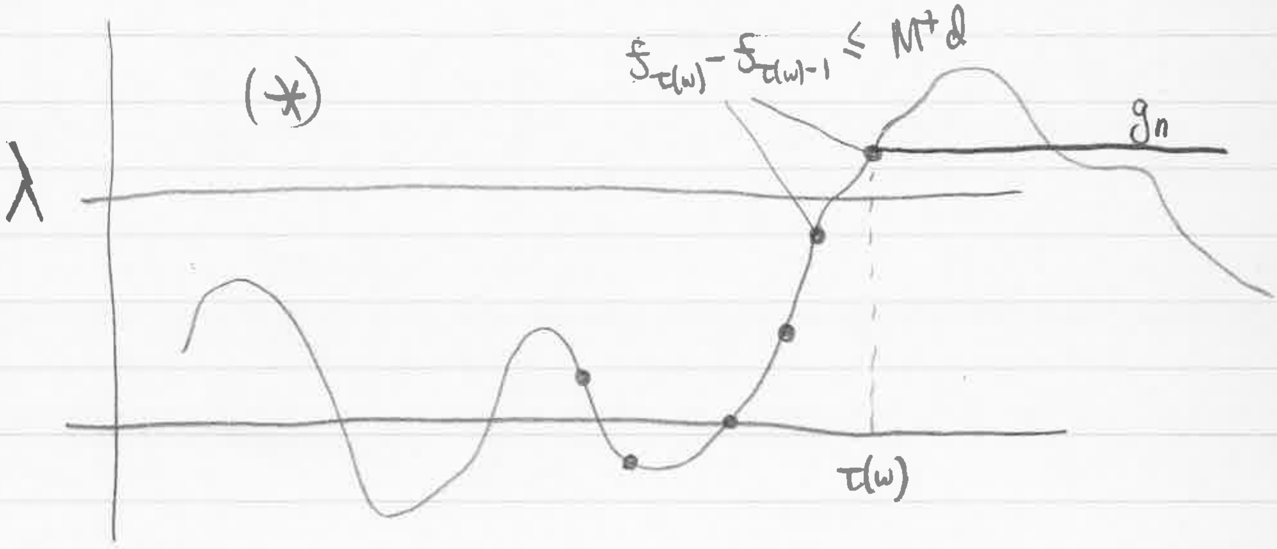
Hence $g_n^+ \leq \lambda + M^+ d \Rightarrow E g_n^+ \leq \lambda + E(M^+ d) = K < \infty$

Now

$$\|g_n\|_1 = E|g_n| \leqslant E(2g_n^+ - g_n)$$

$$\leqslant 2K - Eg_n$$

$$\leqslant 2K - Eg_1 \quad \text{(submartingales expectation increasing)}$$



▨

COROLLARY: $u_1, u_2, \dots,$ $\mathcal{a}$-measurable, with $0 \leq u_n \leq 1$.

$$\left\{ \sum_{k=1}^{\infty} u_k < \infty \right\} \overset{a.e.}{=\!=} \left\{ \sum_{k=1}^{\infty} E(u_k \mid \mathcal{a}_{k-1}) < \infty \right\}$$

(Implies Borel-Cantelli lemma)

Proof. Let $d_n = u_n - E(u_n | a_{n-1})$. This is a martingale difference sequence since

$$E(d_n | a_{n-1}) = 0$$

Moreover $|d_n| \leq 1$. Let $f_n = \sum_{k=1}^{n} d_k = \sum_{k=1}^{n} u_k - \sum_{k=1}^{n} E(u_k | a_{k-1})$

By the theorem f converges a.e. on $\{M^+ f < \infty\}$.

$$M^+ f \leq \sum_{k=1}^{\infty} u_k$$

so if $\sum_{k=1}^{\infty} u_k < \infty$, then $M^+ f < \infty \Rightarrow f$ converges $\Rightarrow \sum_{k=1}^{n} E(u_k | a_{k-1})$ converges.

▨

Another approach to the convergence theorem.

STATEMENTS

① If f is an L^1-bounded martingale, then f converges a.e.

② If f is a uniformly integrable martingale, then f converges a.e.

③ If f is a uniformly integrable martingale, then f converges in L^1 norm.

Note (iii) $\Rightarrow$ (ii) Consider the martingale $f^n = (f_1, f_2, \dots, f_n, f_n, f_n, \dots)$ (this is f^τ where $\tau \equiv n$). Then

$$\underbrace{\| f - f^n \|_1}_{\text{martingale}} = \sup_{k \geq n} \| f_k - f_n \|_1 < \varepsilon^2$$

for sufficiently large n.

$$P\left((f - f^n)^* > \varepsilon \right) \leq \frac{\| f - f^n \|_1}{\varepsilon} < \varepsilon \quad \left(\text{use martingale property here} \right)$$

$$\Rightarrow P\left(\limsup_{m,n \to \infty} |f_m - f_n| > 2\varepsilon \right) < \varepsilon$$

$$\Rightarrow \limsup_{m,n \to \infty} |f_m - f_n| = 0 \quad \text{a.e.}$$

Proof of (iii). Define

$$\varphi(A) := \lim_{k \to \infty} \int_A f_k$$

for $A \in \bigcup_{k=1}^{\infty} \mathcal{a}_k$ (field) If $A \in \bigcup_{k=1}^{\infty} \mathcal{a}_k$, $A \in \mathcal{a}_n$ for some n.

$$\int_A f_k = \int_A f_n \quad \forall k \geq n$$

Note (ii) $\Rightarrow$ (i) Assume f is L^1-bounded. Set

$$\tau := \inf\{n : |f_n| > \lambda\}$$

and $g = f^\tau$. CLAIM - g is uniformly integrable (in fact $g^* \in L^1$)
So by (ii), g converges a.e. But $f = g$ on the set

$$\{\tau = \infty\} = \{f^* \le \lambda\} \uparrow \{f^* < \infty\}$$

But $P(f^* \ge n) \le \|f\|_1/n \to 0$ as $n \to \infty$, so $P\{f^* < \infty\} = 1$.
Hence f converges a.e.

To show claim -

$$|g_n| = |f_{\tau \wedge n}| \le \lambda + |f_\tau|$$

$$\Rightarrow \quad g^* \le \lambda + |f_\tau|$$

$$\Rightarrow \quad Eg^* \le \lambda + \|f_\tau\|_1 \le \lambda + \|f\|_1 < \infty$$

Hence g^* integrable $\Rightarrow$

$$\forall n \quad \int_{|g_n| > b} |g_n| \le \int_{g^* > b} g^* \to 0 \text{ as } b \to \infty$$

9/29 MARTINGALES

(iii) $\mathfrak{f}$ u.i. martingale $\Rightarrow$ $\mathfrak{f}$ converges in L^1

LEMMA: Suppose $\mathcal{B}$ is a field and $\mathcal{A} = \sigma(\mathcal{B})$. Then $\mathcal{B}$ is "dense" in $\mathcal{A}$, i.e. if $\varepsilon > 0$ and $A \in \mathcal{A}$, then there exists $B \in \mathcal{B}$ such that $P(A \triangle B) \leq \varepsilon$

Proof. Let $\mathcal{M} := \{ A \in \mathcal{A} : \exists B \in \mathcal{B} \text{ s.t. } P(A \triangle B) \leq \varepsilon \} \supset \mathcal{B}$
Then $\mathcal{M}$ is a monotone class and so $\mathcal{M} \supset \sigma(\mathcal{B}) = \mathcal{A}$. Hence $\mathcal{A} = \mathcal{M}$.

(So if F $\mathcal{A}$-meas., int. $\exists$ G $\mathcal{B}$-meas, simple with $\|F - G\|_1 < \varepsilon$) ▨

Proof of (iii) Let

$$\varphi(B) := \lim_{k \to \infty} \int_B \mathfrak{f}_k$$

for all $B \in \mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ (field). (limit exists since if $B \in \mathcal{A}_n$, $\int_B \mathfrak{f}_k = \int_B \mathfrak{f}_n \ \forall k \geq n$)
φ is a real-valued set function

CLAIM: φ is countably additive on $\mathcal{B}$.

Proof. Assume $B_n \downarrow 0$. We want to show $\varphi(B_n) \to 0$.
$B_n \downarrow 0 \Rightarrow P(B_n) \downarrow 0$.

$$|\varphi(B_n)| \underset{\substack{\uparrow \\ \text{for } k\geqslant n}}{=} \left|\int_{B_n} f_k\right| \leqslant \int_{B_n\cap\{|f_k|>b\}} |f_k| + \int_{B_n\cap\{|f_k|\leq b\}} |f_k|$$

$$\leqslant \int_{|f_k|>b} |f_k| + bP(B_n)$$

$$\leq \sup_j \int_{|f_j|>b} |f_j| + bP(B_n) < \varepsilon$$

if b is large enough and n is large enough ▨

Extend φ to $\mathcal{A}$. φ is absolutely continuous w.r.t. P on $\mathcal{A}$. By the Radon-Nikodym theorem, $\exists$ $\mathcal{A}$-measurable integrable F s.t.

$$\varphi(A) = \int_A F \quad \forall A\in\mathcal{A}$$

So $\varphi(A_m) = \int_{A_m} F$ $\forall A_m \in \mathcal{A}_m$. But $\varphi(A_m) = \int_{A_m} f_m$

Hence $f_n = E(F|\mathcal{A}_n)$

Approx F by G, $\mathcal{B}$-meas. and simple. Then G is $\mathcal{A}_k$-mea. for some k, so $E(G|\mathcal{A}_n) = G$ for all large n. Hence $E(G|\mathcal{A}_n) \longrightarrow G$ a.e. in L^1. Now show $f_n = E(F|\mathcal{A}_n) = E(G|\mathcal{A}_n)$ ▨

Fix, as usual, $\mathcal{a}_1 \subset \mathcal{a}_2 \subset \dots$. Consider

① $UI := \{ f : f = (f_1, f_2, \dots)$ is a u.i. martingale $\}$

$$\|f\|_1 := \sup_n \|f_n\|_1$$

CLAIM: $UI \cong L^1(\Omega, \mathcal{a}_\infty, P)$ (isometrically isomorphic)

$$f \longmapsto f_\infty \quad (L^1\text{- limit})$$

$$f_\infty \longmapsto (f_n = E(f_n | \mathcal{a}_n))_{n \in \mathbb{N}}$$

② $M^p := \{ f : f = (f_1, \dots)$ L^p- bounded martingale $\}$

$$\|f\|_p := \sup \|f_n\|_p < \infty$$

CLAIM: $M^p \cong L^p(\Omega, \mathcal{a}_\infty, P) \quad 1 < p \leq \infty$

Let $f^* := \sup_n |f_n|$. Then $\|f\|_p \leq \|f^*\|_p \leq q \|f\|_p \; (1 < p \leq \infty)$

(If f is a u.i. martingale or an L^p- bounded martingale $(1 < p \leq \infty)$ then f can not converge to $f_\infty = 0$ unless $f = 0$.)

FLEXIBILITY OF MARTINGALES OVER L^p SPACES

Take $\Omega = [0,1)$. There exist σ-fields $\mathcal{a}_1 \subset \mathcal{a}_2 \subset \ldots$ which converge to $\mathcal{a}_\infty = \mathcal{B}$ (Borel sets) (e.g. dyadic intervals)

Write $f_n = \sum_{k=1}^{n} d_k$

nice functions –

SQUARE FUNCTION $S(f) = \left(\sum_{k=1}^{\infty} d_k^2\right)^{1/2}$

MAXIMUM FUNCTION $f^* = \sup |f_n|$

CONDITION SQUARE FNCT $s(f) = \left(\sum_{k=1}^{\infty} E(d_k^2 | \mathcal{a}_{k-1})\right)^{1/2}$

10/2 MARTINGALES

For $0<p<\infty$, let

$$\|f\|_p = \sup_n \|f_n\|_p = \sup_n \left(E|f_n|^p \right)^{1/p}$$

If $p=1$, we define

$$H^1 := \{ f : f \text{ martingale with } f^* \text{ integrable} \}$$

$$\|f\|_{H^1} := \|f^*\|_1$$

For arbitrary p, $0<p<\infty$, we can define

$$H^p := \{ f : f \text{ martingale}, \|f^*\|_p < \infty \}$$

(Banach space for $p \geq 1$).

$$H^p \cong L^p(\Omega, \mathcal{A}_\infty, P)$$

(since $\|f\|_p \leq \|f^*\|_p \leq q \|f\|_p$)

We now consider the square function

$$S(f) := \left(\sum_{k=1}^{\infty} d_k^2 \right)^{1/2}$$

① $\|S(f)\|_2 = \|f\|_2$

Proof. $f_n = \sum_{k=1}^{n} d_k \ , n \geq 1.$

Case 1. At least one of the d_k's is not square integrable.

$$d_n \notin L^2 \Rightarrow f_n \notin L^2 \Rightarrow \|f_n\|_2 = \infty \Rightarrow \|f\|_2 = \infty$$

But $S(f) \geq |d_n|$, so $d_n \notin L^2 \Rightarrow S(f) \notin L^2 \Rightarrow \|S(f)\|_2 = \infty.$

Case 2. All $d_n \in L^2$.

$$\|f_n\|_2^2 = E f_n^2 = E \sum_{j=1}^{n} \sum_{k=1}^{n} d_j d_k$$

$$= E \sum_{k=1}^{n} d_k^2 + 2E \sum_{k=2}^{n} \underbrace{\sum_{j=1}^{k-1} d_j}_{f_{k-1}} d_k$$

But

$$E f_{k-1} d_k = E\left[E(f_{k-1} d_k \mid a_{k-1})\right]$$

$$\underset{\substack{\uparrow \\ f_{k-1} \text{ indep of } d_k}}{=} E\left[f_{k-1} E(d_k \mid a_{k-1})\right] = 0$$

Hence $\|f_n\|_2^2 = E\, S_n^2(f) = \|S_n(f)\|_2^2$.

(For non-negative submartingales, $\|S(f)\|_2 \le \|f\|_2$)

▨

② $\lambda P\left(S_n(f) > \lambda\right) \le 2\|f_n\|_1$ (f L^1-bdd martingale)

(weak type (1,1))

$$\|S_n(f)\|_2 \le \|f_n\|_2 \quad \text{(strong type (2,2))}$$

for $1<p<2$, Marcinkiewicz inequality

$$\|S_n(f)\|_p \le c_p \|f_n\|_p$$

EXTRAPOLATION

Under certain conditions on f

(*) $c_p \|S(f)\|_p \le \|f^*\|_p \le C_p \|S(f)\|_p$ $\quad 0<p<\infty$

If $g_n := \sum_{k=1}^{n} \varepsilon_k d_k$, where $\varepsilon_k = \pm 1$, then $S(g) = S(f) \Rightarrow \|f^*\|_p \approx \|g^*\|_p$

Then $\|f\|_p \approx \|g\|_p$, so $\left\| \sum_{k=1}^{n} \varepsilon_k d_k \right\|_p \le c_p \left\| \sum_{k=1}^{n} d_k \right\|_p$ $\quad 1<p<\infty$, i.e.

The martingale differences are an unconditional basis for the space they span.

$1 \le p < \infty$: (*) holds for all martingales f

$1 < p < \infty$: 1966 Burkholder

$p = 1$: Burgess Davis 1970

$0 < p < \infty$: Burkholder - Gundy (for special martingales 1970)

"Special martingales" - $f_n = \sum_{k=1}^{n} d_k$, where

$$d_k = \underset{\substack{\uparrow \\ a_{k-1} \text{ measurable}}}{V_k} X_k$$

$X = (x_1, x_2, \dots)$ martingale difference sequence with

(i) $E(x_k^2 \mid a_{k-1}) = 1$ a.e.

(ii) $E(|x_k| \mid a_{k-1}) \geq \alpha$ a.e. $(\alpha > 0)$

Example: ① $|X_k| \equiv 1$ with $EX_k = 0$

i) $d = (d_1, d_2, \dots)$ is a seq. of indep. symmetrically distributed r.v.

ii) $d_n = n^{th}$ block of Walsh series

② $x_1, x_2, \dots$ indep, identically distributed with $EX_k = 0, EX_k^2 = 1$

Conditional square function $s(f) = \left[\sum_{k=1}^{\infty} E(d_k^2 \mid a_{k-1})\right]^{1/2}$

$f_n := \sum_{k=1}^{n} V_k X_k$ X_k as above

$$E(d_k^2 \mid a_{k-1}) = E(V_k^2 X_k^2 \mid a_{k-1}) = V_k^2 \, E(X_k^2 \mid a_{k-1}) = V_k^2$$

So $s(f) = \left(\sum_{k=1}^{\infty} v_k^2\right)^{1/2}$

Suppose $X = (X_1, X_2, \dots)$ martingale, τ stopping time

$f = X^{\tau}$.

$$f_n = \sum_{k=1}^{n} \underbrace{I(\tau \geq k)}_{v_k} x_k$$

Then $s(f) = \tau^{1/2}$

10/4 MARTINGALES

LEMMA: Let $\beta > 1$, $0 < \delta < \sqrt{\beta^2 - 1}$ Let $\varepsilon = \varepsilon(\beta, \delta)$ have the property that

$$\varepsilon \longrightarrow 0 \quad \text{as} \quad \beta \longrightarrow \infty$$
$$\varepsilon \longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow 0$$

For example, can take

$$\varepsilon = \frac{9\delta^2}{\beta^2 - \delta^2 - 1}$$

Then

$$P\left(S(f) > \beta\lambda,\ f^* \overset{\overset{\text{max}}{\downarrow}}{\vee} w^* \leq \delta\lambda\right) \leq \varepsilon P\left(S(f) > \lambda\right) \qquad \forall \lambda > 0$$

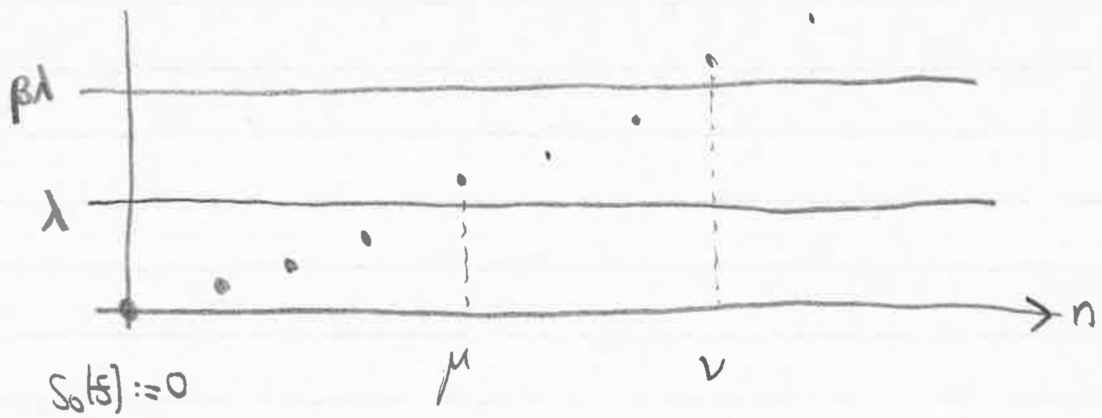
where w_n is $\mathcal{A}_{n-1}$-measurable and $|d_n(\omega)| \leq w_n(\omega)$ $\left(w^* = \sup |w_n|\right)$

⌈ Sometimes $|d_n|$ is $\mathcal{A}_{n-1}$-measurable. Then $w_n = |d_n|$ and $w^* = d^*$

$$|d_n| = |f_n - f_{n-1}| \leq 2f^*$$

so $f^* \vee w^* \leq 2f^*$ ⌋

Proof. Let $\mu := \inf\{n : S_n(f) > \lambda\}$, $\nu := \inf\{n : S_n(f) > \beta\lambda\}$



Let $\sigma = \inf\{n \geq 0 : |f_n| > \delta\lambda \text{ or } |w_{n+1}| > \delta\lambda\}$

(Recall $S_n^2(f) = \sum_{k=1}^{n} d_k^2$, so $\{\mu \leq n\} = \{\sum_{k=1}^{n} d_k^2 > \lambda^2\} \in \mathcal{A}_n$)

μ, ν, σ are stopping times with values in $\{0, 1, 2, \ldots, \infty\}$

$$d_\sigma^* \leq w_\sigma^* \leq \delta\lambda$$

Consider

$$g = {}^{\mu}f^{\nu \wedge \sigma}$$

(martingale started at μ, stopped at $\nu \wedge \sigma$, i.e.

$$g_n = \sum_{k=1}^{n} I(\mu < k \leq \nu \wedge \sigma)\, d_k$$)

g is a martingale since $I(\cdot)$ is $\mathcal{A}_{k-1}$ measurable.

Properties of martingales to be used:

(1) martingale property preserved under starting and stopping

(2) $\|S(f)\|_2 \le \|f\|_2$

CLAIM: $\text{LHS} \le P\left(S^2(g) > (\beta^2 - \delta^2 - 1)\lambda^2 \right)$

$f^* < w^* \le \delta\lambda$ corresponds to $\sigma = \infty$. If $\sigma = \infty$, then $g = {}^{\mu}f^{\nu}$

If $\nu < \infty$

$$S^2(g) = \sum_{k=1}^{\infty} I(\mu < k \le \nu)\, d_k^2$$

$$= S_\nu^2(f) - S_\mu^2(f)$$

$$\ge \beta^2\lambda^2 - (\lambda^2 + \delta^2\lambda^2)$$

$$\swarrow$$

since

$$S_\mu^2(f) = S_{\mu-1}^2(f) + d_\mu^2 \le \lambda^2 + \delta^2\lambda^2$$

Hence by Tchebyshev's inequality

$$\text{LHS} \le \frac{\|S(g)\|_2^2}{(\beta^2-\delta^2-1)\lambda^2} \le \frac{\|g\|_2^2}{(\beta^2-\delta^2-1)\lambda^2}$$

Case 1: $\sigma < \nu$ then $g^* \le 2\lambda\delta$

Case 2: $\sigma \ge \nu$ then $|g_n| \le 3\lambda\delta$

Hence $|g_n| \le 3\lambda\delta\, I(\mu < \infty)$ (if $\mu = \infty$, $g_n = 0$). So

$$\|g\|_2^2 \leq 9\delta^2\lambda^2 \, P(\mu < \infty)$$

$$= 9\delta^2\lambda^2 \, P(S(f) > \lambda)$$

Therefore LHS satisfies

$$\text{LHS} \leq \frac{9\delta^2\lambda^2}{(\beta^2-\delta^2-1)\lambda^2} P(S(f) > \lambda)$$

$$= \frac{9\delta^2}{\beta^2-\delta^2-1} P(S(f) > \lambda)$$

▨

COROLLARY: If $w^* \leq c d^*$, then

$$P(S(f) > \beta\lambda, \, f^* \leq \delta\lambda) \leq \underset{\substack{\uparrow \\ \text{different from before}}}{\varepsilon_0} P(S(f) > \lambda)$$

Let $\beta \to 0$

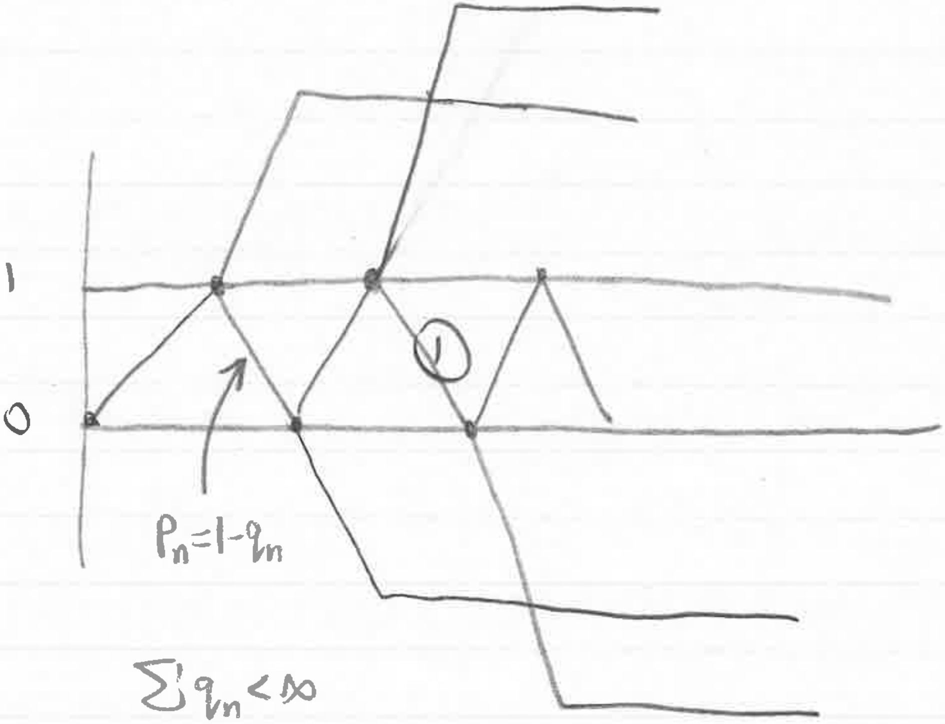
$$P(S(f) = \infty, \, f^* \leq \delta\lambda) = 0$$

Let $\lambda \to \infty$

$$P\left(S(f) = \infty,\ f^* < \infty \right) = 0$$

Then $\{f^* < \infty\} \subset \{S(f) < \infty\}$ a.e.

Can't drop w* arbitrarily in inequality



$\sum q_n < \infty$

$f^* < \infty$ everywhere
$S(f) = \infty$ for path ①

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dual inequality to previous one

$$P(f^* > \beta\lambda, S(f) \vee w^* \leq \delta\lambda) \leq \varepsilon P(f^* > \lambda), \quad \lambda > 0$$

(proof later) $\qquad (\varepsilon = \varepsilon(\beta, \delta) \to 0 \text{ as } \delta \to 0)$

Let $\Phi : [0,\infty] \to [0,\infty]$ be continuous and non-decreasing with $\Phi(0) = 0$. Suppose

$$\Phi(2\lambda) \leq c\,\Phi(\lambda) \quad \forall \lambda$$

examples -

(1) $\Phi(\lambda) = \lambda^p \quad 0 < p < \infty$
(2) $\Phi(\lambda) = \log(1+\lambda)$
(3) $\Phi(\lambda) = \lambda \log(1+\lambda)$
(4) any concave function (concave down)

LEMMA: f, g nonnegative measurable functions on a probability space (or finite measurable space) $(\Omega, \mathcal{A}, P)$. Suppose $\beta > 1, \delta > 0, \varepsilon > 0$ such that

$$P(g > \beta\lambda, f \leq \delta\lambda) \leq \varepsilon P(g > \lambda) \quad \lambda > 0$$

Let γ, η satisfy

$$\left.\begin{array}{l}\Phi(\beta\lambda) \le \gamma\, \Phi(\lambda) \\ \Phi(\delta^{-1}\lambda) \le \eta\, \Phi(\lambda)\end{array}\right. \quad (\forall \lambda > 0)$$

(If $\beta \le 2^k$, then $\Phi(\beta\lambda) \le \Phi(2^k\lambda) \le c^k\, \Phi(\lambda)$. Similarly for δ^{-1})

Assume also that $\gamma\varepsilon < 1$. Then

$$E\, \Phi(g) \le \frac{\gamma\eta}{1-\gamma\varepsilon}\, E\, \Phi(f)$$

To Show: $E\, \Phi(S(f)) \le c_1\, E\, \Phi(f^* \vee w^*)$

$$E\, \Phi(f^*) \le \underset{\underset{\text{depend on } \Phi}{\uparrow}}{c_2}\, E\, \Phi(S(f) \vee w^*)$$

Fix β, so then γ is fixed. Choose δ so small that $\varepsilon < 1/\gamma$. Then the two inequalities follow from the lemma

Proof of lemma:

$$(*) \quad E\, \Phi(f) = \int_0^\infty P(f > \lambda)\, d\Phi(\lambda)$$

Riemann-Stieltjes

or Lebesgue-Stieltjes

Let Φ denote the measure on the Borel subsets of $[0,\infty)$ satisfying

$$\Phi([a,b)) = \Phi(b) - \Phi(a)$$

Then

$$\Phi(f(w)) = \int_0^\infty I(f(w) > \lambda)\, d\Phi(\lambda)$$

Using Fubini's theorem we get (*)
By assumption

$$P(g > \beta\lambda) = P(g > \beta\lambda, f \leq \delta\lambda) + P(g > \lambda\beta, f > \delta\lambda)$$

$$\leq \varepsilon P(g > \lambda) + P(g > \lambda\beta, f > \delta\lambda)$$

$$\leq \varepsilon P(g > \lambda) + P(f > \delta\lambda)$$

Hence

$$E\,\Phi(\beta^{-1}g) \leq \varepsilon E\,\Phi(g) + E\,\Phi(\delta^{-1}f)$$

$$\leq \varepsilon E\,\Phi(g) + \eta E\Phi(f)$$

and so

$$E\Phi(g) = E\,\Phi(\beta\beta^{-1}g) \leq \gamma E\,\Phi(\beta^{-1}g) \leq \gamma\varepsilon E\Phi(g) + \gamma\eta E\Phi(f)$$

Then

$$(**) \quad (1-\gamma\varepsilon)\, E\,\Phi(g) \le \gamma\eta\, E\,\Phi(f)$$

(provided $E\,\Phi(g) < \infty$. But if $E\,\Phi(g) = \infty$, replace g by $g \wedge n$, obtaining $(**)$ with $g \wedge n$. Then let $n \to \infty$)

▨

(Consider case $\Phi(\lambda) = \lambda$)

Proof of dual inequality

Define
$$\mu := \inf\{n : |f_n| > \lambda\} \qquad f^* > \lambda \Leftrightarrow \mu < \infty$$
$$\nu := \inf\{n : |f_n| > \beta\lambda\} \qquad f^* > \beta\lambda \Leftrightarrow \nu < \infty$$
$$\sigma := \inf\{n \ge 0 : S_n(f) \ge \delta\lambda \text{ or } w_{n+1} > \delta\lambda\}$$

Let $g := {}^{\mu} f^{\nu \wedge \sigma}$.

$$S^2(g) \le S_\sigma^2(f) = \overbrace{S_{\sigma-1}^2(f)}^{\le \delta^2\lambda^2} + \overbrace{d_\sigma^2}^{\le w_\sigma^2} \le 2\delta^2\lambda^2$$

$$\text{LHS} \le P(\mu \le \nu < \infty,\ \sigma = \infty) \le P(g^* > (\beta - \delta - 1)\lambda)$$

$$\left(g_n = \sum_{k=1}^{n} I(\mu < k \le \nu \wedge \sigma)\, d_k \right)$$

$$\leq \frac{\|g\|_2^2}{(\beta-\delta-1)^2\lambda^2} = \frac{\|S(g)\|_2^2}{(\beta-\delta-1)^2\lambda^2}$$

weak L^2

$$\leq \frac{2\delta^2\lambda^2 P(\mu<\infty)}{(\beta-\delta-1)^2\lambda^2}$$

$$= \frac{2\delta^2}{(\beta-\delta-1)^2} P(S^* > \lambda)$$

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The Φ used in the previous inequalities has the growth condition

$$\Phi(f \vee g) \leq \Phi(f) + \Phi(g)$$

Therefore

$$E\,\Phi(S(f)) \leq c\,E\,\Phi(f^* \vee w^*) \leq c\,E\,\Phi(f^*) + c\,E\,\Phi(w^*)$$

THEOREM: Φ is as before plus convex (e.g. $\Phi(\lambda) = \lambda^p$, $1 \leq p < \infty$ or $\Phi(\lambda) = (\lambda+1)\log(\lambda+1)$). Then for any martingale f

$$c_1\,E\,\Phi(S(f)) \leq E\,\Phi(f^*) \leq c_2\,E\,\Phi(S(f))$$

In particular, for $1 \leq p < \infty$

$$c_p \|S(f)\|_p \leq \|f^*\|_p \leq C_p \|S(f)\|_p$$

LEMMA: Suppose $z_1, z_2, \ldots$ are non-negative measurable functions. If Φ is as before plus convex, then

$$E\,\Phi\left(\sum_{k=1}^{\infty} E(z_k \mid a_{k-1})\right) \leq \underset{\substack{\uparrow \\ \text{depends only upon growth constant of } \Phi}}{c}\,E\,\Phi\left(\sum_{k=1}^{\infty} z_k\right)$$

B. DAVIS DECOMPOSITION : d_k martingale diff. seq. of f.

$$y_k := d_k I(|d_k| \leqslant 2 d^*_{k-1})$$

$$z_k := d_k I(|d_k| > 2 d^*_{k-1})$$

$(d^*_0 := 0)$. Then $y_k + z_k = d_k$. Let

$$a_k := y_k - E(y_k | a_{k-1})$$

$$b_k := z_k + E(y_k | a_{k-1})$$

CLAIM: $$d = \underset{\uparrow}{a} + b$$

martingale difference seq $\Rightarrow$ b also martingale diff.

Let g, h be the martingales determined by a, b respectively. Then $f = g + h$.

$$|a_k| \leqslant |y_k| + E(|y_k| \,|\, a_{k-1}) \underset{\uparrow}{\leqslant} 4 d^*_{k-1} =: w_k \text{ (for } g \text{ mart.)}$$

$|y_k| \leqslant 2 d^*_{k-1}$ $\qquad$ $w^* = 4d^*$

Then

$$E(\Phi(S(g))) \leqslant c E \Phi(g^*) + c E \Phi(4d^*)$$
$$E(\Phi(g^*)) \leqslant c E \Phi(S(g)) + c E \Phi(4d^*)$$

If $d_k > 2d^*_{k-1}$, then

$$|d_k| + 2d^*_{k-1} \leq |d_k| + |d_k| = 2|d_k| \leq 2d^*_k$$

$$\Rightarrow |d_k| \leq 2(d^*_k - d^*_{k-1})$$

Hence

$$|z_k| \leq 2(d^*_k - d^*_{k-1})$$

and so $\sum_{n=1}^{\infty} |z_k| \leq 2d^*$ (telescoping seq))

Now $b_1 = z_1$ and for $k \geq 2$

$$b_k = z_k - E(z_k | a_{k-1})$$

$$\Rightarrow |b_k| \leq |z_k| + E(|z_k| \,|\, a_{k-1})$$

$$\Rightarrow \sum_{n=1}^{\infty} |b_n| \leq \sum_{n=1}^{\infty} |z_n| + \sum_{n=1}^{\infty} E(|z_n| \,|\, a_{n-1})$$

Hence

$$E\Phi(S(h)) \leq E\Phi\left(\sum_{k=1}^{\infty} |b_k|\right) \leq cE\Phi\left(\sum_{k=1}^{\infty} |z_k|\right) + cE\Phi\left(\sum_{k=1}^{\infty} E(|z_k| \,|\, a_{k-1})\right)$$

$$\underset{\underset{\text{lemma}}{\uparrow}}{\leq} c' E\Phi\left(\sum_{k=1}^{\infty} |z_k|\right) \leq cE\Phi(2d^*)$$

Also $h^* \leq \sum_{k=1}^{\infty} |b_k|$, so $E\Phi(h^*) \leq E\Phi\left(\sum_{k=1}^{\infty} |b_k|\right)$

Use: $S(f) \leq S(g) + S(h)$

$f^* \leq g^* + h^*$

Proof of theorem:

$$E\Phi(f^*) \leq E\Phi(g^*+h^*) \leq cE\Phi(g^*) + cE\Phi(h^*)$$

$$\leq c\left[cE\Phi(S(g)) + cE\Phi(4d^*)\right] + c_0 E\Phi(d^*)$$

$$\leq c_1 E\Phi(S(f)) + c_1 E\Phi(S(h)) + c_0 E\Phi(d^*)$$

$$\leq c_2 E\Phi(S(f)) + c_2 E\Phi(d^*)$$

$$\leq c_3 E\Phi(S(f))$$

LEMMA: (Neveu) $f, g \geq 0$ measurable. $\int_{g>\lambda} (g-\lambda)\,dP \leq \int_{g>\lambda} f$

$$\Longrightarrow E\Phi(g) \leq c\, E\Phi(f)$$

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LEMMA: $(z_1, z_2, \dots)$ non-negative, measurable functions. Φ convex with usual growth condition. Then

$$E\,\Phi\left(\sum_{k=1}^{\infty} E(z_k \mid a_{k-1})\right) \leq c\,E\,\Phi\left(\sum_{k=1}^{\infty} z_k\right)$$

Proof. Let $w_k := E(z_k \mid a_{k-1})$ and

$$Z_n := \sum_{k=1}^{n} z_k \quad \forall n \in \mathbb{N}^{\infty}$$

$$W_n := \sum_{k=1}^{n} w_k \quad \forall n \in \mathbb{N}^{\infty}$$

① To show - $\displaystyle\int_{W_\infty > \lambda} (W_\infty - \lambda) \leq \int_{W_\infty > \lambda} Z_\infty$

Define $\tau := \inf\{n \geq 0 : W_{n+1} > \lambda\}$. Note $W_\tau \leq \lambda$.

$$\int_{W_\infty > \lambda} (W_\infty - \lambda) \leq \int_{\tau < \infty} W_\infty - W_\tau \underset{\uparrow}{=} E\left[W_\infty - W_\tau\right]$$

$\tau = \infty \Rightarrow W_\infty = W_\tau$

$$\underset{\uparrow}{=} E\left[Z_\infty - Z_\tau\right]$$

$$E\left[W_\infty - W_\tau\right] = E\left[\sum_{k=1}^{\infty} I(\tau < k) w_k\right] = E\left[\sum_{k=1}^{\infty} E(I(\tau<k) z_k \mid a_{k-1}\right]$$

$$= E\left[\sum_{k=1}^{\infty} I(\tau < k) z_k\right] = E\left[Z_\infty - Z_\tau\right]$$

$$= \int_{\{\tau<\infty\}} Z_\infty - Z_\tau \leq \int_{\tau<\infty} Z_\infty$$

i.e. $$\int_{W_\infty>\lambda} (W_\infty - \lambda) \leq \int_{\tau<\infty} Z_\infty \leq \int_{W_\infty>\lambda} Z_\infty$$

② To show - $f, g \geq 0$ with $\int_{g>\lambda} g-\lambda \leq \int_{g>\lambda} f$ implies

$$E\,\Phi(g) \leq c\, E\,\Phi(f)$$

⌜ L^p case : $\underset{1<p<\infty}{p=1}$ obvious (let $\lambda \downarrow 0$) $\Phi(\lambda) = \lambda^p$

$$\lambda P(g>2\lambda) \leq \int_{g>2\lambda} g-\lambda \leq \int_{g>\lambda} g-\lambda \leq \int_{g>\lambda} f$$

⌟

convex, monotone increasing

$\Phi(0)=0$

③ LEMMA : $\Phi(\lambda) = \int_0^\lambda \varphi(t)\,dt$ where φ is non-negative, non-decreasing

Proof. Define for $x \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$, $0 \leq x < 1$,

$$f_n(x) = \frac{1}{2^n}\left[\Phi\left(\frac{k}{2^n}\right) - \Phi\left(\frac{k-1}{2^n}\right)\right]$$

Then $f = (f_1, f_2, \ldots)$ is a martingale on $[0,1)$ where

$$\mathcal{D}_n := \sigma\left(\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) ; \ k=1,2,\dots, 2^n \right)$$

Then $0 \leq f_n(x) \leq f_n(y)$ if $0 \leq x < y < 1$ (by convexity of Φ) and all are bounded above by $\Phi(2)-\Phi(1)$. Hence f is uniformly integrable so $f_n \to f_\infty$ a.e. and in L_1

$$\int_D f_n = \int_D f_\infty \qquad D \in \mathcal{D}_n$$

In particular

$$\Phi\left(\frac{k}{2^n}\right) = \int_0^{\frac{k}{2^n}} f_n = \int_0^{\frac{k}{2^n}} f_\infty$$

and so if $0 \leq \lambda < 1$,

$$\Phi(\lambda) = \int_0^{\lambda} f_\infty$$

Let $\varphi(t) = \limsup_{n\to\infty} f_n(t)$. Then φ is non-negative, non-decreasing, and $\varphi = f_\infty$ a.e.

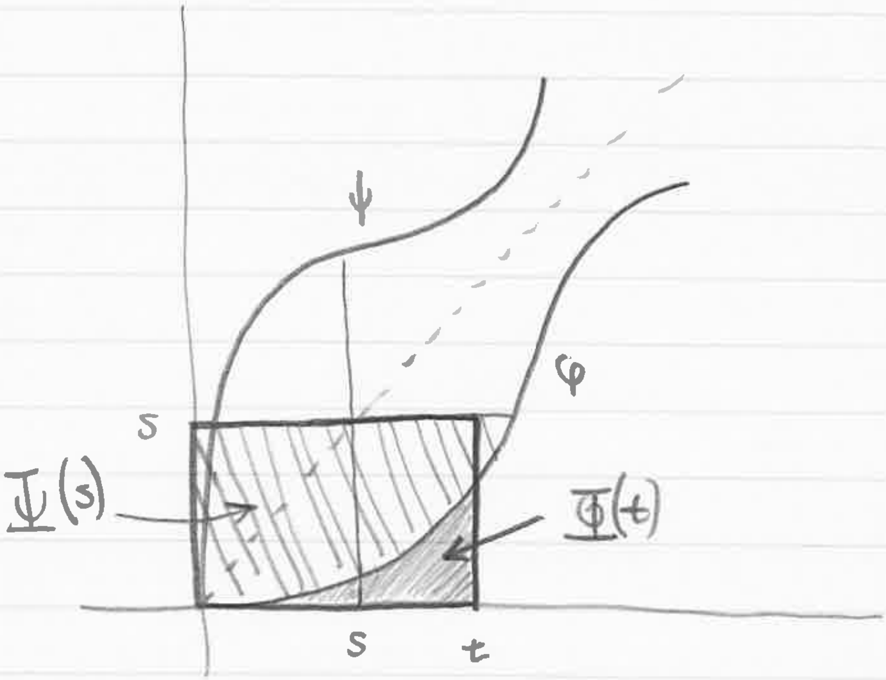
▨

(Same sort of argument will work for Φ absolutely continuous on $[0,1)$ with $\Phi(0)=0$)

Assume (WLOG) that φ is continuous, strictly increasing from $[0,\infty)$ onto $[0,\infty)$. Let ψ be the inverse of φ. Let

$$\Psi(\lambda) := \int_0^{\lambda} \psi(t)\,dt$$

W.H. Young : $st \leq \Phi(t) + \Psi(s) \quad \forall s,t \geq 0$



example : $\varphi(t) = t^{p-1} \qquad \psi(t) = t^{1/(p-1)}$

$$\Phi(\lambda) = \frac{\lambda^p}{p} \qquad \Psi(\lambda) = \frac{\lambda^q}{q} \qquad \left(1/p + 1/q = 1\right)$$

$$st \leq \frac{1}{p}t^p + \frac{1}{q}s^q$$

④ CLAIM: If $\varepsilon > 0$ then there is an $\alpha > 0$ s.t.

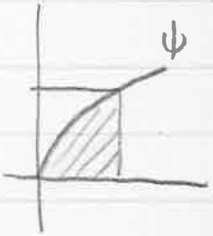
$$s\varphi(t) \leq \varepsilon\, \Phi(t) + \alpha\, \Phi(s)$$

Proof - Let $\varepsilon > 0$. [Claim: $\varphi(2t) \leq a\,\varphi(t)$ since Φ satisfies growth condition]

$$s\varphi(t) \leq a^n\left(\frac{s\varphi(t)}{a^n}\right) \leq a^n\left[\Phi(s) + \Psi\left(\frac{\varphi(t)}{a^n}\right)\right]$$

$$\leq a^n\left[\Phi(s) + \Psi\left(\varphi\left(\frac{t}{2^n}\right)\right)\right] \qquad \left(\varphi(t) \leq a^n\varphi\left(\frac{t}{a^n}\right)\right)$$

$$\leq a^n\left[\Phi(s) + \varphi\left(\frac{t}{2^n}\right)\psi\left(\varphi\left(\frac{t}{2^n}\right)\right)\right]$$



$$= a^n\left[\Phi(s) + \frac{t}{2^n}\,\varphi\left(\frac{t}{2^n}\right)\right]$$

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CLAIM: If $\varepsilon > 0$, then there is an $\alpha > 0$ such that

$$s\varphi(t) \le \varepsilon\, \Phi(t) + \alpha\, \Phi(s)$$

Proof (continued): φ has a growth condition similar to Φ, i.e. $\varphi(2\lambda) \le a\,\varphi(\lambda)$ –

$$2\lambda\,\varphi(2\lambda) \le \int_{2\lambda}^{4\lambda} \varphi(t)\,dt \le \Phi(4\lambda) \le c^2\,\Phi(\lambda)$$

$$\le c^2 \lambda\varphi(\lambda)$$

Hence $\varphi(2\lambda) \le {}^{c^2}/_{2\lambda}\, \varphi(\lambda)$

Now

$$s\varphi(t) = a^n\left[s\,\frac{\varphi(t)}{a^n}\right] \le a^n\left[\Phi(s) + \Psi\left(\frac{\varphi(t)}{a^n}\right)\right]$$

$$\le a^n\left[\Phi(s) + \frac{\varphi(t)}{a^n}\,\psi\left(\frac{\varphi(t)}{a^n}\right)\right]$$

$$\le a^n\,\Phi(s) + \varphi(t)\,\psi\left(\varphi\left(\frac{t}{2^n}\right)\right)$$

$$= a^n\,\Phi(s) + \varphi(t)\,\frac{t}{2^n}$$

$$\le a^n\,\Phi(s) + \frac{2a}{2^n}\,\frac{t}{2}\,\varphi\left(\frac{t}{2}\right)$$

$$\leq \quad a^n \Phi(s) + \frac{a}{2^{n-1}} \Phi(t)$$

$$\leq \quad a^n \Phi(s) + \varepsilon \Phi(t)$$

if n is chosen s.t. $a/2^{n-1} < \varepsilon$.

▨

Want to show

$$\int_{g>\lambda} (g-\lambda) \leq \int_{g>\lambda} f \Rightarrow E\Phi(g) \leq c\, E\, \Phi(f)$$

Now

$$\Phi(\lambda) = \int_0^\lambda \varphi(t)\,dt \overset{\text{integration by parts}}{=} t\varphi(t)\Big|_0^\lambda - \int_0^\lambda t\, d\varphi(t)$$

$$= \lambda\varphi(\lambda) - \int_0^\lambda t\, d\varphi(t)$$

$$= \int_0^\lambda (\lambda - t)\, d\varphi(t)$$

and so we have

$$\Phi(g) = \int_0^\infty (g-t)\, I(g>t)\, d\varphi(t)$$

$$E\,\Phi(g) = E\int_0^\infty (g-t)\, I(g>t)\, d\varphi(t) = \int_0^\infty \int_{g>t} (g-t)\,dP\, d\varphi(t)$$

$$\underset{\nwarrow\,\text{assumption}}{\le} \int_0^\infty \int_{g>t} f \, d\varphi(t) = \int_\Omega f \int_0^g d\varphi(t)$$

$$\le E\left[f \varphi(g) \right]$$

$$\overset{\text{lemma}\searrow}{\le} \varepsilon E \Phi(g) + \alpha E \Phi(f)$$

If $E\Phi(g) < \infty$, then we get

$$(1-\varepsilon) E \Phi(g) \le \alpha E \Phi(f)$$

If $E\Phi(g) = \infty$, replace g by $g \wedge n$ and use monotone convergence theorem

▨

Application of $\underline{\Phi}$-inequality

Suppose $X = (X_t : 0 \le t \le 1)$ is a right-continuous martingale, i.e.

(i) X_t is integrable, $\mathcal{A}_t$-measurable

(ii) $E(X_t \mid \mathcal{A}_s) = X_s$ a.e. $0 \le s < t \le 1$

(iii) For all ω, the mapping $t \longrightarrow X_t(\omega)$ is a right continuous function.

Consider a partition of $[0,1]$

$$0 = t_{j0} \le t_{j1} \le \dots \le t_{jn} \le 1 \quad \left(\begin{array}{l} \forall n, \text{but } t_{jn_0} = 1 \\ \text{for some } n_0 \text{ and hence} \\ \text{all } n > n_0 \end{array} \right)$$

Want the norm of the partition $\to 0$ as $j \to \infty$. Let

$$f_{jn} := X_{t_{jn}}$$

$f_j := (f_{j1}, f_{j2}, \dots)$ is a martingale. Let $S_j := S(f_j)$

Cath. Doléan 1969: $\{S_j\}$ converges in probability

THEOREM: $\{S_j\}$ converges in L^1 if and only if $X^* \in L^1$

(Recall $X^* = \sup\limits_{0 \le t \le 1} |X_t| \underset{\substack{\uparrow \\ \text{right continuity}}}{=} \sup\limits_{t \in \mathbb{Q}} |X_t|$ (hence measurable))

Proof. Assume $X^* \in L^1$. Then there is a Φ with the properties

(1) $$\Phi(\lambda) = \int_0^\lambda \varphi(t)\,dt$$

where φ is continuous strictly increasing from $[0,\infty)$ onto $[0,\infty)$

(2) $\Phi(2\lambda) \le c\,\Phi(\lambda)$

(3) $\frac{1}{\lambda}\Phi(\lambda) \to \infty$ as $\lambda \to \infty$

(4) $E\,\Phi(X^*) < \infty$

(Choose φ such that

$$E\,\Phi(X^*) = \int_0^\infty P(X^* > \lambda)\,\varphi(\lambda)\,d\lambda < \infty$$

[assuming $\int_0^\infty P(X^*>\lambda)\,d\lambda < \infty$] with the growth condition)

Now

$$E\,\Phi(S_j) \leq c\,E\,\Phi(f_j^*) \underset{\overset{\uparrow}{f_j^* \leq X^*}}{\leq} c\,E\,\Phi(X^*) < \infty$$

and so

$$\sup_j E\,\Phi(S_j) < \infty$$

$\Longrightarrow$ (S_j) is uniformly integrable

But U.I. with convergence in probability implies L^1 convergence.

Now suppose (S_j) converges in L^1. Then $\sup E S_j =: K < \infty$

$$E f_j^* \leq c\,E S_j \leq cK < \infty$$

$$\Longrightarrow E \liminf f_j^* \leq \liminf E f_j^* < \infty$$

claim: $X^* = \liminf f_j^*$

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$$s(f) := \left[\sum_{k=1}^{\infty} E(d_k^2 \mid \mathcal{A}_{k-1}) \right]^{1/2}$$ conditional square function

Suppose $x = (x_1, x_2, \ldots)$ is a martingale difference sequence with

$$E(x_k^2 \mid \mathcal{A}_{k-1}) = 1 \quad \text{a.e.}$$

(e.g. $x_1, x_2, \ldots$ independent with $EX_k = 0$, $EX_k^2 = 1$). $X_n := x_1 + \ldots + x_n$
Let $f = X^{\tau}$ $\left(f_n = X_{\tau \wedge n} = \sum_{k=1}^{n} \underbrace{I(\tau \geq k) x_k}_{d_k} \right.$. Then

$$s(f) = \tau^{1/2}$$

$$\left(E(d_k^2 \mid \mathcal{A}_{k-1}) = I(\tau \geq k) E(x_k^2 \mid \mathcal{A}_{k-1}) = I(\tau \geq k) \right)$$

THEOREM: (1) $\| S(f) \|_p \leq c_p \| s(f) \|_p \qquad 0 < p \leq 2$

(2) $\| s(f) \|_p \leq C_p \| S(f) \|_p \qquad 2 \leq p < \infty$

(3) $\| s(f) \|_p \leq c_p \| f \|_p \qquad 2 \leq p < \infty$

(4) $\| f^* \|_p \leq c_p \| s(f) \|_p \qquad 0 < p \leq 2$

(for $p = 2$, $\| S(f) \|_2 = \| s(f) \|_2 = \| f \|_2$)

COROLLARY 1: Assume $E(x_k | a_{k-1}) = 0$ $\forall k$ and $E(x_k^2 | a_{k-1}) = 1$. If τ is a stopping time and if $E\tau^{1/2} < \infty$, then $EX_\tau = 0$

Proof. $E\tau^{1/2} < \infty \Rightarrow \tau$ is finite a.e.

$$0 = Ef_n = EX_{\tau \wedge n} \longrightarrow EX_\tau$$

$\uparrow$ $\|f^*\|_1 \leq c_p \|s(f)\|_1 = c_p \|\tau^{1/2}\|_1 < \infty$
(so Dominated Convergence Th^m holds)

COROLLARY 2: Under the same assumptions with

$$\tau := \inf\{n : X_n > 0\}$$

then $E\tau^{1/2} = 0$.

Proof. $EX_\tau > 0$ by definition of τ

Recall - $$E\,\Phi\left(\sum_{k=1}^{\infty} E(z_k | a_{k-1})\right) \leq c\,E\,\Phi\left(\sum_{k=1}^{\infty} z_k\right)$$

$\uparrow$ convex, growth condition

Proof of theorem:

$$E\,\Phi(s^2(f)) = E\,\Phi\left(\sum_{k=1}^{\infty} E(d_k^2 | a_{k-1})\right)$$

Take $\Phi(\lambda)=\lambda^p$ for $p \geq 1$. This gives (2). By previous inequalities (2) $\Rightarrow$ (3).

Lemma: $\Phi : [0,\infty) \to [0,\infty)$ concave with $\Phi(0+)=\Phi(0)=0$. Let $\Phi(\infty) := \lim_{\lambda\to\infty} \Phi(\lambda)$. (Since concave growth condition is automatically satisfied -

$$\frac{\Phi(2\lambda)+\Phi(0)}{2} \leq \Phi(\lambda)$$

$$\Rightarrow \Phi(2\lambda) \leq 2\,\Phi(\lambda)$$)

Then

$$(*) \quad E\,\Phi\Big(\sum_{k=1}^{\infty} z_k\Big) \leq 2\,E\,\Phi\Big(\sum_{k=1}^{\infty} E(z_k \mid \mathcal{A}_{k-1})\Big)$$

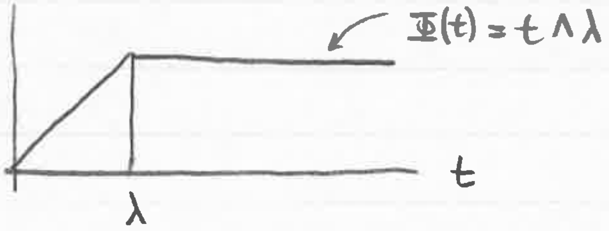
($z_k \geq 0$ measurable)

$$Z_n = \sum_{k=1}^{n} z_k$$

Proof. $w_k := E(z_k \mid \mathcal{A}_{k-1})$. $W = W_\infty = \sum_{k=1}^{\infty} w_k$, $Z = Z_\infty = \sum_{k=1}^{\infty} z_k$

Let $\lambda > 0$.

Special case of Φ



To show: $E(Z \wedge \lambda) \leq 2\,E(W \wedge \lambda)$

Let $\tau := \inf\{n \geq 0 : W_{n+1} > \lambda\}$ (stopping time). Note that $W_\tau \leq \lambda$. Now

$$Z \wedge \lambda \leq Z_\tau + \lambda\, I(\tau < \infty)$$

As before, $E Z_\tau = E W_\tau \underset{\substack{\uparrow \\ W_\tau \leq W}}{\leq} E[W \wedge \lambda]$. Also

$$E[\lambda\, I(\tau < \infty)] \underset{\substack{\uparrow \\ \{\tau < \infty\} \Leftrightarrow \{W > \lambda\}}}{\leq} E[W \wedge \lambda]$$

Hence $E[Z \wedge \lambda] \leq E[W \wedge \tau] + E[W \wedge \lambda] = 2 E[W \wedge \lambda]$.

CLAIM - $\Phi(\lambda) = \int_0^\lambda \varphi(t)\,dt$ (where $\varphi \geq 0$ and non-increasing)

$$\underset{\substack{\uparrow \\ \text{later}}}{=} \lambda \varphi(\infty) + \int_0^\infty (\lambda \wedge t)\, \underset{\substack{\uparrow \\ \text{positive measure}}}{d[-\varphi(t)]}$$

If the claim holds, then

$$E\, \Phi(Z) = \varphi(\infty)\, E(Z) + \int_0^\infty E(Z \wedge t)\, d[-\varphi(t)]$$

Then (*) holds by using the special case

To show: $\Phi(\lambda) = \lambda \varphi(\infty) + \int_0^\infty (\lambda \wedge t)\, d[-\varphi(t)]$

WLOG $\varphi(\infty) = 0$. Then

$$\Phi(\lambda) = \lim_{\substack{a\to 0\\ b\to\infty}} \int_a^b \varphi(t)\, d(\lambda\wedge t)$$

$$= \lim_{\substack{a\to 0\\ b\to\infty}} \left[\varphi(t)(\lambda\wedge t)\Big|_a^b + \int_a^b (\lambda\wedge t)\, d[-\varphi(t)] \right]$$

$$= \lim_{\substack{a\to 0\\ b\to\infty}} \left[\varphi(b)\lambda - \varphi(a)(\lambda\wedge a) + \int_a^b (\lambda\wedge t)\, d[-\varphi(t)] \right]$$

$(\varphi(a)\, a \leq \Phi(a) \to 0$ as $a \to 0$; $\varphi(b) \to \varphi(\infty) = 0)$

$$= \int_0^\infty (\lambda\wedge t)\, d[-\varphi(t)]$$

(Assumption $\varphi(\infty)=0$ not really necessary)

(To show (4) want to show $E[(S^*)^2\wedge\lambda] \leq 5\, E[S^2(S)\wedge\lambda]$
and use $\Phi(p) = \lambda^p$ for $p \leq 1$)

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7 homework $\Rightarrow E\,\underset{\nwarrow\text{ concave}}{\Phi}\left((f^*)^2\right) \le 5\,E\,\Phi\left(s^2(f)\right)$

$$\Rightarrow \|f^*\|_p \le c_p \|s(f)\|_p \qquad 0<p\le 2$$

Can use 8. to show that for any Φ (with growth condition) cont. non-decreasing $\Phi(0)=0$

(*) $$E\,\Phi(f^*) \le c\,E\,\Phi\left(s(f)\vee d^*\right)$$

$$\Rightarrow \|f^*\|_p \le c_p \|s(f)\vee d^*\|_p \qquad 0<p<\infty$$

Rosenthal (1970 Is.J.Math) : $d=(d_1,d_2,\dots)$ independent seq., $Ed_k=0$, $Ed_k^2<\infty$. For $2\le p<\infty$

$$\|f\|_p^p \le c_p\left(\sum_{k=1}^{\infty} E d_k^2\right)^{p/2} + c_p\sum_{k=1}^{\infty} E|d_k|^p$$

$$\|f\|_p^p \ge C_p\left(\sum_{k=1}^{\infty} E d_k^2\right)^{p/2} + C_p\sum_{k=1}^{\infty} E|d_k|^p$$

Proof: By (*) $E\,\Phi(f^*) \le c\,E\,\Phi(s(f)) + c\,E\,\Phi(d^*)$

Also

$$\Phi(d^*) \le \sum_{k=1}^{\infty} \Phi(|d_k|)$$

(since $\Phi(|d_n|) \le \sum_{k=1}^{\infty} \Phi(|d_k|$, now take sup). Then

$$E\,\Phi(f^*) \le c\,E\,\Phi(s(f)) + c \sum_{k=1}^{\infty} E\,\Phi(|d_k|)$$

Now use the fact that by independence $E(d_k^2 | a_{k-1}) = E d_k^2$ with $\Phi(\lambda) = \lambda^p$

▨

(Doob 1954 TAMS)

ANALOGUES

f martingale	u harmonic function	$u(B_t)$ harmonic fct Brownian motion
f^* maximal function		u^* Brownian maximal fct.
$S(f)$ square function		$S(u)$ Brownian square fct
g transform of f	g conjugate harmonic fct.	

The idea of Brownian motion connects the theory of martingales to the theory of harmonic functions.

DEFINITION: R open, connected in $\mathbb{R}^n$. u is harmonic if it has continuous second partial derivatives and

$$\Delta u = 0$$

$$\left(\text{i.e. } \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0 \right) \quad x = (x_1, x_2, \ldots, x_n)$$

examples:

(1) $ax_1 + b$ harmonic in $\mathbb{R}^n$ $(n \geq 1)$

(2) $x_1^2 - x_2^2$ harmonic in $\mathbb{R}^n$ $(n \geq 2)$

(3) $\log |x|$ harmonic in $\mathbb{R}^2 - \{0\}$

(4) real part of an analytic function harmonic in $\mathbb{R}^2$

(5) $\frac{1}{|x|^{n-2}}$ harmonic in $\mathbb{R}^n - \{0\}$ $(n \geq 3)$

(6) any derivative of a harmonic function is harmonic

(7) any Poisson integral

notation: $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty) = \{(x,y) : x \in \mathbb{R}^n, y > 0\}$ $n \geq 1$

Suppose f is integrable on $\mathbb{R}^n$. The Poisson integral of f is

$$u(x,y) := c_n \int_{\mathbb{R}^n} \frac{y f(s)\, ds}{\left(|x-s|^2 + y^2\right)^{\frac{n+1}{2}}} \qquad (x,y) \in \mathbb{R}^{n+1}_+$$

where c_n is chosen so that

$$\int_{\mathbb{R}^n} \frac{c_n y}{\left(|x|^2 + y^2\right)^{\frac{n+1}{2}}}\, dx = 1$$

$n=1$ case - $u(x,y)$ is harmonic

well-defined since f integrable

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \underbrace{\frac{y}{(x-s)^2+y^2}}_{} f(s)\, ds$$

This is harmonic for fixed s if we can show y/x^2+y^2 is harmonic

CLAIM:

$$\Delta u = \frac{1}{\pi}\int_{-\infty}^{\infty} \underset{\overset{\shortparallel}{0}}{\left[\Delta\left(\frac{y}{(x-s)^2+y^2}\right)\right]} f(s)\,ds = 0$$

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$$\Delta |x|^p = p(p+n-2)\,|x|^{p-2}$$

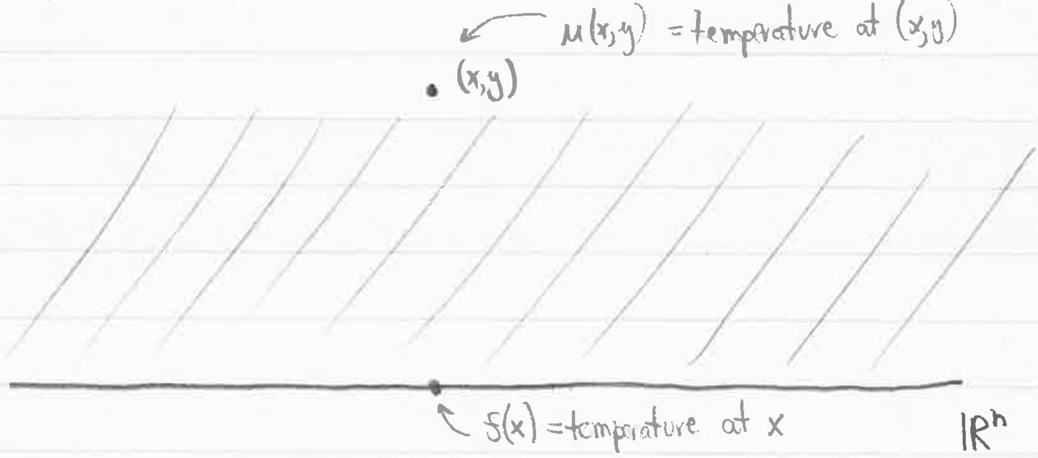
in $\mathbb{R}^n - \{0\}$ for $n \geq 1$, $p \in \mathbb{R}$

So if $n \geq 3$ and $p = -(n-2)$, then

$$\Delta \frac{1}{|x|^{n-2}} = 0$$

Note that $\Delta |x|^p > 0$ if $x = 0$ and $n \geq 2$, $p > 0$.

Physical interpretation



$$u(x,y) = \text{P.I. of } f = \int_{\mathbb{R}^n} \frac{c_n\, y\, f(s)}{(|x-s|^2+y^2)^{n+1/2}}\, ds \qquad c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}$$

= temperature at (x,y)

$$\| u(\cdot,y) \|_{L^1} = \int_{\mathbb{R}^n} |u(x,y)|\, dx \leq \int_{\mathbb{R}^n} |f(s)| \underbrace{\int_{\mathbb{R}^n} \frac{c_n y}{(|x-s|^2+y^2)^{n+1/2}}\, dx}_{=1}\, ds$$

$$= \|f\|_{L^1} \quad \left(\text{In fact } \| u(\cdot,y) - f \|_{L^1} \to 0 \right)$$

Poisson integral corresponds to the uniform integrable martingale (indexed by y)

Conjugate harmonic functions

$$R \subset \mathbb{R}^2 \approx \mathbb{C}$$

unique up to additive constant

If u is harmonic in $\mathbb{R}$, then v is a conjugate of u if $u+iv$ is analytic in $\mathbb{R}$ (always exists if R is simply connected)

Kolmogorov - Showed that if u,v are conjugate in D (unit disk) with $v(0)=0$, then for $0<r<1$

$$\forall \lambda > 0 \qquad \lambda\, m\left(\{\theta : |v(re^{i\theta})| > \lambda\}\right) \leq K \int_0^{2\pi} |u(re^{i\theta})|\, d\theta$$

Lebesgue measure on $[0,2\pi)$ independent of u,v,r,λ

(weak-type inequality) Another version of this is as follows: Take $f \in L^1(0,2\pi)$

$$\tilde{f}(\theta) := \frac{1}{\pi}\int_0^{2\pi} f(t) \cot\left(\frac{\theta-t}{2}\right) dt$$

$$= \lim_{\varepsilon \to 0} \int_{|\theta - t| > \varepsilon} f(t) \cot\left(\frac{\theta - t}{2}\right) dt$$

(limit exists a.e.) $\tilde{f}$ is called the conjugate function of f.

$$u(re^{i\theta}) = PI\, f = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} f(t)\, dt$$

$$\begin{tikzcd} f \arrow[r] & u \arrow[r] & v \arrow[r] & \tilde{f} \\ & \text{harmonic conjugate with } v(0)=0 \arrow[ur] & & \end{tikzcd}$$

$$\tilde{f} = \lim_{r \to 1} v(re^{i\theta})$$

v is not necessarily the Poisson integral of $\tilde{f}$

Kolmogorov inequality becomes

$$\lambda\, m(\{\theta : |\tilde{f}(\theta)| > \lambda\}) \leq K \int_0^{2\pi} |f(\theta)|\, d\theta$$

Davis -

$$K = \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} - \frac{1}{9^2} - \dots}$$

Square function for u = square function for v

Martingale transform inequalty

$g_n := \sum_{k=1}^{n} v_k d_k \qquad \lambda P(g^* > \lambda) \le 2\|f\|_1$

$\uparrow$ a_{k-1} meas. $|v_k| \le 1$

(d_k Martingale diff. seq of f)

If $v_k = \pm 1$, then $S(g) = S(f)$

M. Riesz - $1 < p < \infty$

$$\int_0^{2\pi} |v(re^{i\theta})|^p d\theta \le c_p \int_0^{2\pi} |u(re^{i\theta})|^p d\theta$$

(doesn't hold for $0 < p \le 1$)

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$$\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n , y > 0\}$$

M. Riesz 1949
Horvath 1953
Stein-Weiss 1960 (Acta. Math)

Suppose $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ is harmonic. Suppose $v_1, \dots, v_n$ are also harmonic on $\mathbb{R}^{n+1}_+$. We say $v = (v_1, \dots, v_n)$ is conjugate to u if

Generalized Cauchy-Riemann equations

$$\begin{cases} \dfrac{\partial u}{\partial x_k} = \dfrac{\partial v_k}{\partial y} & 1 \leq k \leq n \\[2ex] \dfrac{\partial v_k}{\partial x_j} = \dfrac{\partial v_j}{\partial x_k} & 1 \leq j \leq k \leq n \\[2ex] \dfrac{\partial u}{\partial y} + \dfrac{\partial v_1}{\partial x_1} + \dots + \dfrac{\partial v_n}{\partial x_n} = 0 \end{cases}$$

Let $U : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ be harmonic. Let $u = \dfrac{\partial U}{\partial y}$. Let

$$v_i = \frac{\partial U}{\partial x_i}$$

So

$$\nabla U = \text{grad } U = \left(\frac{\partial U}{\partial y}, \frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_n} \right)$$

Then $(v_1, \dots, v_n)$ is harmonic to u.

Suppose $D = \{ z = x+iy = re^{i\theta} : r<1 \}$. Let $F = u+iv$ be analytic in D. Define for $0<p<\infty$

$$\|F\|^p_{H^p} = \sup_{0\le r<1} \int_0^{2\pi} |F(re^{i\theta})|^2 \, d\theta$$

Now set

$$\underset{\substack{\uparrow \\ \text{Hardy}}}{H^p} = \{ F : F \text{ analytic in } D, \ \|F\|_{H^p} < \infty \}$$

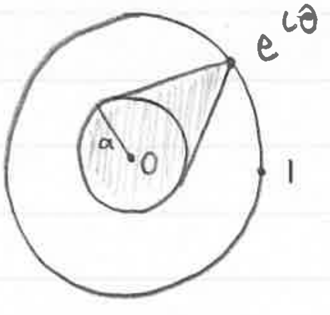
Suppose u is "L^p-bounded" harmonic function, i.e.

$$\sup_{r<1} \int_0^{2\pi} |u(re^{i\theta})|^p \, d\theta < \infty$$

Unfortunately, for $p<1$, this does not imply that $u(re^{i\theta})$ converges in L^p-norm as $r \to 1$. However, if $F \in H^p$, then $F(re^{i\theta})$ does converge in L^p-norm as $r \to 1$. If both u and its conjugate v is L^p-bounded, then $F \in H^p$ (and vice-versa)

Maximal function of u

$\Gamma_a(\theta)$ = interior of the smallest convex set containing $e^{i\theta}$ and $|z| = a$



$$N_a(u)(\theta) := \sup\{|u(z)| : z \in \Gamma_a(\theta)\}$$

(non-tangential maximal function of u)   Function on $[0, 2\pi)$

<u>**THEOREM**</u>: If u is harmonic in D and v is conjugate to u with $v(0) = 0$, then for $0 < p < \infty$

$$(*) \qquad \|N_a(v)\|_p \le c_{p,a} \|N_a(u)\|_p$$

More generally, if Φ is a general Φ-function (i.e. $\Phi : [0,\infty] \to [0,\infty]$ is non-decreasing, continuous, $\Phi(0) = 0$, $\Phi(2\lambda) \le c\,\Phi(\lambda)$, $\lambda > 0$) then

$$\int_0^{2\pi} \Phi(N_a(v)(\theta))\,d\theta \le C_a \int_0^{2\pi} \Phi(N_a(u)(\theta))\,d\theta$$

(also holds for $\mathbb{R}^{n+1}_+$)   [Unit disk, $\mathbb{R}^2_+$ TAMS 1971 – $\mathbb{R}^{n+1}_+$, Φ Studia 1972]

$$N_0(u) := \sup\{|u(re^{i\theta})| : 0 \le r < 1\}$$

(radial maximal function). The above theorem is true for $a=0$ (ineq. $*$) but the Φ inequality is not generally true for $a=0$ ↖ Fefferman-Stein 1972

THEOREM: If $F = u + iv$ is analytic in D, then for $0<a<1$

$$F \in H^p \Leftrightarrow N_a(u) \in L^p$$

for every $0<p<\infty$

Proof

$$|F(re^{i\theta})| \leq |u(re^{i\theta})| + |v(re^{i\theta})|$$

$$\leq N_a(u)(\theta) + N_a(v)(\theta)$$

$$\Rightarrow |F(re^{i\theta})|^p \leq 2^{p-1}\left(N_a^p(u)(\theta) + N_a^p(v)(\theta)\right)$$

$$\Rightarrow \sup_{r<1}\int_0^{2\pi} |F(re^{i\theta})|^p \leq 2^{p-1}\left(\|N_a(u)\|_p^p + \|N_a(v)\|_p^p\right)$$

$$\leq c\,\|N_a(u)\|_p^p$$

Hence

$$\|F\|_{H^p} \leq c_{p,a}\,\|N_a(u)\|_p$$

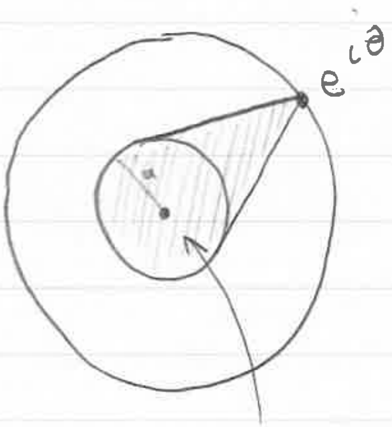
But moreover,

$$C_{p,a}\|N_a(u)\|_p \underset{\nearrow}{\leq} \|F\|_{H^p}$$

Hardy-Littlewood Acta Math 1930

10/25 MARTINGALES

$N_a(u)$ corresponds to f^*. What corresponds to $S(f)$? $A_a(u)$



(Lusin Bull. Calcutta Math Soc 1930)

$\Gamma_a(\theta)$ (Stoltz Domain)

$$A_a(u)(\theta) := \left[\iint_{\Gamma_a(\theta)} |\nabla u(x,y)|^2 \, dx\, dy\right]^{1/2} \quad \forall\theta$$

(possibly $= +\infty$)

$\left(\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}\right)\right)$

Note that $A_a(u+c) = A_a(u)$ where c is a constant.
Suppose v is conjugate to u. Then $A_a(u) = A_a(v)$
For by the Cauchy-Riemann eq.

$$|\nabla u|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2 = |\nabla v|^2$$

References: TAMS 1971
Acta 1972
Studia Math 1972

If $F = u + iv$ is analytic, then

$$A_a(u)(\theta) = \Big(\iint_{\Gamma_a(\theta)} |F'(z)|^2 \, dz \Big)^{1/2}$$

Since

$$F'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \Longrightarrow |F'(z)| = \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial x}\Big)^2$$

$$= \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2 = |\Delta u|^2$$

If F is also univalent on $\Gamma_a(\theta)$, then $A_a^2(u)(\theta)$ is the area of the image of $\Gamma_a(\theta)$ under F. So if F is univalent and bounded, then $F(\Gamma_a(\theta))$ has finite area, so $A_a^2(u)(\theta)$ is finite.

THEOREM: If Φ is a Φ-function as before, then

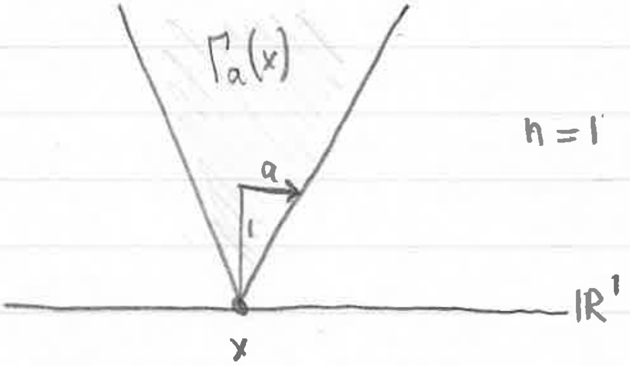
$$c_a \int_0^{2\pi} \Phi\big(A_a(u)(\theta)\big)\, d\theta \underset{\substack{\uparrow \\ \text{Always true}}}{\leq} \int_0^{2\pi} (N_a(u)(\theta))\, d\theta \underset{\substack{\uparrow \\ \text{Assume here that } u(0)=0}}{\leq} C_a \int_0^{2\pi} \Phi\big(A_a(u)(\theta)\big)\, d\theta$$

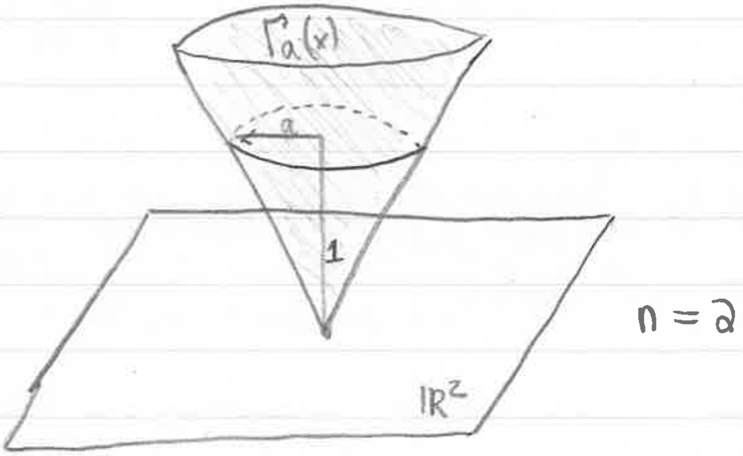
(Same for $\mathbb{R}^{n+1}_+$) The choice of c_a, C_a depends only on the growth constant for Φ and on a. Hence

$$\| A_a(u) \|_p \approx \| N_a(u) \|_p \qquad 0 < p < \infty$$

$$\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$$

$$\Gamma_a(x) := \{ (s,y) : |x-s| < ay \} \quad (a \text{ fixed constant})$$





If u is defined on $\mathbb{R}^{n+1}_+$, we define for each $x \in \mathbb{R}^n$

$$A_a(u)(x) = \left[\int_{\Gamma_a(x)} |\nabla u(s,y)|^2 \, y^{1-n} \, ds \, dy \right]^{1/2}$$

$$N_a(u)(x) = \sup \{ |u(s,y)| : (s,y) \in \Gamma_a(x) \}$$

Note that $\{ N_a(u) > \lambda \}$ and $\{ A_a(u) > \lambda \}$ are open sets.

What corresponds to w^* in this case?

(Recall $w_n \geq |d_n|$ and was $\mathcal{O}_{n-1}$-measurable)

Define

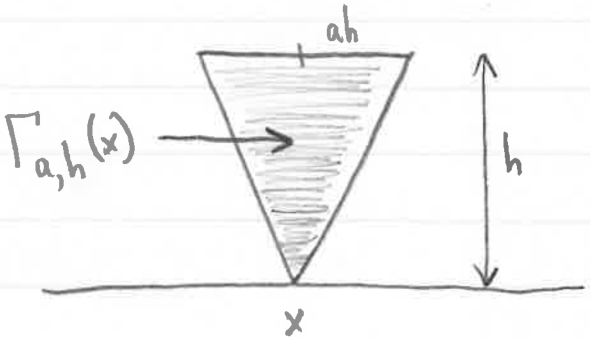
$$D_a(u)(x) = \sup_{(s,y) \in \Gamma_a(x)} y \, |\nabla u(s,y)|$$

If m is Lebesgue measure on $\mathbb{R}^n$ and $Q \subset \mathbb{R}^n$ is measurable, we define

$$m_Q(E) = m(E \cap Q)$$

(m_Q is a finite measure if Q has finite measure)

$$\Gamma_{a,h}(x) = \{ (s,y) : |x-s| < ay, \ 0 < y < h \}$$



THEOREM (Distribution function inequality for $A_a(u)$)

Let Q be a cube in $\mathbb{R}^n$: $Q = I_1 \times I_2 \times \dots \times I_n$ where $I_i, 1 \leq i \leq n$ are intervals of $\mathbb{R}$ of the same finite length. Let u be harmonic in $\mathbb{R}^{n+1}_+$. Then for all $\lambda > 0$

$$m_Q\left(A_{a,h}(u) > \beta\lambda,\ N_{a,h}(u) \vee D_{a,h}(u) \leq \delta\lambda\right)$$

$$\leq \varepsilon\, m_Q\left(A_{a,h}(u) > \lambda\right)$$

where $\beta > 1$, $\delta > 0$, and $\varepsilon = \varepsilon(\beta, \delta, n, a) \begin{array}{ll} \rightarrow 0 & \delta \rightarrow 0 \\ \rightarrow 0 & \beta \rightarrow \infty \end{array}$

$$\left(\text{Can take } \varepsilon = c_{n,a} \frac{\delta^2}{\beta^2 - 1}\right)$$

10/27 MARTINGALES

LEMMA : u harmonic in $\mathbb{R}^{n+1}_+$. Q cube (as before) in $\mathbb{R}^n$, diam $Q = 2ah$
Then with $\beta > 0$, $\delta > 0$

$$m_Q\left(A_{a,h}(u) > \beta\lambda\ ,\ N_{a,h}(u) \vee D_{a,h}(u) \leq \delta\lambda\right) \leq \underset{\substack{\uparrow \\ \text{looks like } c_{n,a}\frac{\delta^2}{\beta^2}}}{\varepsilon}\, m(Q) \qquad \forall \lambda > 0$$

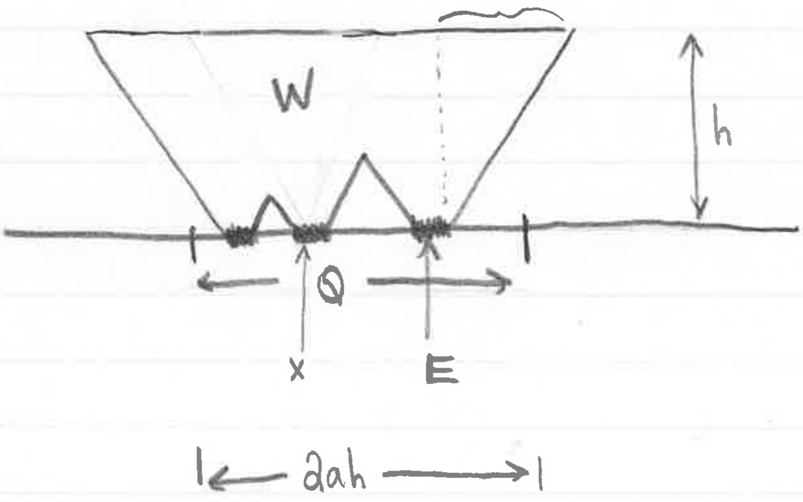
Proof. (Take $n=1$. Basic idea holds for $n>1$) Let

$$E := \{ A_{a,h} \geq \beta\lambda \text{ and } N_{a,h} \vee D_{a,h} \leq \delta\lambda \} \cap Q$$

To show $m(E) \leq \varepsilon m(Q)$ Assume $m(E) > 0$. Let

$$W := \bigcup_{x \in E} \Gamma_{a,h}(x) \quad (\neq \phi)$$

(open set, connected)



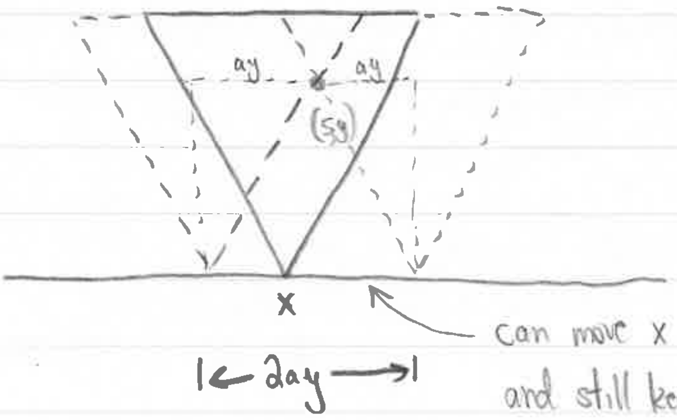
In W $|u| \leq \delta\lambda$ since any $(s,h) \in W$ belongs to some cone at $x \in E$
Also in W $y|\nabla u| \leq \delta\lambda$

CLAIM: $$\beta^2\lambda^2 m(E) \underset{\substack{A_{a,h} > \beta\lambda \\ \text{on } E}}{<} \int_E A^2_{a,h}(x)\,dx = \int_E \iint_{\Gamma_{a,h}(x)} |\nabla u(s,y)|^2\,ds\,dy$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty \chi(x,s,y)\,|\nabla u(s,y)|^2\,dx\,ds\,dy$$

$$\chi_{\Gamma_{a,h}(x)}(s,y) = \begin{cases} 1 & \text{if } |x-s| < ay,\ 0<y<h \\ & \quad x \in E \\ 0 & \text{otherwise} \end{cases}$$

Fubini
$$= \int_{\mathbb{R}}\int_0^\infty |\nabla u(s,y)|^2 \underbrace{\left[\int_{\mathbb{R}} \chi(x,s,y)\,dx\right]}_{\substack{= 0 \text{ if } (s,y) \notin W \ (x \notin E) \\ \leq 2ay\,\chi_W(s,y)}} ds\,dy$$



can move x along this part and still keep $(s,y) \in \Gamma_{a,h}(x)$

$$\leqslant 2a \iint_W y \, |\nabla u(s,y)|^2 \, ds \, dy$$

Check (by differentiation)

$$2 \, |\nabla u|^2 = \Delta u^2 \quad \text{for } u \text{ harmonic}$$

$$\leqslant a \iint_W y \, \Delta u^2 (s,y) \, ds \, dy$$

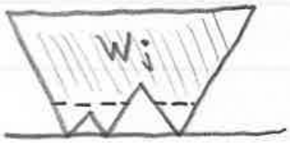
Green's identity :

$$\iint_D u \Delta v - v \Delta u = \int_{\partial D} \left(u \frac{dv}{dn} - v \frac{du}{dn} \right) d\sigma$$

↑ outer normal

↑ length measure (surface measure in higher dimensions)

$$W_j := \{ (s,y) \in W : y > 1/j \}$$



$$W_j \uparrow W \text{ as } j \to \infty$$

[trouble is that u is not defined on all the boundary of W]

~~So~~ for some $j \in \mathbb{N}$,

$$\beta^2\lambda^2 m(E) < a \iint_{W_j} y\, \Delta u^2(s,y)\, ds\, dy$$

$$= a \int_{\partial W_j} \left(y \frac{\partial u^2}{\partial n} - u^2 \frac{\partial y}{\partial n} \right) ds$$

∂W_j is a finite number of line segments (the more components of E, the lower the height of the peaks - only a finite # are above $1/j$)

$$= a \int_{\partial W_j} 2yu \frac{\partial u}{\partial n}\, d\sigma - a \int_{\partial W_j} u^2 \frac{\partial y}{\partial n}\, d\sigma$$

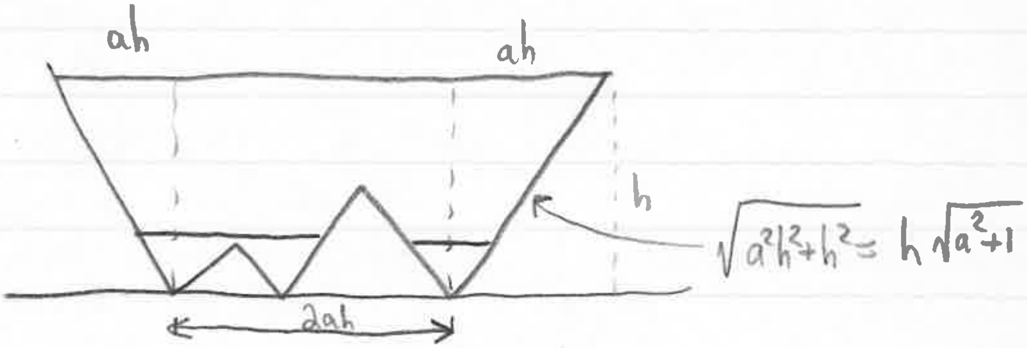
$$\left|\frac{\partial u}{\partial n}\right| = \left|\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)\cdot n\right| \le |\nabla u|$$

$$\left|2yu\frac{\partial u}{\partial n}\right| \le 2y|u|\left|\frac{\partial u}{\partial n}\right| \le 2|u|y|\nabla u| \le 2(\delta\lambda)(\delta\lambda)$$

$$\left|u^2\frac{\partial y}{\partial n}\right| \le (\delta\lambda)^2\left|\frac{\partial y}{\partial n}\right| \le (\delta\lambda)^2$$

$$\le 3a\delta^2\lambda^2\, \sigma(\partial W_j)$$

$$\le 6\left(a+\sqrt{1+a^2}\right)\delta^2\lambda^2\, m(Q)$$



length of ∂W_j on top $\leq 4ah = 2\,m(Q)$

length of other part $\leq 4h\sqrt{a^2+1} = \frac{2\sqrt{a^2+1}}{a}\,m(Q)$

Hence

$$\beta^2 m(E) \leq \underbrace{6\left(a+\sqrt{1+a^2}\right)}_{c_a} \delta^2 m(Q)$$

▨

10/30 MARTINGALES

BASIC DISTRIBUTION FUNCTION INEQUALITY FOR $A_{a,h}(u)$

$$(*) \quad m_Q\left(A_{a,h} > \beta\lambda,\ N_{a,h} \vee D_{a,h} \leq \delta\lambda\right) \leq c_{n,a} \frac{\delta^2}{\beta^2-1} m_Q\left(A_{a,h} > \lambda\right)$$

$\beta > 1,\ \delta > 0,\ a > 0,\ h > 0,\ \lambda > 0,\ \operatorname{diam} Q = 2ah$ ← for $n=1$ $C_a = \overbrace{6(a+\sqrt{a^2+1})}^{c_a}$.

$(2a+1)$

Proof. (For $n=1$) If $Q \subset \{A_{a,h} > \lambda\}$, then the lemma implies the result.

Now assume $Q \not\subset \{A_{a,h} > \lambda\}$ (open set) Let $Q^0 = \operatorname{int}(Q)$ Then

$$\{A_{a,h} > \lambda\} \cap Q^0 = \text{open set} = \sum_{j \in J} \underset{\substack{\uparrow \\ \text{open intervals, disjoint}}}{Q_j^0}$$

Let $\overset{\in Q^0}{x_j}$ be an endpoint of Q_j not in $\{A_{a,h} > \lambda\}$. Then

$$A_{a,h}(x_j) \leq \lambda$$

(since $x_j \notin \{A_{a,h} > \lambda\}$). Let

$$E_j = \{A_{a,h} > \beta\lambda,\ N_{a,h} \vee D_{a,h} \leq \delta\lambda\} \cap Q_j$$

We want to show that $m(E_j) \leq c_a \dfrac{\delta^2}{\beta^2 - 2a\delta^2 - 1} m(Q_j)$

If $C_a \dfrac{\delta^2}{\beta^2-1} \geq 1$, then inequality $(*)$ is trivial. If

$$1 > C_a \frac{\delta^2}{\beta^2-1} \geq (2a+1)\frac{\delta^2}{\beta^2-1}$$

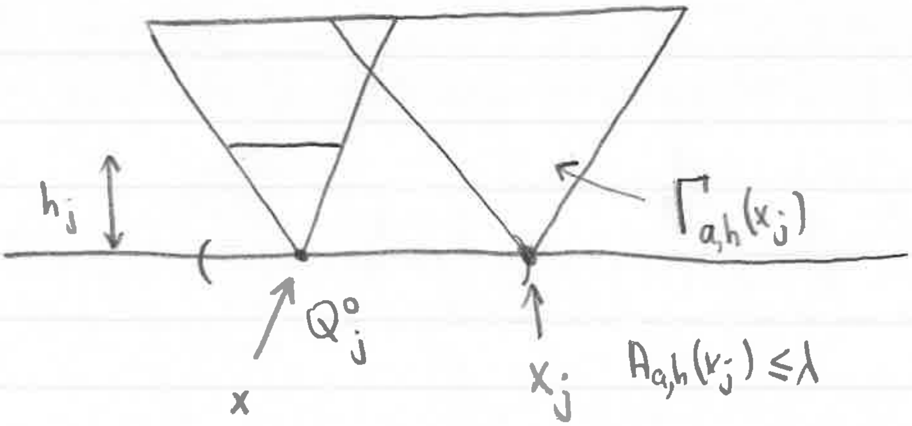
then

$$\beta^2-1 > (2a+1)\delta^2$$

$$\Rightarrow \beta^2 - 2a\delta^2 - 1 > \delta^2 > 0$$

and

$$C_a \frac{\delta^2}{\beta^2-2a\delta^2-1} \leq c_a(2a+1)\frac{\delta^2}{\beta^2-1}$$



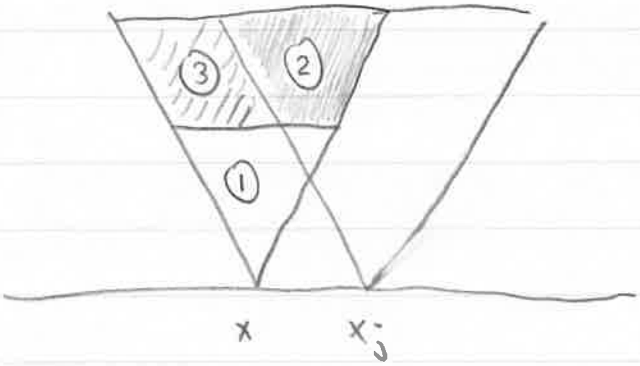
Define h_j by diam $Q_j = 2ah_j$, so $h_j < h$

CLAIM: $x \in E_j \Longrightarrow x \in \{A_{a,h_j} > (\beta^2 - 2a\delta^2 - 1)\lambda, N_{a,h_j} \vee D_{a,h_j} \leq \delta\lambda\} \cap Q$

$$\beta^2\lambda^2 < A^2_{a,h}(x) = \iint_{\Gamma_{a,h}(x)} |\nabla u(s,y)|^2 \, ds\, dy$$

$$= \iint_{\Gamma_{a,h_j}(x)} + \iint_{\substack{\Gamma(a,h)(x) \\ |x_j - s| \leq ay \\ h_j < y < h}} + \iint_{\substack{\Gamma(a,h)(x) \\ |x_j - s| \geq ay \\ h_j < y < h}} |\nabla u(s,y)|^2 \, ds \, dy$$

① + ② + ③



$$\left[\begin{array}{l} D_{a,h} \leq \delta\lambda \Rightarrow \\ \quad y|\nabla u(s,y)| \leq \delta\lambda \\ \qquad (s,y) \in \Gamma_{a,h}(x) \end{array} \right]$$

$$(1) = A^2_{a,h_j}(x)$$

$$(2) \leq A^2_{a,h}(x_j) \leq \lambda^2$$

$$(3) \leq \int_{h_j}^{h} \int_{\substack{|x_j - s| > ay \\ |x - s| < ay}} |\nabla u(s,y)|^2 \, ds \, dy$$

$\left.\begin{array}{l} |x_j - s| > ay \\ |x-s| < ay \end{array}\right\}$ interval of most length $\leq 2ah_j$

$$\leq \int_{h_j}^{h} \frac{\delta^2 \lambda^2}{y^2} \, 2ah_j \, dy$$

$$\leq 2a\delta^2\lambda^2 \, h_j \int_{h_j}^{\infty} \frac{1}{y^2} \, dy = 2a\delta^2\lambda^2$$

Hence

$$\beta^2\lambda^2 < A^2_{a,h}(x) \le A^2_{a,h_j}(x) + \lambda^2 + 2a\delta^2\lambda^2$$

$$\Longrightarrow \beta^2\lambda^2 - 2a\delta^2\lambda^2 - \lambda^2 < A^2_{a,h_j}(x)$$

This shows the claim. Hence

$$m(E_j) \le m_{Q_j}\left(A_{a,h_j} > (\beta^2 - 2a\delta^2 - 1)^{1/2}\lambda,\ N_{a,h_j} \vee D_{a,h_j} \le \delta\lambda\right)$$

$$\underset{\substack{\uparrow \\ \text{lemma}}}{\le} c_a \frac{\delta^2}{\beta^2 - 2a\delta^2 - 1}\, m(Q_j)$$

Now add over j to get desired result.

▨

(NOTE SIMILARITY WITH MARTINGALE INEQUALITY)

Now we would like to remove the $D_{a,h}$ dependence.

11/1 MARTINGALES

LEMMA: (Stein Acta Math 1961 for $n \geq 1$)

$$D_{a,h}(u) \leq c_{n,a} N_{2a,2h}(u)$$

Proof ($n=1$) (Suppress the u)

Recall $D_{a,h}(x) = \sup\{y\,|\nabla u(s,y)| : (s,y) \in \Gamma_{a,h}(x)\}$.

Now

$$|\nabla u(\text{center of a circle})| \leq 4 \frac{\text{max of } |u| \text{ on boundary}}{\text{radius of circle}}$$

(Suppose center $= 0$ and radius $= 1$)

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)u(e^{it})}{1-2r\cos(\theta-t)+r^2}\,dt$$

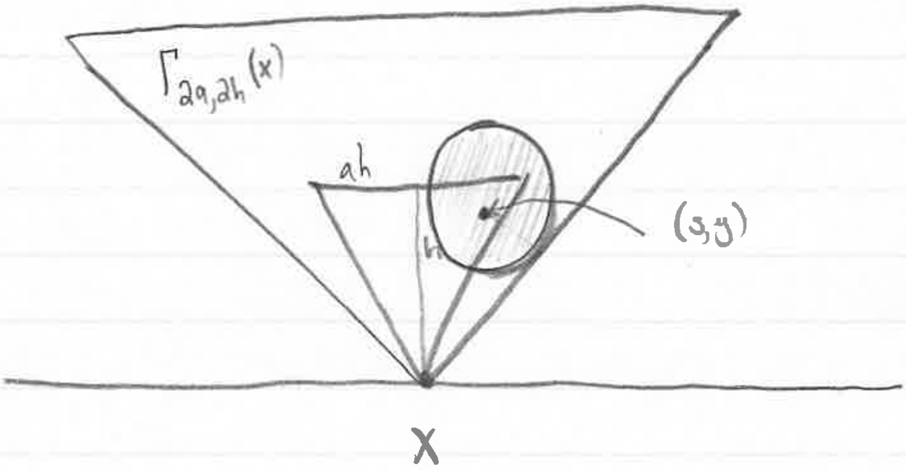
To show $\left|\frac{\partial u}{\partial x}(0)\right| \leq 2$ max of $|u|$ on boundary. With $r>0$, $\theta=0$

$$u(r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos t+r^2}\, u(e^{it})\,dt$$

$$u'(0) = \frac{1}{2\pi}\int_0^{2\pi} 2\cos t\; u(e^{it})\,dt$$

$$\left|\frac{\partial u}{\partial x}(0)\right| = |u'(0)| \leq 2 \max_{\partial} |u|$$

Similarly for $\partial u/\partial y\,(0)$.



radius of circle $\geq c_a y$

$$|\nabla u(s,y)| \leq 4 \frac{N_{2a,2h}(x)}{c_a y}$$

$$\Rightarrow y|\nabla u(s,y)| \leq \frac{4}{c_a} N_{2a,2h}(x)$$

$$\Rightarrow D_{a,h}(x) \leq C_a N_{2a,2h}(x)$$

▨

Remark: Also true that $D_{a,h}(x) \leq c_{n,a} A_{2a,2h}(x)$

(RHS)

THEOREM 1 : u harmonic in $\mathbb{R}^{n+1}_+$

$$m_Q\left(A_{a,h} > \beta\lambda\ ,\ N_{2a,2h} \leq \delta\lambda\right) \leq c_{n,a}\,\frac{\delta^2}{\beta^2-1}\, m_Q\left(A_{a,h} > \lambda\right)$$

where $\lambda > 0$, $\beta > 1$, $\delta > 0$, $a > 0$, $h > 0$, Q any cube with diameter $2ah$

THEOREM 2: Φ general Φ-function (not necessarily convex or concave) u harmonic in $\mathbb{R}^{n+1}_+$

$$\int_{\mathbb{R}^n} \Phi(A_a)\,dx \leq \underset{\substack{\uparrow \\ \text{depends only on } n, a, \text{ growth constant of } \Phi}}{c} \int_{\mathbb{R}^n} \Phi(N_a)\,dx$$

Proof. By the Φ-inequality lemma used for martingales we have

$$\int_{\mathbb{R}^n} \Phi(A_{a,h})\,dm_Q \leq c \int_{\mathbb{R}^n} \Phi(N_{2a,2h})\,dm_Q$$

so

$$\int_{\underset{\substack{\uparrow \\ \text{any cube with} \\ \text{diameter } 2ah}}{Q_h}} \Phi(A_{a,h})\,dx \underset{\substack{\uparrow \\ N_{2a,2h} \leq N_{2a}}}{\leq} c \int_{\mathbb{R}^n} \Phi(N_{2a})\,dx$$

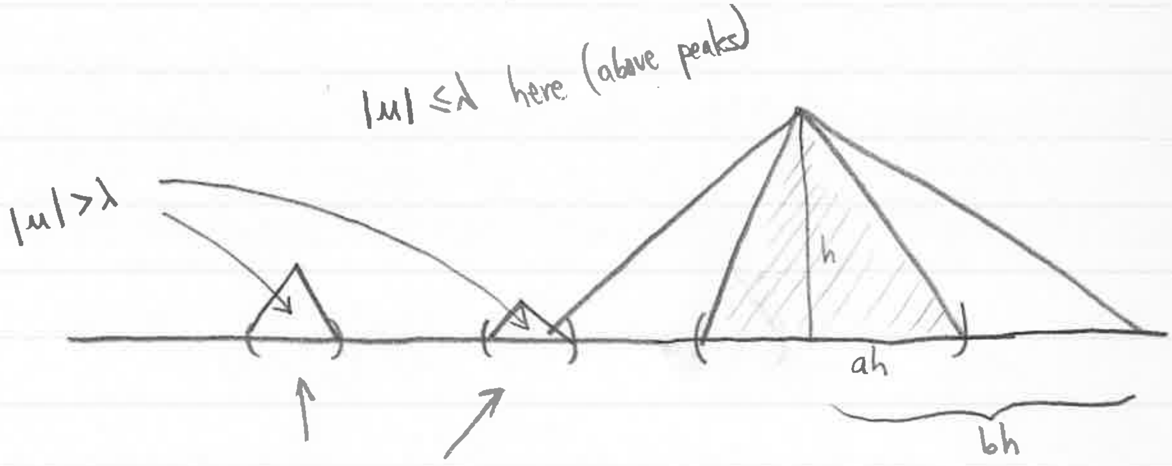
Let $h \to \infty$. $A_{a,h} \uparrow A_a$, $Q_h \uparrow \mathbb{R}^n$. By MCT

$$\int_{\mathbb{R}^n} \Phi(A_a)\,dx \le c \int_{\mathbb{R}^n} \Phi(N_{2a})\,dx$$

(Want to get a's on both sides - use $\int \Phi(N_{2a})\,dx = \int_0^\infty m(N_{2a}>\lambda)\,d\Phi(\lambda)$ and

Lemma: (Studia 1972) $m(N_b>\lambda) \le c\left(\frac{b}{a}\right)^n m(N_a>\lambda)$ $\forall\lambda$ where $0<a<b$

Proof ($n=1$)



$\{N_a>\lambda\} = \bigcup_{j\in J} I_j$ (I_j open interval)

$I_j^* = \frac{b}{a} I_j$

$$m(N_b>\lambda) \le m\Big(\bigcup_{j\in J} I_j^*\Big) \le \sum m(I_j^*)$$

$$\le \frac{b}{a}\sum m(I_j) = \frac{b}{a} m(N_a>\lambda)$$ ▨

(LHS)

THEOREM 1' : u harmonic in $\mathbb{R}^{n+1}_+$

$$m_Q\left(N^{(u)}_{a,h} > \beta\lambda,\ A^{(u)}_{2a,2h} \le \delta\lambda\right) \le \varepsilon\, m_Q\left(N^{(u)}_{a,h} > \lambda\right)$$

$\left(\varepsilon = \varepsilon(n,a,\beta,\delta) \to 0 \text{ as } \delta \to 0 \text{ or } \beta \to 0 \right)$ diam $Q = 2ah$ PROVIDED
$u(q,h) = 0$ where q is the center of Q (otherwise replace u by $u - u(q,h)$)

THEOREM 2' : Φ as before

$$\int_{\mathbb{R}^n} \Phi(N_a)\,dx \le c \int_{\mathbb{R}^n} \Phi(A_a)\,dx$$

PROVIDED $\lim\limits_{y\to\infty} u(0,y) = 0$

Remark: Assume $\Phi(\lambda) > 0$ for $\lambda > 0$. Then $\int \Phi(A_a)\,dx < \infty$ implies that $\lim\limits_{y\to\infty} u(0,y)$ exists and is finite. So can normalize

Proof of Thm 2': Assume Q centered at 0

$$\int_{\mathbb{R}^n} \Phi\left(N_{a,h} \left(u - u(0,q)\right)\right)^{dm_Q} \le c \int_{\mathbb{R}^n} \Phi(A_{2a,2h})\,dm_Q$$

$$\int_{\mathbb{R}^n} \Phi\left(N_a \quad (u)\right) \le \liminf_{n\to\infty} \int_{Q_h} \Phi\left(N_{a,h}(u - u(0,h)\right) \le C \int_{\mathbb{R}^n} \Phi(A_{2a})\,dx$$

But

$$\int_{\mathbb{R}^n} \Phi(N_{2a})\,dx \leq c \int_{\mathbb{R}^n} \Phi(N_a)\,dx$$

11/3 MARTINGALES

If $\int_{\mathbb{R}^n} \Phi(A_a(u))\, dx < \infty$ and $\Phi(\lambda) > 0$ for $\lambda > 0$, then

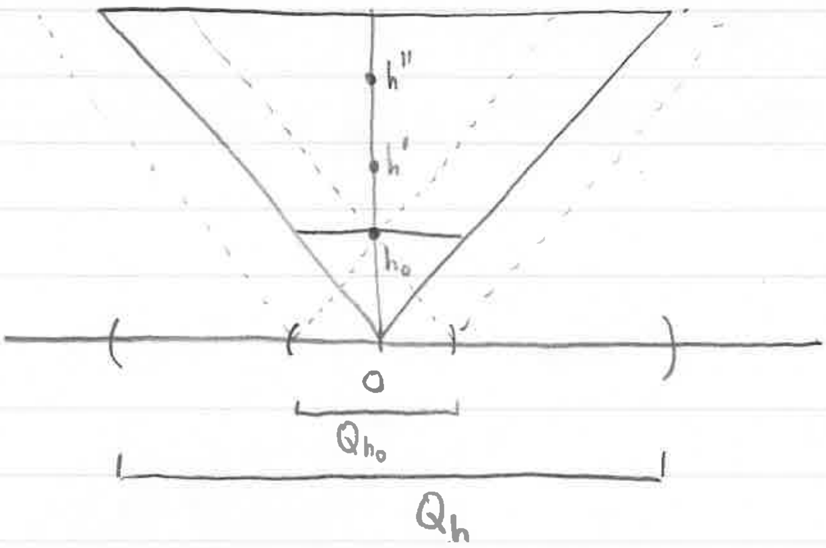
$$\lim_{y \to \infty} u(0,y)$$

exists and is finite [Indeed $\lim_{y\to\infty} (x,y)$ exists and is finite $\forall x$ - all limits equal]

Proof. Recall

$$\int_{Q_h} \Phi\left(N_{\frac{a}{2},h}(u - u(0,h))\right) dx \leq c \underbrace{\int_{\mathbb{R}^n} \Phi(A_a)\, dx}_{K}$$

(Q center 0, diameter ah)



Take $h_0 < h' < h'' < h$.

$$\int_{Q_{h_0}} \Phi\left(\frac{1}{2} \sup_{h_0 < h' < h'' < h} |u(0,h') - u(0,h'')|\right) \leq \int_{Q_{h_0}} \Phi\left(N_{\frac{a}{2},h}(u - u(0,h))\right)$$

Let $h \to \infty$, we get

$$\int_{Q_{h_0}} \Phi\Big(\frac{1}{2} \sup_{h_0<h'<h''} |u(0,h') - u(0,h'')| \le K < \infty$$

Then

$$\int_{Q_{h_0}} \Phi\left(\frac{1}{2} \overset{\leftarrow LS \rightarrow}{\limsup_{\substack{h'\to 0\\ h''\to 0}} |u(0,h') - u(0,h'')|}\right) \le \int \{\ \} \le K$$

$$\Longrightarrow \Phi(LS) \le \frac{K}{m(Q_{h_0})} \quad \text{as } h_0 \to 0$$

Hence $LS = 0$, so $\lim_{y\to\infty} u(0,y)$ by Cauchy criterion

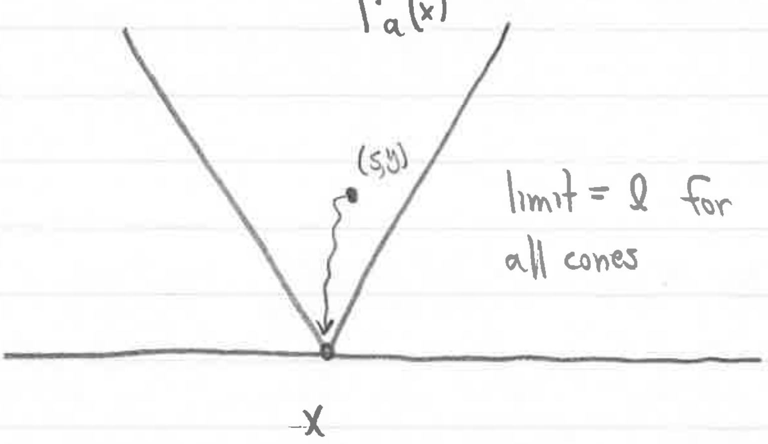
▨

<u>**DEFINITION:**</u> $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ converges nontangentially at x if there is an $l \in \mathbb{R}$ s.t.

$$\lim_{\substack{(s,y)\in\Gamma_a(x)\\ (s,y)\to x}} u(s,y) = l \qquad \forall a > 0$$

u converges radially at x if

$$\lim_{y\to 0} u(x,y) = l$$



Example : $u(x,y) := \dfrac{y}{x^2+y^2}$

$u(0,y) = 1/y$, so no radial convergence at 0

$$u(x,y) := \frac{x}{x^2+y^2}$$

radial convergence at 0, but $u(y,y) = 1/2y$, so no nontangential convergence at 0

DEFINITION: u is nontangentially bounded at x if there is some $a > 0$ and some $h > 0$ such that u is bounded on $\Gamma_{a,h}(x)$ (So $N_{a,h}(u)x < \infty$)

Nontangential convergence surely implies nontangential boundedness

CALDERÓN (1950 TAMS $n>1$)
PRIVALOV (1919 $n=1$)

THEOREM: Suppose u is harmonic in $\mathbb{R}^{n+1}_+$. If u is nontangentially bounded at each x in a measurable set $E \subset \mathbb{R}^n$, then u converges nontangentially at almost all $x \in E$.

Trick #1 - Reduce to the assumption that for some fixed (a,h)

$$E \subset \{ N_{a,h}(u) < \infty \}$$

THEOREM: Suppose u is harmonic in $\mathbb{R}^{n+1}_+$. The following sets are equal a.e.

(1) $\{ x \in \mathbb{R}^n : u \text{ converges nontangentially at } x \}$

(2) $\{ x \in \mathbb{R}^n : N_{a,h}(u)x < \infty \}$

(3) $\{ x \in \mathbb{R}^n : A_{a,h}(u)x < \infty \}$

Proof. Recall for diam $Q = 2ah$

$$m_Q(A_{a,h} > \beta\lambda, N_{2a,2h} \leq \delta\lambda) \leq \varepsilon m(Q)$$

Let $\beta \to \infty$. Then $\varepsilon \to 0$, so

$$m_Q\left(A_{a,h}=\infty,\ N_{2a,2h}\le \delta\lambda\right)=0$$

Let $\lambda \to \infty$.

$$m_Q\left(A_{a,h}=\infty,\ N_{2a,2h}<\infty\right)=0$$

Now $\mathbb{R}^n \subset \bigcup_j Q_j$ where diameter of each Q_j is $2ah$, so

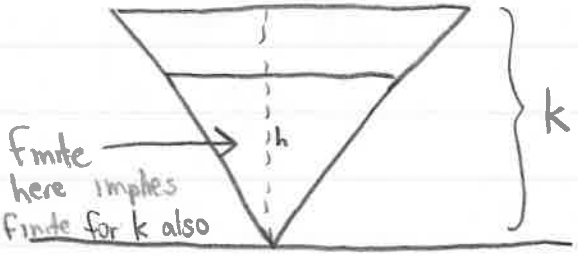
$$m\left(A_{a,h}=\infty,\ N_{2a,2h}<\infty\right)=0$$

Hence

(*) $\{N_{2a,2h}<\infty\}\subset\{A_{a,h}<\infty\}$ a.e.

We use the following

Lemma - $\{N_{a,h}<\infty\}=\{N_{b,k}<\infty\}$ a.e.



May assume $h=k$ and $a<b$. To show LHS $\subset$ RHS a.e.
So let $E\subset\{N_{a,h}<\infty\}$, $m(E)<\infty$.

To Show - $E\subset\{N_{b,h}<\infty\}$, i.e. $m(N_{b,h}=\infty, E)=0$

Let G be an open set containing E. Claim

$$m(N_{b,h} > \lambda, G) \leq c_{n,a,b}\, m(N_{a,h} > \lambda, G)$$

$\uparrow$ for $n=1$ this is b/a

Proof is the same as before (write $G = \bigcup I_j$ and $I_j^* = b/a\, I_j$ ($n=1$ case))

Let $G \downarrow E$; we get

$$m(N_{b,h} > \lambda, E) \leq c\, m(N_{a,h} > \lambda, E)$$

Let $\lambda \to \infty$. Then

$$m(N_{b,h} = \infty, E) \leq c\, m(N_{a,h} = \infty, E) \underset{\substack{\uparrow \\ E \subset \{N_{a,h} < \infty\}}}{=} 0$$

$\boxtimes$

So in $(*)$ we get

$$\{N_{a,h} < \infty\} = \{N_{aa,ah} < \infty\} \subset \{A_{a,h} < \infty\} \text{ a.e.}$$

11/6 MARTINGALE

(Proof continued)

$$m_Q\left(N_{a,h}(u - \underset{\substack{\uparrow \\ \text{center of } Q}}{u(q,h)}) > \beta\lambda,\ A_{2a,2h} \le \delta\lambda \right) \le \underset{\substack{\uparrow \\ \varepsilon \to 0 \text{ as } \beta \to \infty \text{ or } \delta \to 0}}{\varepsilon}\, m(Q)$$

To Show - $\{A_{2a,2h} < \infty\} \subset \{u \overset{\text{through } \Gamma_a(x)}{\text{converges}}\}$ a.e.

$\uparrow$ nontangential convergence

Define $LS_a(x) := \limsup\limits_{\substack{(s,y) \in \Gamma_a(x) \\ (s',y') \in \Gamma_a(x) \\ (s,y) \to x \\ (s',y') \to x}} |u(s,y) - u(s',y')|$. Want $LS_a(x) = 0$

Now

$$m_Q\left(LS_a > 2\beta\lambda,\ A_{2a,2h} \le \delta\lambda\right) \le m_Q\left(N_{a,h}(u - u(q,h)) > \beta\lambda,\ A_{2a,2h} \le \delta\lambda\right)$$

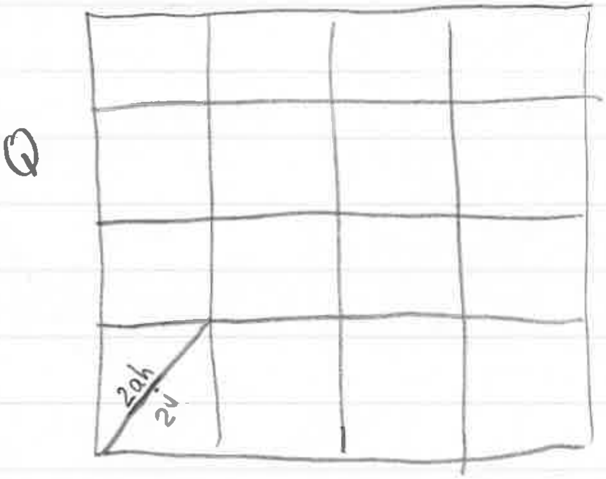
(triangle inequality), and so

$$m_Q\left(LS_a > 2\beta\lambda,\ A_{2a,2h} \le \delta\lambda\right) \le \varepsilon m(Q)$$

This last inequality holds for all cubes of diameter $2ah$ or $4ah$ or $8ah$, etc ($2^n ah$) So

$$m_Q\left(LS_a > 2\beta\lambda,\ A_{2a,\frac{2h}{2^j}} \le \delta\lambda\right) \le \varepsilon\, m(\underset{\substack{\uparrow \\ \text{diameter } 2ah}}{Q})$$

for all $j \in \mathbb{N}^0$.



Let $j \to \infty$

$$m_Q\left(LS_a > 2\beta\lambda,\ A_{2a,2h} < \infty\right)$$

$$\le \lim_{j\to\infty} m_Q\left(LS_a > 2\beta\lambda,\ A_{2a,\frac{2h}{2^j}} \le \delta\lambda\right) \le \varepsilon\, m(Q)$$

(If $A_{2a,2h} < \infty$, then $A_{2a,\,2h/2^j} \to 0$ as $j \to \infty$)

Now let $\delta \to 0$. Then $\varepsilon \to 0$, so

$$m_Q\left(LS_a > 2\beta\lambda,\ A_{2a,2h} < \infty\right) = 0$$

Let $\lambda \to 0$. Then $m_Q\left(LS_a > 0,\ A_{2a,2h} < \infty\right) = 0$

Hence

$$\{A_{2a,2h}\} \cap Q \subset \{LS_a = \infty\} \cap Q \quad \text{a.e.}$$

Taking unions over such Q we get

$$\{A_{2a,2h}\} \subset \{LS_a = \infty\} \quad \text{a.e.}$$

Summary

$$\{A_{2a,2h} < \infty\} \subset \left\{x : \begin{matrix} u \text{ converges} \\ \text{through } \Gamma_a(x) \end{matrix}\right\} \subset \{N_{a,h} < \infty\}$$

$$= \{N_{4a,4h} < \infty\} \subset \{A_{2a,2h} < \infty\}$$

Hence

$$\{u \text{ converges}\} = \bigcap_{\substack{a>0 \\ a\in\mathbb{Q}}} \{u \text{ converges through } \Gamma_a(x)\}$$

$$= \bigcap_{\substack{a>0 \\ a\in\mathbb{Q}}} \{N_{a,h} < \infty\}$$

$$= \{N_{a_0,h} < \infty\} \quad \text{a.e.}$$

CALDERON'S THEOREM: If u is nontangentially bounded on E, then u converges nontangentially a.e. on E

Proof

$$\{x : u \text{ nontangenially bounded at } x\} = \bigcup_{\substack{a>0\\ h>0}} \{N_{a,h}(u)x < \infty\}$$

$$= \{N_{a_0,h_0}(u)x < \infty\} \supseteq \{x : u \text{ converges nontangentially}\} \text{ a.e.}$$

▨

THEOREM: Suppose u is harmonic in $\mathbb{R}^2_+$ and v is conjugate to u there. Then

$$\{x : u \text{ converges nontangentially at } x\} = \{x : v \text{ converges nont. at } x\} \text{ a.e.}$$

Proof. $\text{LHS} = \{N_{a,h}(u) < \infty\} = \{A_{a,h}(u) < \infty\} \underset{\substack{\uparrow\\ |\nabla u| = |\nabla v|}}{=} \{A_{a,h}(v) < \infty\}$

$= \{N_{a,h}(v) < \infty\} = \text{RHS}$ a.e.

▨

CARLESON (1962 Ark. Math): If u is nontangentially bounded from below on a measurable set E, then u converges nontangentially a.e. on E

COROLLARY: If u is non-negative on $\mathbb{R}^{n+1}_+$, then u converges nontangentially a.e.

The proof of the theorem uses the following inequality

$$m_Q\big(N_{a,h}(u) > \beta\lambda,\ \underset{\substack{\uparrow \\ u^- = -(u\wedge 0)}}{N_{b,k}(u^-)} \le \delta\lambda\big) \le \underset{\substack{\uparrow \\ \varepsilon = c_{n,a}\left(\frac{1+\delta}{\beta+\delta} + \frac{1}{b} + \frac{h}{k}\right)}}{\varepsilon\, m(Q)} \qquad \forall \lambda > u(\overset{\swarrow \text{ center of } Q}{q},h)$$

$$\operatorname{diam} Q = 2ah$$

Letting $\beta \to \infty$

$$m_Q\big(N_{a,h}(u) = \infty,\ N_{b,k}(u^-) \le \delta\lambda\big) \le c_{n,a}\left(\frac{1}{b} + \frac{h}{k}\right) m(Q)$$

Letting $\lambda \to \infty$

$$\underbrace{m_Q\big(N_{a,h}(u) = \infty,\ N_{b,k}(u^-) < \infty\big)}_{\text{independent of } b,k \text{ since } N_{b,k}(u^-) = N_{1,1}(u^-)} \le c_{n,a}\left(\frac{1}{b} + \frac{h}{k}\right) m(Q)$$

Let $b,k \to \infty$

$$\big(N_{b,k}(u^-) < \infty\big) \cap Q \subset \{N_{a,h}(u) < \infty\} \cap Q$$

Now union over Q

11/8 MARTINGALES

Correction to $m(N_{b,h} > \lambda, G) \leq c_{n,b,a}\, m(N_{a,h} > \lambda, G)$. This is incorrect since if G is small enough we could get $G \subset \{N_{b,h} > \lambda\}$. Change to

$$m(\lim_{k \to 0} N_{b,k} > \lambda, G) \leq c_{n,b,a}\, m(N_{a,h} > \lambda, G)$$

Then

$$\{N_{a,h} < \infty\} \subset \{\lim_{k\to 0} N_{b,k} < \infty\}$$

$$\qquad\qquad\qquad\qquad \parallel$$

$$\{N_{b,k_0} < \infty\} \quad \text{for some } k_0$$

BROWNIAN MOTION

Probability space (Ω, a, P). $X_t : \Omega \to \mathbb{R}^n$ measurable
We write

$$X = (X_t)_{t \geq 0}$$

for this family of functions and assume

(i) $P(X_t \in B) = \frac{1}{(2\pi t)^{n/2}} \int_B e^{-|x|^2/2t} \, dx$ $\quad t > 0$
$B \subset \mathbb{R}^n$ Borel

(ii) If $k > 1$ and $0 \leq t_0 < t_1 < \dots < t_k$, then

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_k} - X_{t_{k-1}}$$

are independent

(iii) If $\omega \in \Omega$, then the mapping $t \mapsto X_t(\omega) : [0,\infty) \to \mathbb{R}^n$ is continuous

(iv) $\forall \omega \in \Omega$, $X_0(\omega) = 0$.

Such an $X = (X_t)_{t \geq 0}$ is called a standard Brownian motion in $\mathbb{R}^n$ starting at 0.

THEOREM (Wiener ~1923) Such a stochastic process exists

THEOREM: Brownian motion is rotationally invariant.

Proof. Suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation s.t.

$$|Tx| = |x| \quad \forall x \in \mathbb{R}^n$$

[eg. in $n=1$ $Tx = \pm x$; $n=2$ $Tz = e^{i\theta} z$ for some θ]. Let

$$Y_t := T X_t$$

CLAIM - $Y = (Y_t)_{t \geq 0}$ is a Browning motion.

(i) $P(Y_t \in B) = P(X_t \in T^{-1} B)$

$$= \frac{1}{(2\pi t)^{n/2}} \int_{T^{-1}B} e^{-|x|^2/2t} \, dx$$

$$= \frac{1}{(2\pi t)^{n/2}} \int_{B} e^{-|Tx|^2/2t} \, dx$$

$\uparrow$ $m(T^{-1}B) = m(B)$

$$= \frac{1}{(2\pi t)^{n/2}} \int_{B} e^{-|x|^2/2t} \, dx$$

(ii), (iii), and (iv) follow immediately

☑

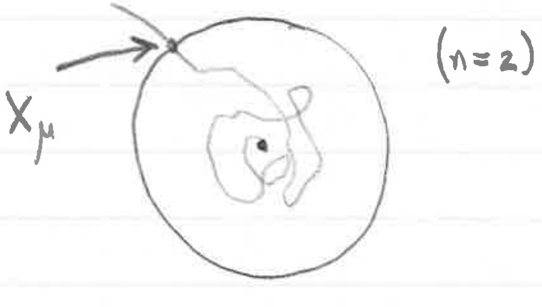
Let $\mu(\omega) := \inf \{ t \geq 0 : |X_t(\omega)| = 1 \}$.

Claim - $P(\mu < \infty) = 1$. For

$$P(\mu = \infty) \leq P(|X_t| < 1) = \frac{1}{(2\pi t)^{n/2}} \int_{|x|<1} e^{-|x|^2/2t} \, dx$$

$$\leq \frac{1}{(2\pi t)^{n/2}} \int_{|x|<1} dx \longrightarrow 0 \quad \text{as } t \to \infty$$

Define $X_\mu(\omega) := X_{\mu(\omega)}(\omega)$ (defined except on $\{\mu = \infty\}$ which has prob 0)



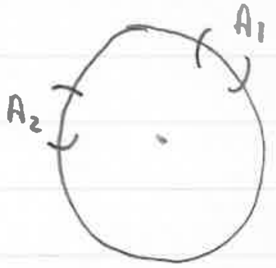
THEOREM: X_μ has normalized Lebesgue measure as its distribution on $\{|X|=1\}$, i.e. if σ is (Lebesgue) surface measure, then

$$P(X_\mu \in A) = \frac{\sigma(A)}{\sigma(\{|x|=1\})}$$

Proof (n=2)

$$P(Z_\mu \in A_1) = P(e^{i\theta} Z_\mu \in A_1)$$

↑ Z, $e^{i\theta}Z$ have same distribution by rotational invariance



$$= P(Z_\mu \in e^{-i\theta} A_1)$$

$$= P(Z_\mu \in A_2) \quad (\text{choose } \theta \text{ s.t. } e^{-i\theta}A_1 = A_2)$$

(A_1 and A_2 have the same surface measure) This shows we have a uniform distribution

▨

Write X_t as a vector $X_t = (X_{1,t}, X_{2,t}, \dots, X_{n,t})$

Claim- $(X_{1,t})_{t \geq 0}$ is a standard Brownian motion on $\mathbb{R}$

(i) $P(X_{1,t} \in B_1) = P(X_t \in B_1 \times \mathbb{R} \times \dots \times \mathbb{R})$

$$= \int_{B_1} \frac{e^{-x_1^2/2t}}{\sqrt{2\pi t}}\,dx_1 \underbrace{\int_{\mathbb{R}} \frac{e^{-x_2^2/2t}}{\sqrt{2\pi t}}\,dx_2}_{=1} \cdots \underbrace{\int_{\mathbb{R}} \frac{e^{-x_n^2/2t}}{\sqrt{2\pi t}}\,dx_n}_{=1}$$

$$= \int_{B_1} \frac{e^{-x_1^2/2t}}{\sqrt{2\pi t}}\,dx_1$$

The component processes of the X_t vector are independent.

This means that we can get n-dimensional Brownian motion by putting together 1-dimensional Brownian motion.

11/10 MARTINGALES

Assume that $(\Omega, \mathcal{A}, P)$ is a probability space on which is defined an independent sequence $Z_1, Z_2, \ldots$ of real-valued normal $(0,1)$ random variables

$$P(Z_k \le \lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-x^2/2}\, dx = F(\lambda)$$

⌈For example let $[0,1) = \Omega$. For $w \in [0,1)$ write

$$w = .\, b_1(w)\, b_2(w) \cdots b_n(w) \cdots$$

use primes

Let $U_1 := \sum_{k=1}^{\infty} b_{2^k}/2^k$, $U_2 = \sum_{k=1}^{\infty} b_{3^k}/2^k$, $U_3 = \sum_{k=1}^{\infty} \frac{b_{5^k}}{2^k}$, ...

$U_1, U_2, \ldots$ are independent since the b_n's are. Moreover, U_k is uniformly distributed on $[0,1)$. Let

$$Z_k = F^{-1}(U_k)$$

⌋

Facts about $Z_1, Z_2, \ldots$

$(a_k : k \in \mathbb{N}) \in \ell_2$

(i) The series $\sum_{k=1}^{\infty} a_k Z_k$ converges a.s. and in L_2-norm to a normal random variable $(0, \sum_{k=1}^{\infty} a_k^2)$

(Partial sums $S_n = \sum_{k=1}^{n} a_k Z_k$ martingale with $\|S_n\|_2^2 = \sum_{k=1}^{n} a_k^2$

S converges a.s. and in L_2-norm)

(ii) $\sum a_k^2 < \infty$, $\sum b_k^2 < \infty$

$$\left(\sum_{k=1}^{\infty} a_k Z_k , \sum_{k=1}^{\infty} b_k Z_k \right) = \sum_{k=1}^{\infty} a_k b_k$$

↑

inner product in L_2

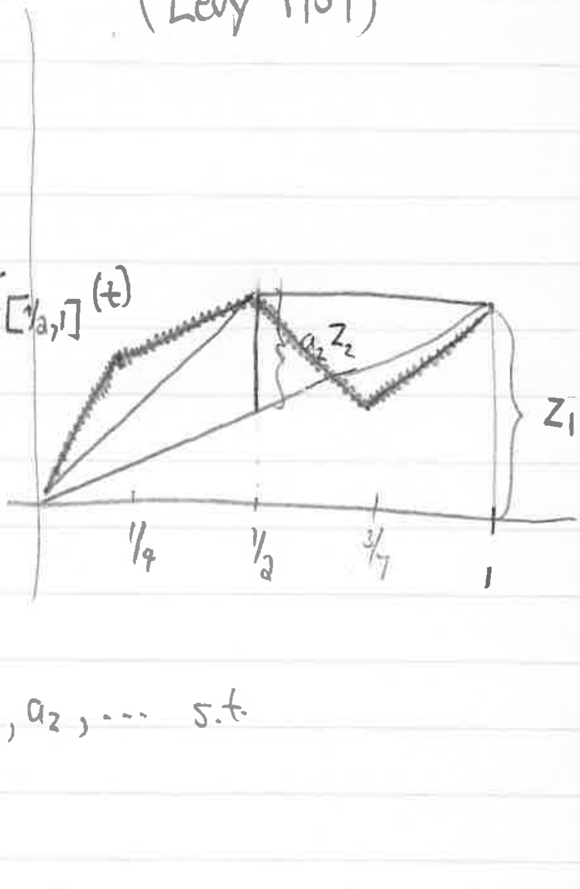
Construction of Browman Motion for $0 \leq t \leq 1$ ($n=1$)

(Levy 1939)

① $X_t^{(0)} := 0$

$X_t^{(1)} := Z_1 t$

$X_t^{(2)} := a_2 Z_2 t \, I_{[0,1/2]}(t) + a_2 Z_2 \, I_{[1/2,1]}(t)$



At each stage we chose coefficients $a_1 = 1, a_2, \dots$ s.t.

$$\left(X_{j/2^k} \right)_{j=0,1,\dots,2^k}$$

is like Brownian motion restricted. Then $\sum_{n=1}^{\infty} \sup_{0 \leq t \leq 1} | X_t^{(n)} - X_t^{(n+1)} | < \infty$ a.s.

$X_t^{(n)} \longrightarrow X_t$ uniformly in t

② Assume $\varphi_1, \varphi_2,$ form a complete orthonormal system of functions in $L_2[0,1]$. Then

$$\int_0^1 \varphi_1(x)\,\varphi_2(x)\,dx = \delta_{ij}$$

(18) p129
Walsh PAMS 1967
Shepp Annals 1966
Math Stat

and if $f \in L^2[0,1]$

$$\|f\|_2^2 = \sum_{k=1}^{\infty} a_k^2 \qquad \left(a_k = \int_0^1 f(x)\, a_\varphi(x)\, dx\right)$$

Then $\sum_{k=1}^{n} a_k \varphi_k$ converges to L^2 in f, and $(f|g) = \sum_{k=1}^{\infty} a_k b_k$

<u>LEMMA</u>: If $Z_1, Z_2, \ldots$ are independent normal $(0,1)$ and $\varphi_1, \varphi_2, \ldots$ is a complete orthonormal system on $[0,1]$, then

$$\sum_{k=1}^{\infty} a_k(t)\, Z_k \longrightarrow \text{(uniformly on } t \in [0,1]\text{])}$$

$a_k(x) = (X_{[0,t]}, \varphi_k) = \int_0^t \varphi_k(x)\,dx$.

11/13 MARTINGALES

LEMMA:

$$\sum_{k=1}^{\infty} \underset{\substack{\uparrow \\ \text{i.i.d} \\ \text{normal } (0,1)}}{Z_k^{(\omega)}} \int_0^t \underset{\substack{\uparrow \\ \text{complete orthonormal} \\ \text{system on } [0,1]}}{\varphi_k(x)}\, dx \quad \text{converges uniformly w.r.t. } t \quad \text{a.s.} \ (\text{on } \Omega - N)$$

Consequences: Define

$$X_t(\omega) := \begin{cases} 0 & 0 \leq t \leq 1 \quad \text{if } \omega \in N \\ \sum_{k=1}^{\infty} Z_k(\omega) \int_0^t \varphi_k & 0 \leq t \leq 1 \quad \text{if } \omega \in \Omega - N \end{cases}$$

X_t is continuous by the uniform convergence.

$$X_t - X_s = \sum_{k=1}^{\infty} Z_k \int_s^t \varphi_k \qquad (0 \leq s < t \leq 1)$$

$$= \sum_{k=1}^{\infty} a_k Z_k$$

where $a_k = (\chi_{[s,t]}, \varphi_k)$ are the Fourier coefficients of $\chi_{[s,t]}$. Hence

$$\begin{array}{c} \|\chi_{[s,t]}\|_2^2 = \sum_{k=1}^{\infty} a_k^2 \\ \| \\ \int_s^t dx = t - s \end{array}$$

Hence $X_t - X_s = \sum a_k Z_k$ is normal $(0, \sum a_k^2)$ = normal $(0, t-s)$
Then for $s=0$, X_t is normal $(0,t)$

[STATIONARY PROPERTY - $P(X_t - X_s \in B) = P(X_{t-s} \in B)$]

If $t_0 < t_1 < \dots < t_n$, then $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are orthogonal. If

$$X_{t_1} - X_{t_0} = \sum_{k=1}^{\infty} Z_k \overbrace{\int_{t_0}^{t_1} \varphi_k}^{a_k}$$

$$X_{t_2} - X_{t_1} = \sum_{k=1}^{\infty} Z_k \underbrace{\int_{t_1}^{t_2} \varphi_k}_{b_k}$$

Then

$$(X_{t_1} - X_{t_0} \mid X_{t_2} - X_{t_1}) = \sum_{k=1}^{\infty} a_k b_k = \int_0^1 \chi_{[t_0, t_1]}(x) \chi_{[t_1, t_2]}(x) \, dx$$

$$= \int_0^1 0 \, dx = 0$$

In this way we construct a sequence $(X_t^{(n)} : 0 \leq t \leq 1)$ of independent Brownian motions. Then for Brownian motion on $0 \leq t < \infty$, we define

$$X_t := \begin{cases} X_t^{(1)} & 0 \leq t \leq 1 \\ X_t^{(1)} + X_t^{(2)} & 1 \leq t \leq 2 \\ X_t^{(1)} + X_t^{(2)} + X_t^{(3)} & 2 \leq t \leq 3 \\ \vdots & \vdots \end{cases}$$

Proof of Lemma: Let $D_{ij} := [\,{}^{i-1}/_{2^j} \,,\, {}^{i}/_{2^i}\,) \quad 1 \leq i \leq 2^j$

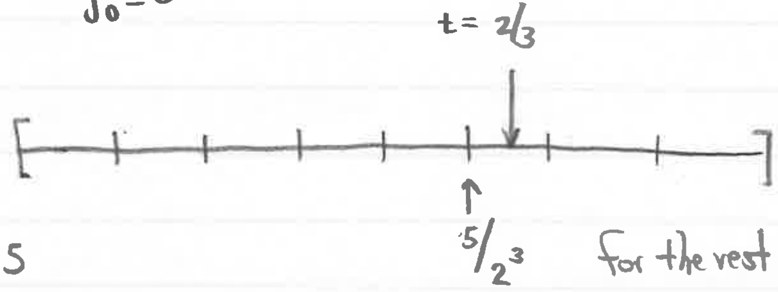
Then $[0,t)$ can be written (though not uniquely) as

$$[0,t) = [0, {}^{i_0}/_{2^{j_0}}) \dot{\cup} D_{i_1 j_1} \dot{\cup} D_{i_2, j_2} \dot{\cup} \ldots \quad (j_0 < j_1 < j_2 < \ldots)$$

Choose any j_0. Then i_0 satisfies

$$\frac{i_0}{2^{j_0}} < t \leq \frac{i_0 + 1}{2^{j_0}}$$

example - $j_0 = 3$



Take $i_0 = 5$

For the rest we look at the binary expansion of $2/3 - 5/8$

Show Cauchy uniformness

$$\left| \sum_{m \leq k \leq n} Z_k \int_0^t \varphi_k \right| \leq \sum_{k=m+1}^{n} Z_k \int_{[0, {}^{i_0}/_{2^{j_0}})} \varphi_k + \left| \sum_{k=m+1}^{n} Z_k \int_{D_{i_1 j_1}} \varphi_k \right| + \ldots$$

$$\leq \max_{0 \leq i \leq 2^{j_0}} \left| \sum_{k=m+1}^{n} Z_k \int_0^{i/2^{j_0}} \varphi_k \right| + \sum_{j > j_0} \sup_{0 \leq i \leq 2^j} \sup_{1 \leq m' < n' < \infty} \left| \sum^{m' < k \leq n'} Z_k \int_{D_{ij}} \varphi_k \right|$$

Define a martingale $\mathcal{F}_{ij}$ = martingale of partial sums of

$$\sum_{k=1}^{\infty} Z_k \int_{D_{ij}} \varphi_k$$

CLAIM: $\sum_{j=1}^{\infty} \sup_{0 \le i \le 2^j} \mathcal{F}^*_{ij} < \infty$ a.s. (N_1 exceptional set)

Choose $\varepsilon > 0$. Let $\omega \notin N_1$. Then choose $j_0 = j_0(\varepsilon, \omega)$ so that

$$\sum_{j > j_0}^{\infty} \sup_{0 \le i \le 2^j} \mathcal{F}^*_{ij} < \varepsilon/4$$

Then

$$\sum_{j > j_0} \sup_{0 \le i \le 2^j} \sup_{1 \le m' < n' \le \infty} \left| \sum_{m' < k \le n'} Z_k \int_{D_{ij}} \varphi_k \right| \le \sum_{j > j_0} \sup_{0 \le i \le 2^j} (2\mathcal{F}^*)$$

$$\le 2 \cdot \varepsilon/4 = \varepsilon/2$$

Now choose exceptional set N_2 so that for all j

$$\sum_{k=1}^{\infty} Z_k(\omega) \int_0^{i/2^{j_0}} \varphi_k \text{ converges if } \omega \notin N_2$$

for each $i \le 2^j$. Choose n_0 s.t. if $n > m > n_0$, then

$$\max_{0 \leq i \leq 2^{j_0}} \left| \sum_{m<k\leq n} Z_k \int_0^{i/2^{j_0}} \varphi_k \right| < \varepsilon/2$$

11/15 MARTINGALES

f_{ij} = martingales of partial sums :

$$f_{ij,n} = \sum_{k=1}^{n} Z_k \int_{D_{ij}} \varphi_k(x)\,dx$$

where $D_{ij} = \left[\frac{i-1}{2^j}, \frac{i}{2^j}\right]$

To Show: $\displaystyle\sum_{j=1}^{\infty} \sup_{1 \le i \le 2^j} f^*_{ij} < \infty$ a.s.

(Shall use $\|g^*\|_p \le q \|g\|_p$) For $p=4$

$$\|f^*_{ij}\|_4^4 \le \left(\frac{4}{3}\right)^4 \|f_{ij}\|_4^4 = \left(\frac{4}{3}\right)^4 E\Big(\overbrace{\sum_{k=1}^{\infty} Z_k \int_{D_{ij}} \varphi_k}^{Y}\Big)^4$$

f_{ij} L^2-bounded $\Rightarrow$ uniformly integrable

Y normal $\left(0, \displaystyle\sum_{k=1}^{\infty}\Big(\int_{D_{ij}} \varphi_k\Big)^2\right)$

$$\uparrow \int_0^1 \chi^2_{D_{ij}}(x)\,dx = 1/2^j$$

So Y is normal $(0, 1/2^j) \Rightarrow Y/\sqrt{1/2^j}$ is normal $(0,1)$

Then

$$EY^4 = \frac{1}{2^{2j}} E\left(\frac{Y}{\sqrt{1/2^j}}\right)^4 = \frac{3}{2^{2j}}$$

So

$$\| f^*_{ij} \|_4^4 \leq \frac{c}{2^{2j}}$$

Now

$$\sup_{1\leq i\leq 2^j} f^*_{ij} \leq \left(\sum_{i=1}^{2^j} (f^*_{ij})^4 \right)^{1/4}$$

$$\Rightarrow E\left(\sup_{1\leq i\leq 2^j} f^*_{ij}\right) \leq E\left(\sum_{i=1}^{2^j} (f^*_{ij})^4\right)^{1/4}$$

$$\leq \left[E\left(\sum_{i=1}^{2^j} (f^*_{ij})^4 \right)\right]^{1/4}$$

$$\leq \left[2^j \frac{c}{2^{2j}} \right]^{1/4} = \frac{c^{1/4}}{2^{1/4 j}}$$

Hence

$$E\left(\sum_{j=1}^{\infty} \sup_{1\leq i\leq 2^j} f^*_{ij} \right) \leq \sum_{j=1}^{\infty} \frac{\hat{c}}{(2^{1/4})^j} < \infty$$

▨

EXERCISE 16. Recall the following fact. There is a stochastic process $X = \{X_t\}_{0 \le t < \infty}$ with the following properties: (i) X_t is normally distributed with expectation zero and variance t, $0 \le t < \infty$; (ii) X has independent increments: if $0 \le t_0 < t_1 < \cdots < t_n$ and $n \ge 2$, then

$$X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$$

are independent; (iii) X has continuous paths: if $\omega \in \Omega$, then the map $t \to X_t(\omega)$ is continuous on $[0, \infty)$; (iv) X starts from 0: if $\omega \in \Omega$, then $X_0(\omega) = 0$.

Such a process, called Brownian motion, may be constructed on any nonatomic probability space. It is easy to show that there is an independent sequence $(Z_1, Z_2, \ldots)$ of normal random variables with zero expectations and unit variances and that finite linear combinations of the Z's are independent if and only if they are orthogonal. Assuming this much, fill in the details and prove the existence of Brownian motion for $0 \le t \le 1$ (this is enough; why?) following Paul Lévy [AJM 1940]

Define $X(0) = 0$ and $X(1) = Z_1$.

If $X\left(\frac{k}{2^n}\right) = X\left(\frac{2k}{2^{n+1}}\right)$, $0 \le k \le 2^n$,

have been defined, let

$$X\left(\frac{2k-1}{2^{n+1}}\right) = \frac{1}{2^{\frac{n+2}{2}}} Z_{2^n+k} + \frac{1}{2}\left[X\left(\frac{k-1}{2^n}\right) + X\left(\frac{k}{2^n}\right)\right]$$

for $k = 1, \ldots, 2^n$, and show by induction on n that the processes $\left\{X\left(\frac{k}{2^n}\right)\right\}_{0 \le k \le 2^n}$ have independent increments and that $X\left(\frac{k}{2^n}\right)$ is normal with expectation zero and variance $\frac{k}{2^n}$. Define $\{X_n(t)\}_{0 \le t \le 1}$ to be the process such that, for $\omega \in \Omega$, $X_n(\cdot, \omega)$ is the polygonal path determined by the points $\left(\frac{k}{2^n}, X\left(\frac{k}{2^n}, \omega\right)\right)$, $0 \le k \le 2^n$. Show that

$$\sup_{0 \le t \le 1} \left| X_n(t) - X_{n+1}(t) \right| \le \sup_{1 \le k \le 2^n} \left| Z_{2^n+k} \right| / 2^{\frac{n+}{2}}$$

$$= Y_n, \text{ say,}$$

satisfies $\sum_{n=0}^{\infty} Y_n < \infty$ a.e.* Let $\Omega_0 = \{ \sum_{n=0}^{\infty} Y_n < \infty \}$. For $\omega \in \Omega_0$, the functions $X_n(\cdot, \omega)$ converge uniformly to a function $X(\cdot, \omega)$. Show that this limiting process $\{X(t)\}_{0 \le t \le 1}$ restricted to Ω_0 satisfies the properties (i) – (iv) of Brownian motion.

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* See next page for a hint.

HINT. If $a > 0$, then

$$\int_a^\infty e^{-\frac{t^2}{2}}\,dt \le \int_a^\infty \frac{t}{a} e^{-\frac{t^2}{2}}\,dt \le a^{-1} e^{-\frac{a^2}{2}}.$$

Use this inequality and

$$P(Y_n > \lambda) \le \sum_{k=1}^{2^n} P\left(|Z_{2^n+k}| > 2^{\frac{n+2}{2}}\lambda\right)$$

$$= 2^{n+1} \int_{2^{\frac{n+2}{2}}\lambda}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt$$

to show that there is a sequence $\lambda_0, \lambda_1,$ of positive numbers satisfying $\sum_{n=0}^\infty \lambda_n < \infty$ and $\sum_{n=0}^\infty P(Y_n > \lambda_n) < \infty$.

EXERCISE 17. Let X be Brownian motion. Show that

$$\sum_{k=1}^{2^n} \left[X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right) \right]^2 , \quad n \geq 1$$

is a reversed martingale converging to 1 a.e.

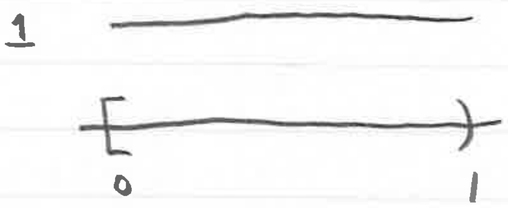
EXERCISE 18. Use the above exercise to show that for almost all ω, $X(\cdot, \omega)$ is not of bounded variation on $[0, 1]$.

$$\overset{\nearrow 1}{\sum_{k=1}^{2^n} \left(X\left(\frac{k-1}{2^n}\right) - X\left(\frac{k}{2^n}\right) \right)^2}$$

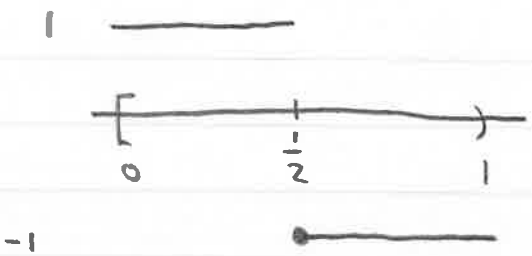
$$\leq \underbrace{\max_{1 \leq j \leq 2^n} \left| X\left(\frac{j-1}{2^n}\right) - X\left(\frac{j}{n}\right) \right|}_{\downarrow \, 0} \underbrace{\sum_{k=1}^{2^n} \left| X\left(\frac{k-1}{2^n}\right) - X\left(\frac{k}{2^n}\right) \right|}_{\Rightarrow \, \downarrow \, \infty}$$

HAAR SYSTEM

χ_0 :



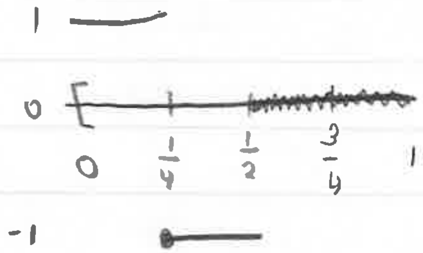
χ_1 :



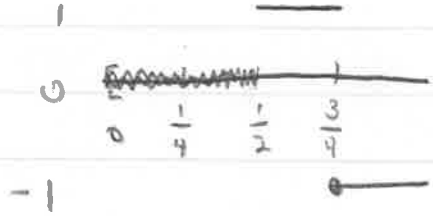
(Used in Levy construction of Brownian Motion)

χ_2 : $\sqrt{2}$ ·

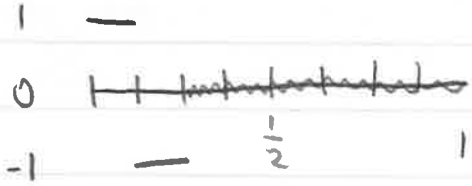
normalizing factor



χ_3 : $\sqrt{2}$ ·



χ_4 : 2 ·



CLAIM: $(\chi_0, \chi_1, \chi_2, \cdots \quad)$ is a martingale difference sequence w.r.t.

$$\mathcal{A}_n = \sigma(\chi_0, \dots, \chi_n)$$

Since

$$\int_A \chi_{n+1} = 0 \qquad \forall A \in \mathcal{A}_n$$

Then $f_n = \sum_{k=0}^{n} \underbrace{a_k \chi_k}_{d_k}$ $\quad n = 0, 1, 2, \dots$ is a martingale with

$|d_k|$ being $\mathcal{A}_{k-1}$-measurable (since $|\chi_k|$ is $\mathcal{A}_{k-1}$-measurable)

Let $f \in L^p[0,1]$ for $1 \leq p < \infty$ and define

$$a_k := \int_0^1 f(y)\, \chi_k(y)\, dy$$

Then

$$\sum_{k=0}^{2^n-1} a_k \chi_k(x) = \int_0^1 \underbrace{\sum_{k=0}^{2^n-1} \chi_k(x)\, \chi_k(y)}_{= \begin{cases} 2^n & \text{if } x,y \text{ are in same dyadic interval of length } 2^n \\ 0 & \text{otherwise} \end{cases}} f(y)\, dy$$

Therefore

$$\sum_{k=0}^{2^n-1} a_k \chi_k(x) = \frac{\int_{\frac{j-1}{2^n}}^{j/2^n} f(y)\,dy}{1/2^n}$$

$$= E\left[f \mid D_n\right] \longrightarrow f \text{ in } L^p \text{ norm}$$

↑ σ-field generated by dyadic intervals of length $1/2^n$

$$\xrightarrow[n\to\infty]{} E[f \mid D_\infty]$$

$$= E[f \mid \mathcal{B}] = f \quad \text{a.e. and in } L^p \text{ norm}$$

If $f \in L^2$, then

$$\|f\|_2^2 = \lim_{n\to\infty} \left\| \sum_{k=0}^{2^n-1} a_k \chi_k \right\|_2^2 \overset{\text{orthonormality}}{=} \sum_{k=0}^{\infty} a_k^2$$

which gives the completeness of the Haar system in L^2

[Reference for Îto integral McKean Stochastic Integrals]

11/17 MARTINGALES

ITÔ INTEGRAL

Let $B = (B_t)_{0 \leq t < \infty}$ be a Brownian motion in $\mathbb{R}$ starting at 0. Let $\mathcal{B}_t = \sigma(B_s : 0 \leq s \leq t)$

Want to talk about something of the form

$$\int_0^t \varphi \, dB$$

↑ trouble - this is not of bounded variation

We would like this to be $\mathcal{B}_t$-measurable, continuous in t a.s. and

$$\int_0^t (C\varphi_1 + \varphi_2)\, dB = c\int_0^t \varphi_1 \, dB + \int_0^t \varphi_2 \, dB$$

Sort of a version of a martingale transform. Brownian motion is a martingale:

$$E(B_t | \mathcal{B}_s) = \mathcal{B}_s \text{ a.s.} \quad \forall\, 0 \leq s < t$$

Since

$$E(B_t | \mathcal{B}_s) = E(B_s + B_t - B_s \,|\, \mathcal{B}_s)$$

$$= E(B_s | \mathcal{B}_s) + E(\underbrace{B_t - B_s}_{\text{independent}} | \mathcal{B}_s)$$

$$= B_s$$

independent, so this is $E(B_t - B_s) = EB_t - EB_s = 0$

We take $\varphi : [0,\infty) \times \Omega \longrightarrow \mathbb{R}$ which are measurable relative to the Borel sets of $[0,\infty) \times \mathcal{B}_\infty$ and for which $\varphi(t,\cdot)$ is $\mathcal{B}_t$-measurable.

(φ is called nonanticipating Brownian functional)

An elementary nonanticipating Brownian functional ψ is one for which there exists a sequence $0 < t_0 < t_1 < \dots < t_n \to \infty$ s.t. $\psi(\cdot,\omega)$ is constant on each interval $[t_{k-1}, t_k]$, $k \in \mathbb{N}$
For such a ψ we define

$$\int_0^t \psi \, dB := \sum_{\{t_k \leq t\}} \psi(t_{k-1})(B_{t_k} - B_{t_{k-1}}) + \psi(t_k)(B_t - B_{t_k})$$

This satisfies all 4 conditions we wanted. (Note that the definition is independent of the partition)

DEFINITION (Itô integral for any nonanticipating Brownian functional φ)
Suppose $X = (X_t)_{0 \leq t < \infty}$ is a family of functions on Ω satisfying

1. X_t is $\mathcal{B}_t$-measurable $\forall t$
2. $t \mapsto X_t(\omega)$ is continuous on $[0,\infty)$ $\forall \omega \in \Omega$
3. If $\varepsilon > 0$ then $\exists \, \delta > 0$ s.t. if ψ is an elementary nonanticipating Brownian functional satisfying

↑ assume this always exists

$$P\left(\left[\int_0^\infty (\varphi - \psi)^2 \, dt \right]^{1/2} > \delta \right) < \delta$$

then

$$P\left(\sup_{0 \le t < \infty} \left| X_t - \int_0^t \psi \, dB \right| > \varepsilon \right) < \varepsilon$$

Then we call X the Itô integral of φ and write

$$X_t = \int_0^t \varphi \, dB$$

Note: If X' also satisfies 1, 2, 3, then $X_t = X_t' \ \forall t \ge 0$ a.s.

THEOREM: φ nonanticipating. The Itô integral exists (an X exists satisfying 1, 2, 3) if and only if

$$\int_0^t \varphi^2 \, ds < \infty, \quad \forall \, 0 \le t < \infty, \quad \text{a.s.}$$

LEMMA: Let ψ be elementary nonanticipating. Then

$$P\left(\sup_{0 \le t < \infty} \left| \int_0^t \psi \, dB \right| > b, \left(\int_0^\infty \psi^2 dt \right)^{1/2} \le a \right) \le \frac{a^2}{b^2}$$

[or could use

$$E\left(\sup_{0 \le t < \infty} \left| \int_0^t \psi \, dB \right| \wedge 1 \right) \le 5 \, E\left(\left(\left(\int_0^\infty \psi^2 dt \right)^{1/2} \wedge 1 \right) \right)$$]

Proof : Suppose $0 = t_0 < t_1 < \dots$ is a partition for ψ. Let

$$f_n := \sum_{k=1}^{n} \underbrace{\underbrace{\psi(t_{k-1})}_{v_k}\underbrace{(B_{t_k} - B_{t_{k-1}})}_{b_k}}_{d_k} = \int_0^{t_n} \psi \, dB$$

$$= \sum_{k=1}^{n} d_k = \sum_{k=1}^{n} v_k b_k$$

martingale transform
not necessarily integrable

Let $\mathcal{A}_k = \mathcal{B}_{t_k}$. v_k is $\mathcal{A}_{k-1}$ measurable. Then

$$f^* = \sup_{1 \le n < \infty} \left| \int_0^{t_n} \psi \, dB \right|$$

$$s(f) = \left(\int_0^{\infty} \psi^2 \, dt \right)^{1/2}$$

$$\uparrow$$

$\ulcorner$

$$s(f) = \sum_{k=1}^{\infty} E(d_k^2 \mid \mathcal{A}_{k-1}) = \sum_{k=1}^{\infty} v_k^2 E(b_k^2 \mid \mathcal{A}_{k-1})$$

$$= \sum_{k=1}^{\infty} v_k^2 E(b_k^2)$$ (by independence of increments)

$$= \sum_{k=1}^{\infty} v_k^2 (t_k - t_{k-1})$$ (normal $(0, t_k - t_{k-1})$)

$$= \sum_{k=1}^{\infty} \psi(t_{k-1})^2 (t_k - t_{k-1})$$

$$= \sum_{k=1}^{\infty} \int_{[t_{k-1}, t_k]} \psi^2 \, dt \quad (\text{constant on } [t_{k-1}, t_k))$$

$$= \int_0^{\infty} \psi^2 \, dt$$

Now use fact that

$$P(\xi^* \geq b, \Delta(\xi) \leq a) \leq \frac{a^2}{b^2}$$

11/20 MARTINGALES

Suppose $f : [0,1] \to \mathbb{R}$ is square integrable. Define

$$A_n(f)(x) = 2^n \int_{\frac{k-1}{2^n}}^{k/2^n} f(y)\,dy \quad \text{for } x \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$$

$(k=1,\dots,2^n)$. $A_n f \underset{a.e.}{=} E(f \mid \mathcal{B}_n) \to f$ in L^2. Define

$$T_n(f)(x) = \begin{cases} f(x - 1/2^n) & \text{if } 1/2^n \le x < 1 \\ 0 & \text{if } 0 \le x < 1/2^n \end{cases}$$

CLAIM: $A_n T_n f \to f$ in L^2

Proof. First note that $T_n f \to f$ in L^2. Approximate f by a g uniformly continuous on $[0,1)$ so that $\|f-g\|_2$ is small (application of continuity theorem for conditional expectation). Then

$$\|T_n f - f\|_2 \le \|T_n f - T_n g\|_2 + \|T_n g - g\|_2 + \|g - f\|_2$$

$$\le \|f-g\|_2 + \sup_{\frac{1}{2^n} \le x \le 1} |g(x - 1/2^n) - g(x)| + \|g - f\|_2$$

$$\Rightarrow \limsup_{n\to\infty} \|T_n f - f\|_2 \le 2\|f-g\|_2 \downarrow 0 \text{ as } g \to f$$

Then

$$\|A_nT_nf - f\|_2 \le \|A_nT_nf - A_nf\|_2 + \|A_nf - f\|_2$$

$$\le \|T_nf - f\|_2 + \|A_nf - f\|_2 \longrightarrow 0$$

▨

Now suppose φ is nonanticipating. Assume $\int_0^t \varphi^2 ds < \infty$ $\forall t$ a.s. $(\omega \notin N)$

$$\psi_n(t,\omega) := \left(A_nT_n\varphi(\cdot,\omega)\right)(t) \quad (\mathcal{B}_t\text{-measurable})$$

Then by our claim

$$\int_0^1 (\varphi - \psi_n)^2 dt \longrightarrow 0 \quad \forall \omega \notin N$$

Hence we can find a ψ s.t.

$$P\left(\left[\int_0^1 (\varphi - \psi)^2\right]^{1/2} > \delta_1\right) < \delta_1$$

no matter how small δ_1 is. Similar, we can find another ψ s.t.

$$P\left(\left[\int_1^2 (\varphi - \psi)^2\right]^{1/2} > \delta_2\right) < \delta_2$$

Choose $\delta_1, \delta_2, \dots$ s.t. $\delta = \delta_1 + \delta_2 + \dots < \infty$. Then putting all the ψ's together we have

$$P\left(\int_0^\infty (\varphi - \psi)^2 > \delta_1^2 + \delta_2^2 + \dots\right) < \delta_1 + \delta_2 + \dots = \delta$$

Hence

$$P\left(\left[\int_0^\infty (\varphi-\psi)^2\right]^{1/2} > \sqrt{\delta_1^2+\delta_2^2+\cdots} \right) \leq \delta_1+\ldots = \delta$$

Hence we have shown that

LEMMA: $\int_0^t \varphi^2 ds < \infty$, $0 \leq t < \infty$, a.s. $\Rightarrow$ ψ exists

Recall $P\left(\sup_t \left|\int_0^t \psi \, dB\right| > b, \left(\int_0^\infty \psi^2 dt\right)^{1/2} \leq a \right) \leq \frac{a^2}{b^2}$

Let $b=\varepsilon$, $a=\varepsilon^2$. Then

$$P\left(\sup_t \left|\int_0^t \psi \, dB\right| > \varepsilon \right) \leq \frac{\varepsilon^4}{\varepsilon^2} + P\left(\left(\int_0^\infty \psi^2 dt\right)^{1/2} > \varepsilon^2 \right)$$

So if $P\left(\left(\int_0^\infty \psi^2 dt\right)^{1/2} > \varepsilon^2\right) < \varepsilon^2$, then

(*) $$P\left(\sup_t \left|\int_0^t \psi \, dB\right| > \varepsilon \right) < 2\varepsilon^2$$

Existence Theorem: Itô integral exists (X exists) if and only if $\int_0^t \varphi^2 ds < \infty$ $\forall t < \infty$ a.s.

To construct X choose $2 = \varepsilon_0 \geqslant \varepsilon_1 \geqslant \varepsilon_2 \geqslant \ldots$ s.t.

$$\sum_{k=0}^\infty \varepsilon_k < \infty \; ; \; \sum_{k=0}^\infty \varepsilon_k^2 < \infty$$

Choose ψ_n elementary so that $\psi_0 = 0$ and

$$P\left(\left[\int_0^\infty (\varphi - \psi_n)^2 \, dt \right]^{1/2} > \frac{1}{2} \varepsilon_n^2 \right) < \frac{1}{2} \varepsilon_n^2 \qquad n = 0, 1, \dots$$

$$\Rightarrow P\left(\left[\int_0^\infty (\psi_{n+1} - \psi_n)^2 \, dt \right]^{1/2} > \varepsilon_n^2 \right) < \varepsilon_n^2$$

Apply (*) to $\psi_{n+1} - \psi_n$ to get

$$P\left(\sup_t \left| \int_0^t \psi_{n+1} \, dB - \int_0^t \psi_n \, dB \right| > \varepsilon_n \right) < 2 \varepsilon_n^2$$

Then

$$\sum_{n=0}^\infty P\left(\sup_t \left| \int_0^t \psi_{n+1} \, dB - \int_0^t \psi_n \, dB \right| > \varepsilon_n \right) < 2 \sum \varepsilon_n^2 < \infty$$

so by Borel - Cantelli

$$\sum_{n=0}^\infty \sup_t \left| \int_0^t \psi_{n+1} \, dB - \int_0^t \psi_n \, dB \right| < \infty \quad \text{a.s.}$$

Let N be the set where the series is infinite $(P(N) = 0)$ Let

$$N_t := \left\{ \sum_{n=0}^\infty \left(\int_0^t \psi_{n+1} \, dB - \int_0^t \psi_n \, dB \right) \text{ diverges} \right\}$$

CLAIM: ① N_t is $\mathcal{B}_t$-measurable
② $N_t \subset N$

$$\text{Let } X_t = \begin{cases} \displaystyle\sum_{n=0}^\infty \left(\int_0^t \psi_{n+1} \, dB - \int_0^t \psi_n \, dB \right) \overset{\text{telescoping series}}{=} \lim_{n \to \infty} \int_0^t \psi_n \, dB & \text{off } N_t \\ 0 & \text{on } N_t \end{cases}$$

Then X_t is $\mathcal{B}_t$-measurable and $t \mapsto X_t$ is continuous off N

Let $\varepsilon > 0$. Choose n so that

i) $P\left(\sup_t \left| X_t - \int_0^t \psi_n \, dB \right| > \frac{\varepsilon}{2} \right) < \frac{\varepsilon}{2}$

ii) $\varepsilon_n < \frac{\varepsilon}{2}$ + $\varepsilon_n^2 < \varepsilon/4$

Then

$$P\left(\left[\int_0^\infty (\psi_n - \psi)^2 \right]^{1/2} > \varepsilon_n^2 \right) < \varepsilon_n^2$$

$$\Rightarrow P\left(\sup_t \left| \int_0^t \psi_n dB - \int_0^t \psi dB \right| > \frac{\varepsilon}{2} \right) < 2\varepsilon_n^2 < \varepsilon/2$$

$$\Rightarrow P\left(\sup_t \left| X_t - \int_0^t \psi dB \right| > \varepsilon \right) < \varepsilon$$

11/22 MARTINGALES

Is $\psi_n(t,\cdot)$ $\mathcal{B}_t$-measurable?

Alternative 1: $\overline{\mathcal{B}_t}$ = σ-field generated by $\mathcal{B}_t$ and all null sets in $\mathcal{B}_\infty$
Can get by with assuming $\int_0^t \varphi\, dB$ is $\overline{\mathcal{B}_t}$ measurable by using

$$f \in L^1([0,1]) \implies \int_0^1 f(t)\,dt = \lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{2^n-1} f\left(s+\frac{k}{2^n}\right) \quad \text{almost all } s$$

Alternative 2: $\varphi : [0,\infty) \times \Omega \to \mathbb{R}$ nonanticipating. Assume in addition that the restriction of φ to $[0,t) \times \Omega$ is measurable relative to the product σ-field Borel subsets of $[0,t) \times \mathcal{B}_t$

SPECIAL CASE: $t \to \varphi(t,\omega)$ continuous. Take

$$\psi_n(t,\omega) = \varphi\left(\frac{k-1}{2^n}, \omega\right) \text{ if } t \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$$

Continuation of proof. Now suppose the Itô integral exists. The set

$$\left\{ \psi : P\left(\left[\int_0^\infty (\varphi-\psi)^2\,dt\right]^{1/2} > \delta \right) < \delta \right\}$$

is non-empty for each $\delta > 0$ by definition. Then

$$\int_0^t \varphi^2 \, ds \quad \int_0^t (\varphi - \psi)^2 \, ds$$

EXAMPLE

(1) $\varphi(t,w) = B_t(w)$ (continuous case)

Recall - $\int_a^b f\,dg + \int_a^b g\,df = f(b)g(b) - f(a)g(a)$ in Riemann-Stieltjes sense provided one of the integrals exist. If f is continuous and g is of bounded variation, the were OK.

Suppose f is continuous and of bounded variation on $[0,t]$

Then

$$2\int_0^t f\,df = f^2(t) - f^2(0)$$

Is $2\int_0^t B\,dB = B_t^2$? NO

Take $t=1$. Take for our approximating nonanticipating function

$$2\int_0^1 \psi_n \, dB = \sum_{k=1}^{2^n} 2 \, B_{k-1/2^n} \left(B_{k/2^n} - B_{k-1/2^n} \right)$$

$$= \sum_{k=1}^{2^n} \left[\left(B_{k/2^n} + B_{k-1/2^n} \right) - \left(B_{k/2^n} - B_{k-1/2^n} \right) \right]$$

$$\cdot \left(B_{k/2^n} - B_{k-1/2^n} \right)$$

$$= \sum_{k=1}^{2^n} B^2_{k/2^n} - B^2_{k-1/2^n} - \sum_{k=1}^{2^n} \left(B_{k/2^n} - B_{k-1/2^n} \right)^2$$

$$= B_1^2 - \underbrace{\sum_{k=1}^{2^n} \left(B_{k/2^n} - B_{k-1/2^n} \right)^2}_{\substack{\text{reversed martingale} \\ \text{converging to } 1}} \rightarrow B_1^2 - 1$$

(Note $\int_0^1 \varphi^2 \, dt < \infty$ since φ continuous)

In general, $2\int_0^t B \, dB = B_t^2 - t$

Note $B_t^2 - t$ is a martingale:

$$E\left(B_t^2 - t \mid \mathcal{B}_s \right) = E\left((B_s + B_t - B_s)^2 \mid \mathcal{B}_s \right) - t$$

$$= E\left(B_s^2 \mid \mathcal{B}_s \right) + 2 B_s E\left(B_t - B_s \mid \mathcal{B}_s \right)$$

$$+ E\left((B_t - B_s)^2 \mid \mathcal{B}_s \right) - t$$

$$= B_s^2 + 2B_s E(\overset{0}{\cancel{B_t - B_s}}) + E(B_t - B_s)^2 - t$$

$$= B_s^2 + (t-s) - t = B_s^2 - s$$

ITÔ'S LEMMA: $u : \mathbb{R} \to \mathbb{R}$ continuous second derivative. Then

$$u(B_t) = u(0) + \int_0^t u'(B)\, dB + \frac{1}{2}\int_0^t u''(B)\, ds$$

(example $u(x) = x^2$: $B_t^2 = 2\int_0^t B\, dB + \frac{1}{2}(2t)$)

11/27 MARTINGALES

ITO's LEMMA: $u: \mathbb{R} \to \mathbb{R}$ has continuous second derivatives

Then assuming $|u''(x)| \leq e^{x/2}$

$$\forall t \qquad u(B_t) = u(0) + \underbrace{\int_0^t u'(B)\,dB}_{\text{integral in Ito sense}} + \frac{1}{2}\underbrace{\int_0^t u''(B_s)\,ds}_{\text{ordinary Lebesgue integral}} \quad \text{a.s.}$$

Proof. Take $t=1$. For $1 \leq k \leq 2^n$, consider

$$u\left(B_{k/2^n}\right) - u\left(B_{k-1/2^n}\right)$$

Recall Taylor's theorem

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2$$

Take $a = B_{k-1/2^n}$ and $x = B_{k/2^n}$. Then

$$u\left(B_{k/2^n}\right) - u\left(B_{k-1/2^n}\right) = \overset{(1)}{u'\left(B_{k-1/2^n}\right)\left(B_{k/2^n} - B_{k-1/2^n}\right)}$$
$$+ \overset{(2)}{\tfrac{1}{2}u''\left(B_{k-1/2^n}\right) 1/2^n} + \overset{(3)}{\tfrac{1}{2}u''\left(B_{k-1/2^n}\right.\left[\left(B_{k/2^n} - B_{k-1/2^n}\right)^2 - \frac{1}{2^n}\right]}$$
$$\underset{(4)}{+ \tfrac{1}{2}\left[u''(c) - u''\left(B_{k-1/2^n}\right)\right]\left(B_{k/2^n} - B_{k-1/2^n}\right)^2}$$

where $B_{k-1/2^n} \leq c \leq B_{k/2^n}$

Then

$$u(B_1) - u(0) = \sum_{k=1}^{2^n}{}' \text{ LHS} = \sum{}'(1) + \sum{}'(2) + \sum{}'(3) + \sum{}'(4)$$

① $\sum'(1) \longrightarrow \int_0^1 u'(B)\,dB$ in probability

↑ approximating sum

② $\sum(2) \longrightarrow \frac{1}{2}\int_0^1 u''(B_s)\,ds$ everywhere

③ $\| \sum'(3) \|_2^2 = \frac{1}{4} \sum_{k=1}^{2^n} E\left[u''\left(B_{\frac{k-1}{2^n}}\right)^2 E\left(\left(B_{k/2^n} - B_{k-1/2^n}\right)^2 - \frac{1}{2^n}\right)^2 \middle| B_{k-1/2^n}\right]$

↑ orthogonality

↑ normal$(0, 1/2^n)$ ↖ independent ↗

$$= \frac{1}{4} \cdot \frac{3}{2^n} \sum_{k=1}^{2^n} E\, u''\left(B_{\frac{k-1}{2^n}}\right)^2$$

↑ $E(Y-1)^2 = 3 \cdot \text{Var}(Y)^2$ $\quad$ Y normal $(0, \sigma^2)$

$$\leq 3/4\, M\, 1/2^n \longrightarrow 0 \text{ as } n \to \infty$$

$$\Longrightarrow \sum(3) \to 0 \text{ in propability}$$

④ $\frac{1}{2} \sum_{k=1}^{2^n} \sup\left\{ |u''(y) - u''(x)| \left(B_{k/2^n} - B_{k-1/2^n}\right)^2 : \begin{matrix} 1 \leq k' \leq 2^n \\ x, y \text{ between } B_{\frac{k'}{2^n}}, \frac{k+1'}{2^n} \end{matrix} \right\}$

= modulus of continuity $\sum_{k=1}^{2^n} \left(B_{k/2^n} - B_{k-1/2^n}\right)^2 \to 0$ as $n \to \infty$

$t \to f(t) = u''(B_t(\omega)) \quad 0 \le t \le 1$

f continuous on $[0,1]$

f on $[k-1/2^n, k/2^n]$

$$\delta_n := \sup_{\substack{0 \le x \le y \le 1 \\ |x-y| \le 1/2^n}} |f(x) - f(y)| \to 0 \text{ as } n \to \infty$$

▨

ITO'S LEMMA (for n-space) $u : \mathbb{R}^n \to \mathbb{R}$ has continuous second partials. Then

$$u(B_t) = u(0) + \int_0^t \nabla u(B) \cdot dB + \frac{1}{2}\int_0^t \Delta u(B_s)\, ds$$

(So if u is harmonic, then

$$u(B_t) = u(0) + \int_0^t \nabla u(B) \cdot dB \quad)$$

Suppose ψ is elementary, so $\psi(\cdot,\omega)$ is constant on each $[t_{k-1}, t_k)$. Then

$$Y_t := \int_0^t \psi \, dB = \psi(B_{t_0})(B_{t_1} - B_{t_0}) + \ldots + \psi(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}}) + \psi(B_{t_k})(B_t - B_{t_k})$$

$$t_k \leq t < t_{k+1}$$

$Y := (Y_i : n \in \mathbb{N})$. Suppose Y_t is integrable $\forall\, 0 \leq t < \infty$. do so then Y is a martingale. Let $s < t$ with $t_j \leq s < t_{j+1}$. We may assume $s = t_j$ for some $j = 1, 2, \ldots, k$ (if not so add one more point to the partition)

To show $E(Y_t \mid \mathcal{B}_{t_j}) = Y_{t_j}$ by checking $j = k, j = k-1, \ldots$

(In fact $Y_{t_0}, Y_{t_1}, \ldots, Y_{t_k}, Y_t$ is a martingale)

11/29 MARTINGALES

In proving the last theorem we had an expression of the form

$$2\Sigma(3) = \sum_{k=1}^{2^n} \underbrace{u''\left(B_{\frac{k-1}{2^n}}\right)}_{V_k} \underbrace{\left[\left(B_{k/2^n} - B_{k-1/2^n}\right)^2 - 1\right]}_{d_k}$$

$$(d_\ell = 0 \ \ \forall \ell > 2^n)$$

$$= f_{2^n}$$

Recall for a martingale $P(f^* > b,\ s(f) \leq a) \leq a^2/b^2$. Here

$$s^2(f) = \sum_{k=1}^{2^n} V_k^2\, E\left(d_k^2 \mid \mathcal{B}_{k-1/2^n}\right)$$

$$= \sum_{k=1}^{2^n} V_k^2\, 3 \cdot \frac{1}{2^{2n}}$$

$$\leq \underbrace{\sup_{0\leq s\leq 1} |u''(B_s)|^2}_{<\infty}\ 3\, \frac{1}{2^{2n}}$$

Then

$$P\left(|f_{2^n}| > b\right) \leq \frac{a^2}{b^2} + P\left(\sup_{0\leq s\leq 1} |u''(B_s)| > a\, \frac{2^{n/2}}{\sqrt{3}}\right)$$

$$\Longrightarrow \limsup_{n\to\infty} P\left(|f_{2^n}| > b\right) \leq \frac{a^2}{b^2} \quad \forall a>0, b>0$$

$$\Rightarrow \limsup P\left(|\mathfrak{F}_n| > b \right) = 0 \quad \forall b > 0$$

$$\Rightarrow |\mathfrak{F}_{2n}| \to 0 \text{ in probability}$$

Recall

$$(*) \quad P\left(\sup_{0 \le t < \infty} \left| \int_0^t \varphi \, dB \right| > b, \left(\int_0^\infty \varphi^2 \, ds \right)^{1/2} \le a \right) \le \frac{a^2}{b^2}$$

Applications:

1. $P\left(\int_0^\infty \varphi^2 \, ds = 0 \right) = 1 \Rightarrow P\left(\sup_{0 \le t < \infty} \left| \int_0^t \varphi \, dB \right| = 0 \right) = 1$
2. $\left\{ \int_0^\infty \varphi^2 \, dt < \infty \right\} \underset{a.e.}{\subset} \left\{ \lim_{t \to \infty} \int_0^t \varphi \, dB \text{ exists \& is finite} \right\}$

Another inequality

$$E \underset{\substack{\uparrow \\ \text{concave}}}{\Phi} \left(\sup_{0 \le t < \infty} \left| \int_0^t \varphi \, dB \right|^2 \right) \le 5 \, E \Phi \left(\int_0^\infty \varphi^2 \, dt \right)$$

This is true for elementary functions ψ by the martingale argument. To go from ψ to φ, assume WLOG Φ is bounded. Assume $\varphi = 0$ beyond $t > T$. Then $\int_0^\infty \varphi^2 \, dt < \infty$. Now approximate

$$\int_0^T \varphi^2 \, dt$$

by $\int_0^T \psi^2 \, dt$

Stopping times for continuous time

Want to show

$$u(B_{t\wedge\tau}) = u(0) + \int_0^t u'(B) I(\tau \geq s) \, dB + \frac{1}{2} \int_0^{t\wedge\tau} u''(B) \, ds$$

STOPPING TIME: $\tau : \Omega \to [0,\infty]$ s.t. $\{\tau < t\} \in \mathcal{B}_t$

STRICT STOPPING TIME: $\tau : \Omega \to [0,\infty]$ s.t. $\{\tau \leq t\} \in \mathcal{B}_t$

example: ① F closed in $\mathbb{R}^n$

$$\tau := \inf\{t : B_t \notin F\}$$

Then $$\{\tau < t\} = \bigcup_{\substack{s \text{ rat.} \\ 0 \leq s < t}} \{B_s \in \underset{\uparrow_{F^c}}{\mathcal{O}}\} \in \mathcal{B}_t$$

② R is open region in $\mathbb{R}^n$. Let $G_1 \subset G_2 \subset \cdots \subset \underset{\text{open sets}}{\cup G_n} = R$ s.t. $F_n = \overline{G_n} \subset R$. Let

$$\tau := \inf\{t : B_t \notin R\}$$

CLAIM: $\tau = \sup \tau_n$ where τ_n = first exit time from F_n (stopping time by ①

$$\tau_1 \le \tau_2 \le \tau_3 \le \dots \le \tau_n \le \tau$$

$$\Longrightarrow \sup \tau_n \le \tau$$

Suppose equality does not hold (say for some ω)

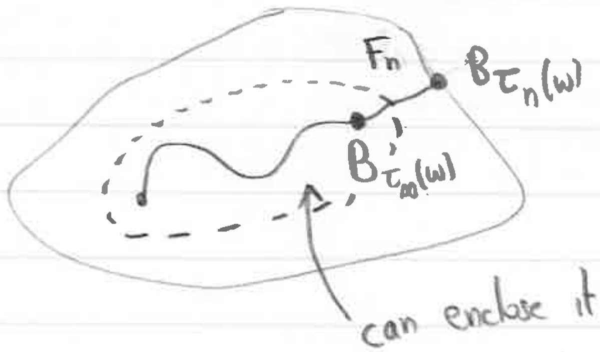
$$\underset{\substack{\uparrow \\ \text{finite}}}{\tau_\infty(\omega)} := \sup \tau_n(\omega) < \tau(\omega)$$

The set $\{B_s(\omega) : \underbrace{0 \le s \le \tau_\infty(\omega)}_{\text{compact interval}}\}$ is compact subset of open set R

The G_n's are an open covering of this compact set, so $\exists\, n(\omega)$ s.t.

$$\{B_s(\omega) : 0 \le s \le \tau_\infty(\omega)\} \subset G_n \subset F_n$$

$$\Longrightarrow \tau_\infty(\omega) < \tau_n(\omega)$$ ↯



can enclose it in an open set still contained in G_n

12/1 MARTINGALES

Recall : Given Brownian motion $B=(B_t)_{0\le t<\infty}$

$$\mathcal{B}_t := \sigma\{ B_s : 0\le s\le t\}$$
$$\mathcal{B}_\infty := \sigma\{ B_s : 0\le s<\infty\}$$

Let $\overline{\mathcal{B}_t}$ be the smallest σ-field containing $\mathcal{B}_t$ and $\{N\in\mathcal{B}_\infty : P(N)=0\}$

$$\overline{\mathcal{B}_t} = \{ C\in\mathcal{B}_\infty : \exists A\in B_t \text{ s.t. } P((A-C)\cup(C-A))=0\}$$

Then g is $\overline{\mathcal{B}_t}$-measurable if and only if $\exists f$ $\mathcal{B}_t$-measurable s.t. $g=f$ a.s.

We looked at non-anticipating functions $\varphi : [0,\infty)\times\Omega\to\mathbb{R}$ which were measurable relative to Borel fields of $[0,\infty)\times\mathcal{B}_\infty$. We allow $\varphi(t,\cdot)$ to be $\overline{\mathcal{B}_t}$-measurable

If W is an elementary non-anticipating function

$$\int_0^t \psi\, dB = \psi(t_0)(B_{t_1}-B_{t_0})+\dots$$

Then $f_n = \int_0^{t_n}\psi\, dB$ is a martingale relative to $\overline{\mathcal{B}_{t_n}}$ $\forall n\ge 1$

Condition of Ito integral to exist: $\int_0^t \varphi^2 ds<\infty$, $0\le t<\infty$, a.s. implies Ito integral X exist : $t\to X_t$ continuous everywhere and

$$X_t = \int_0^t \varphi\, dB$$

CLAIM: $\int_0^t \varphi^2 \, ds$ is $\overline{\mathcal{B}_t}$-measurable

Proof. Take $t=1$. Define

$$C := \left\{ (s,w) : \int_0^1 \varphi^2(s',w)\, ds' = \lim_{n\to\infty} \frac{\sum_{k=1}^{2^n} \varphi\left(s+\frac{k}{2^n}, w\right)}{2^n} \right\}$$

$D :=$ divergence set

By Jessen's theorem

$$\int_0^1 I_D(s,w)\, ds = 0 \qquad \forall w$$

By Fubini

$$\int_0^1 \int_\Omega I_D(s,w)\, dP\, ds = 0$$

$$\Longrightarrow \int_\Omega I_D(s,w)\, dP \qquad \text{a.s. in } s$$

So for one $s \in [0,1]$, $I_D(s,w) = 0$ almost all w

Let

$$J = \begin{cases} \lim & \text{where limit exists} \\ 0 & \text{where it doesn't exist} \end{cases}$$

for fixed s. J is $\mathcal{B}_t$-measurable

$$X_t = \int_0^t \varphi^2 \, dB$$

$$S(x) = \left(\int_0^\infty \varphi^2 dB\right)^{1/2}$$

$$s_t(x) = \left(\int_0^t \varphi^2 ds\right)^{1/2}$$

Let $\tau = \inf\{t : s_t(x) > \lambda\}$. This is a stopping time relative to $\overline{\mathcal{B}_t}$

$$\{\tau < t\} \in \overline{\mathcal{B}_t} \Rightarrow \varphi(s,w) = \begin{cases} 1 & \text{if } 0 \le s \le \tau(w) \\ 0 & \text{if } s > \tau(w) \end{cases}$$

Also, $\varphi(t, \cdot) \in \overline{\mathcal{B}_t}$ and so jointly measurable
Let

$$f(s,y) = \begin{cases} 1 & \text{if } 0 \le s \le y \\ 0 & \text{if } s > y \end{cases}$$

f is Borel measurable on $[0,\infty] \times [0,\infty]$. The map $(s,w) \to (s, \tau(w))$ is measurable on the Borel fields of $[0,\infty) \times \mathcal{B}_\infty$.

$$X_\tau = \int_0^\tau \varphi \, dB = \int_0^t \varphi \chi \, dB$$

Where $y(s,w) = 1\chi_{[0,\tau(w)]}$

LEMMA: $$\underset{\big\uparrow\ \sup\limits_{0\le t<\infty}\|X_t\|_2}{\|X\|_2} = \|s(X)\|_2$$

In particular, if $\|s(X)\|_2 < \infty$, then X_∞ exists and

$$\|X_\infty\|_2 = \|s(X)\|_2$$

i.e. $$E\left(\int_0^\infty \varphi\, dB\right)^2 = E\int_0^\infty \varphi^2\, ds$$

12/4 MARTINGALES

LEMMA: For a non-anticipating function

$$(*) \quad E\int_0^\infty \varphi^2 dt < \infty \Rightarrow E\left(\int_0^\infty \varphi\, dB\right)^2 = E\int_0^\infty \varphi^2 dt$$

Proof Chose ψ_n so close to φ (as in the construction of the integral) that

$$(1) \qquad \int_0^\infty (\varphi-\psi_n)^2 dt \to 0 \text{ a.s.}$$

$$(2) \qquad \int_0^\infty \psi_n^2 dt \leq \int_0^\infty \varphi^2 dt$$

(A_n and T_n are contractions in L^2)

$$(3) \qquad \sup_{0\leq t<\infty} \left| \int_0^t \varphi\, dB - \int_0^t \psi_n dB \right| \to 0 \text{ a.s.}$$

Now $(*)$ holds for ψ_n: let $f_m := \int_0^{t_m} \psi_m dB$. Then $f=(f_1, f_2, \dots)$ is a martingale with $s^2(f) = \int_0^\infty \psi_n^2 dt$ integrable by (2) and $E\int_0^\infty \varphi^2 dt < \infty$. Then $\|f\|_2 = \|s(f)\|_2 < \infty \Rightarrow f$ u.i so $\|f\|_2 = \|f_\infty\|_2$

$$f_\infty = \int_0^\infty \psi_n\, dB$$

$\|f_\infty\|_2^2 = \|s(f)\|_2^2$ is $(*)$ for t_m.
Hence

$$E\left(\int_0^\infty \psi_n\, dB\right)^2 = E\int_0^\infty \psi_n^2 dt \leq E\int_0^\infty \varphi^2 dt$$

By (3) taking limits and Fatou

(4) $$E\left(\int_0^\infty \varphi\, dB\right)^2 \leq E\int_0^\infty \varphi^2\, dt$$

Now apply (4) to $\varphi - \psi_n$!

$$E\left(\int_0^\infty \varphi\, dB - \int_0^\infty \psi_n\, dB\right)^2 \leq E\underbrace{\int_0^\infty (\varphi-\psi_n)^2}_{\leq 2\int_0^\infty \varphi^2 + 2\int_0^\infty \psi_n^2}\, dt \to 0$$

$$\leq 4\int_0^\infty \varphi^2\, dt \text{ integrable}$$

So

$$\left|\left\|\int_0^\infty \varphi\, dB\right\|_2 - \left\|\int_0^\infty \psi_n\, dB\right\|_2\right| \leq \|\cdots\|_2 \to 0$$

Also

$$\left(E\int_0^\infty \varphi^2\, dt\right)^2 \leq \left(E\int_0^\infty (\varphi-\psi_n)^2\, dt\right)^{1/2} + \left(E\int_0^\infty \psi_n^2\, dt\right)^2$$

↑ Minkowski

$$\leq \underbrace{\left(E\int_0^\infty (\varphi-\psi_n)^2\, dt\right)^{1/2}}_{\downarrow 0} + \left(E\int_0^\infty \varphi^2\, dt\right)^{1/2}$$

Hence

$$E\left(\int_0^\infty \varphi^2\, dt\right) = \lim E\int_0^\infty \psi_n^2\, dt = \lim E\left(\int_0^\infty \psi_n\, dB\right)^2$$

$$= E\left(\int_0^\infty \varphi \, dB \right)^2$$

$\boxtimes$

LEMMA: φ non-anticipating with

$$s_t^2(x) := \int_0^t \varphi^2 \, ds < \infty \quad a.s.$$

$X = (X_t)_{t \geq 0}$, $X_t = \int_0^t \varphi \, dB$. $X^* = \sup_t |X_t|$

$$P\left(s(X) > \beta\lambda,\ X^* \leq \delta\lambda \right) \leq \frac{\delta^2}{\beta^2 - 1} P\left(s(X) > \lambda \right), \quad \lambda > 0$$

$(\beta > 1, \delta > 0)$

Proof. Let

$$\mu := \inf\{t : s_t(X) > \lambda)$$

$$\nu := \inf\{t : s_t(X) > \beta\lambda\}$$

$$\sigma := \inf\{t : |X_t| > \delta\lambda\}$$

Stopping times relative to $(\mathcal{B}_t)_{t \geq 0}$
Then

$$P\left(s(X) > \beta\lambda,\ X^* \leq \delta\lambda \right) \leq P\left(\mu \leq \nu < \infty,\ \sigma = \infty \right)$$

Also

$$\{\mu<\infty\} = \{s(X)>\lambda\} = \{s_\mu(X) = \lambda\} = \{s(X^\mu) = \lambda\} \text{ a.s.}$$

$$\int_0^{\mu\wedge t} \varphi\, dB = \int_0^t \varphi\, I(s\le\mu)\, dB \text{ a.s.}$$

Then

$$P(\mu\le\nu<\infty, \sigma=\infty) \le P\Big(\underbrace{s^2(X^{\nu\wedge\sigma})}_{\beta^2\lambda^2} - \underbrace{s^2(X^{\mu\wedge\sigma})}_{\lambda^2} \ge \beta^2\lambda^2 - \lambda^2\Big)$$

$$\le \frac{E\, s^2(X^{\nu\wedge\sigma}) - E\, s^2(X^{\mu\wedge\sigma})}{(\beta^2-1)\lambda^2} \quad \text{(Chebyshev)}$$

$$\left[\begin{array}{c} s^2(X^{\nu\wedge\sigma}) = \int_0^{\nu\wedge\sigma} \varphi^2\, dt \le \int_0^\nu \varphi^2\, dt = s_\nu^2(X) \le \beta^2\lambda^2 \\ \Longrightarrow E\, s^2(X^{\nu\wedge\sigma}) < \infty \end{array}\right]$$

$$= \frac{E X^2_{\nu\wedge\sigma} - E X^2_{\mu\wedge\sigma}}{(\beta^2-1)\lambda^2} \quad \text{(Lemma applied to } \varphi\, I(s\le\mu\wedge\sigma))$$

$$\Big[\begin{array}{c} X^2_{\nu\wedge\sigma} - X^2_{\mu\wedge\sigma} = 0 \text{ if } \mu=\infty.\ X^2_{\nu\wedge\sigma} - X^2_{\mu\wedge\sigma} \le \lambda^2\delta^2 \text{ if } \mu<\infty \\ \Rightarrow X^2_{\nu\wedge\sigma} - X^2_{\mu\wedge\sigma} \le \lambda^2\delta^2\, I(\mu<\infty) \end{array}$$

$$\le \frac{\delta^2\lambda^2 E I(\mu<\infty)}{(\beta^2-1)\lambda^2}$$

$$= \frac{\delta^2\lambda^2 P(s(X)>\lambda)}{(\beta^2-1)\lambda^2} = \frac{\delta^2 P(s(X)>\lambda)}{\beta^2-1}$$

▨

The other inequality is

$$P(X^* > \beta\lambda,\ s(X) \le \delta\lambda) \le \frac{\delta^2}{(\beta-1)^2} P(X^* > \lambda)$$

Proof.

$$\mu = \inf\{t : |X_t| > \lambda\}$$

$$\nu = \inf\{t : |X_t| > \beta\lambda\}$$

$$\sigma = \inf\{t : s_t(X) > \delta\lambda\}$$

Then as before

$$P(X^* > \beta\lambda,\ s(X) \le \delta\lambda) \le P(\mu \le \nu < \infty,\ \sigma = \infty)$$

$$\leq P\left(\left(X_{\nu\wedge\sigma} - X_{\mu\wedge\sigma}\right)^2 \geq (\beta\lambda - \lambda)^2\right)$$

$$\leq \frac{E\left(X_{\nu\wedge\sigma} - X_{\mu\wedge\sigma}\right)^2}{(\beta-1)^2\lambda^2}$$

[Now use $\int \varphi \, I(\mu\wedge\sigma < s \leq \nu\wedge\sigma \; dB = \int_{\mu\wedge\sigma}^{\nu\wedge\sigma} \varphi^2 \, dt$ in lemma]

$$= \frac{E\left[s^2_{\nu\wedge\sigma}(X) - s^2_{\mu\wedge\sigma}(x)\right]}{(\beta-1)^2\lambda^2}$$

$$\leq \frac{\delta^2\lambda^2 \, P(X^* > \lambda)}{(\beta-1)^2\lambda^2}$$

$$= \frac{\delta^2 \, P(X^* > \lambda)}{(\beta-1)^2}$$

☒

12/6 MARTINGALES

Suppose $\Phi : [0,\infty] \to [0,\infty]$ is continuous, non-decreasing, $\Phi(0) = 0$ and $\Phi(2\lambda) \leq c\,\Phi(\lambda)$

THEOREM: $c\,E\,\Phi(s(X)) \leq E\,\Phi(X^*) \leq C\,E\,\Phi(s(X))$

Ito integral

constants depend only on growth constant

(notation $E\,\Phi(X^*) \approx E\,\Phi(s(X))$)

In particular $\|X^*\|_p \approx \|s(X)\|_p \quad 0 < p < \infty$

Proof is same as in discrete martingale theorem

Special case - τ stopping time of B Brownian motion

$$X_t = \int_0^t \varphi\, dB \qquad \varphi = I(\tau \geq s)$$

$$= B_{\tau \wedge t}$$

so $X^* = B^*_\tau$ and $s(X) = \left(\int_0^\infty \varphi^2\, ds \right)^{1/2} = \tau^{1/2}$. Then

$$E\,\Phi(\tau^{1/2}) \approx E\,\Phi(B^*_\tau)$$

Not possible to have inequality with B_τ rather than B^*. For example take $\tau = \inf\{t>1 : B_t = 0\}$. Then $B_\tau = 0$ but $\tau^{1/2} > 1$

Applications to Harmonic functions

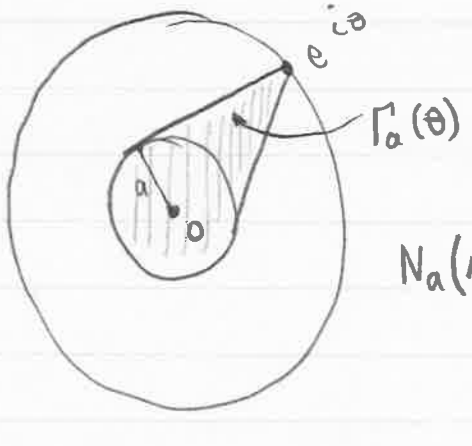
$F = u + iv$ analytic in $D = \{|z|<1\}$
$F(0) = 0$

THEOREM: If Φ is as above

(*) $$\int_0^{2\pi} \Phi(N_a(u))\, d\theta \approx \int_0^{2\pi} \Phi(N_a(v))\, d\theta$$

[Ref. TAMS 1971
Karl Peterson book]

Recall definitions:



$$N_a(u)\theta := \sup_{z \in \Gamma_a(\theta)} |u(z)|$$

Probability proof: We may assume that F is entire: Replace F by F_r, $F_r(z) := F(rz)$ $|z|<1$. Then F_r is analytic in $|z| < 1/r$.

$F_r(z) = a_0 + a_1 z + \dots + a_n z^n + \dots$ converges uniformly in $|z| \le 1$

If theorem holds for u_r and v_r, then it holds for u and v, since

$$\underset{0<r\le s<1}{N_a(u_r)(\theta) \le N_a(u_s)(\theta)} \nearrow N_a(u)$$

Let B be Brownian motion in $\mathbb{C}$. Then

$$u(B_t) = \cancelto{0}{u(0)} + \int_0^t \nabla u(B)\cdot dB + \int_0^t \cancelto{0}{\Delta u(B)}\, ds$$

(harmonic)

Let $\tau = \inf\{t : |B_t| = 1\}$. Then

$$u(B_{\tau\wedge t}) = \int_0^{\tau\wedge t} \nabla u(B)\cdot dB$$

$$S(u) = \text{square function} = \left(\int_0^\tau |\nabla u(B)|^2\, ds\right)^{1/2}$$

(Brownian square function)

$$u^* = \sup_{0\le t<\tau} |u(B_t)|$$

(Brownian maximal function)

Then by the theorem

$$\begin{array}{ccc} E\,\Phi(S(u)) & \approx & E\,\Phi(u^*) \\ \| & & \\ E\,\Phi(S(v)) & \approx & E\,\Phi(v^*) \end{array}$$

Cauchy Riemann equations $|\nabla u| = |\nabla v|$

Hence $E\Phi(u^*) \approx E\Phi(v^*)$ (This actually applies to any region)
Now must show this implies $(*)$

Lemma: $m(N_a(u) > \lambda) \approx P(u^* > \lambda) \quad \forall \lambda > 0$
↑ Lebesgue measure on $[0, 2\pi)$

Assuming lemma we have

$$\int_0^{2\pi} \Phi(N_a(u))\, d\theta = \int_0^\infty m(N_a(u) > \lambda)\, d\Phi(\lambda)$$

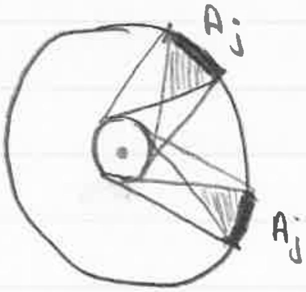
$$\approx \int_0^\infty P(u^* > \lambda)\, d\Phi(\lambda)$$

$$= E\Phi(u^*)$$

Proof of lemma: To show $P(u^* > \lambda) \leq c_a m(N_a(u) > \lambda)$ we use fact that

$$\{e^{i\theta} : N_a(u)\theta > \lambda\} = \text{open set}$$

$$= \bigcup_j A_j$$
← arcs (open)

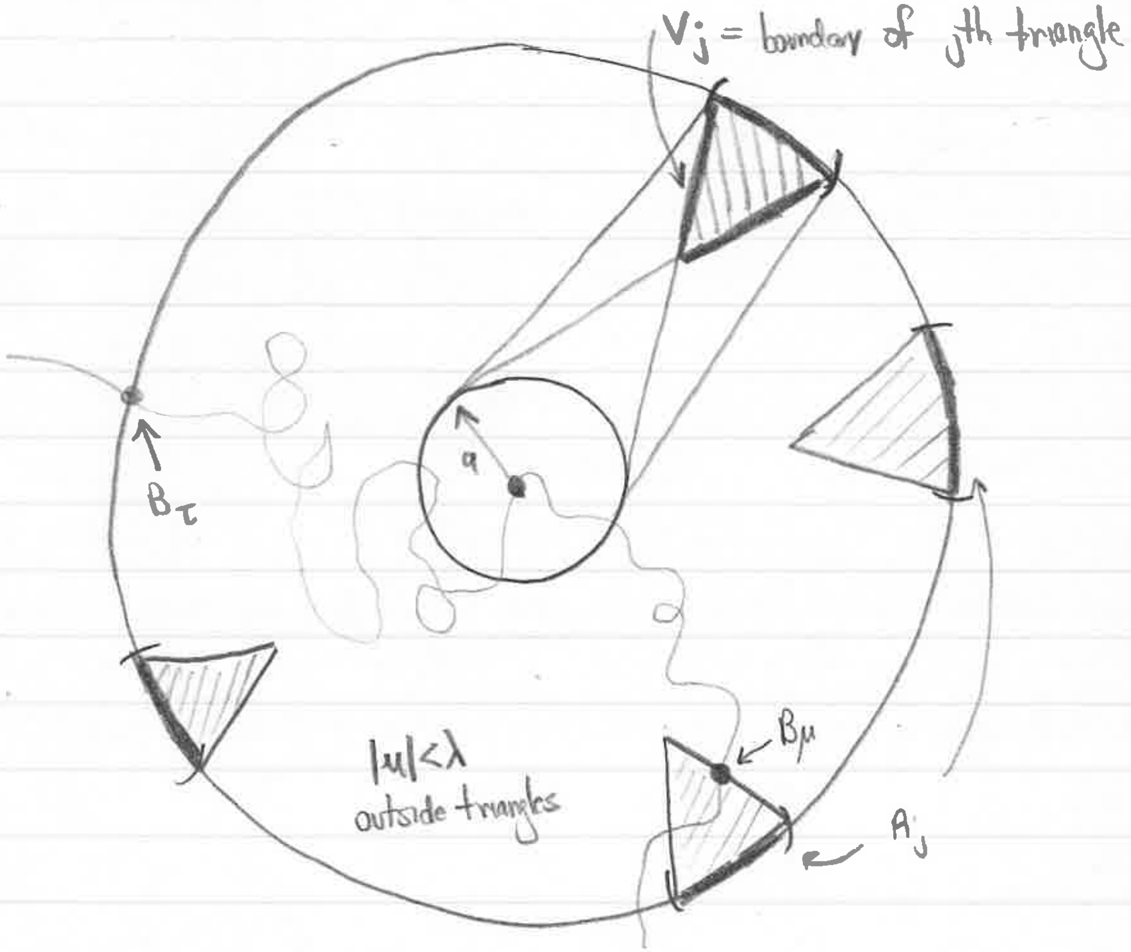


Outside the triangles, $|u| \leq \lambda$

12/8 MARTINGALES

LEMMA: $P(u^* > \lambda) \approx m(N_a(u) > \lambda)$, $\lambda > 0$

Proof. To show $P(u^* > \lambda) \leq C_a \, m(N_a(u) > \lambda)$



$A = \{e^{i\theta} : N_a(u)(\theta) > \lambda\} = \bigcup_j A_j$ ← open arcs

$$\frac{m(N_a(u) > \lambda)}{2\pi} = \frac{m(\{\theta \in [0, 2\pi] : e^{i\theta} \in A\})}{2\pi} = P(B_\tau \in A)$$

↑ 1st time hit ∂

↑ B_τ is uniformly distributed on $|z| = 1$

$$\mu = \inf\{t : B_{\tau \wedge t} \in UV_j\}$$

$$\{\mu < \infty\} = \{B \text{ hits } UV_j \text{ before } \partial D\}$$

Note that $\{\mu < \infty\} \supset \{u^* > \lambda\}$ and $\{\mu < \infty\} \supset \{B_\tau \in A\}$.
Then

$$P(B_\tau \in A) = P(B_\tau \in A, \mu < \infty)$$

[Define $\mathcal{B}_{\mu+} := \{B \in \mathcal{B}_\infty : \{\mu < t\} \cap B \in \mathcal{B}_t \ \forall t > 0\}$. This is a σ-field
Notice $\{\mu < \infty\} \in \mathcal{B}_{\mu+}$]

$$P(B_\tau \in A, \mu < \infty) = E\left[I(\mu<\infty) E(I(B_\tau \in A) \mid \mathcal{B}_{\mu+})\right]$$

$$= E\left[I(\mu<\infty) P(B_\tau \in A \mid \mathcal{B}_{\mu+})\right]$$

conditional probability

$$= E\left[I(\mu<\infty) f(B_\mu)\right]$$

Strong Markov property

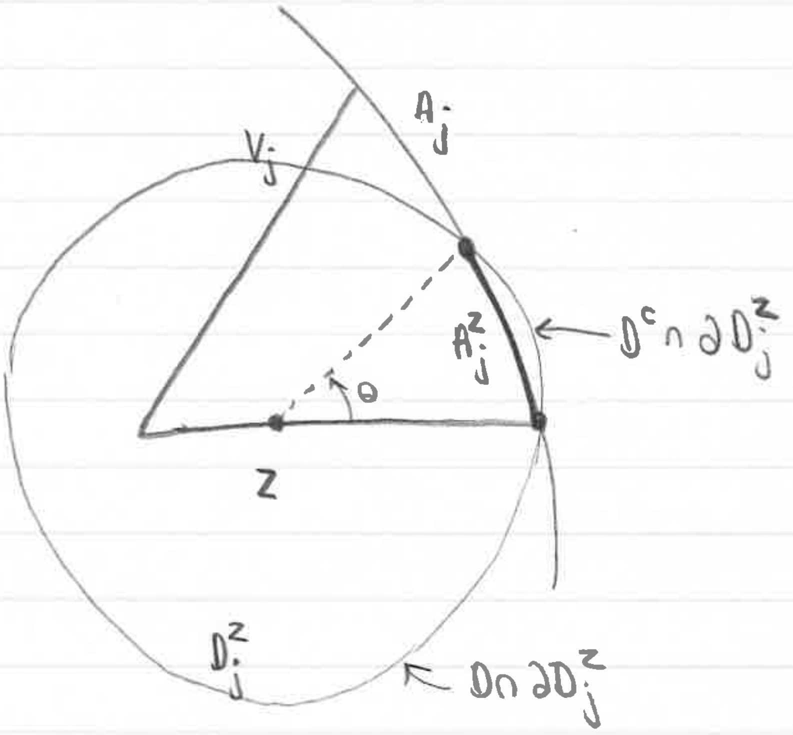
$$f(z) = \begin{cases} P(z+B \text{ hits } A \text{ before other part of } \partial D) = P_z(B_\tau \in A) \\ \text{if } z \in UV_j \end{cases}$$

Brownian motion starting at z

see below

$$\geq \frac{a}{\pi} P(\mu < \infty)$$

$$\geq \frac{a}{\pi} P(\mu^* > \lambda)$$



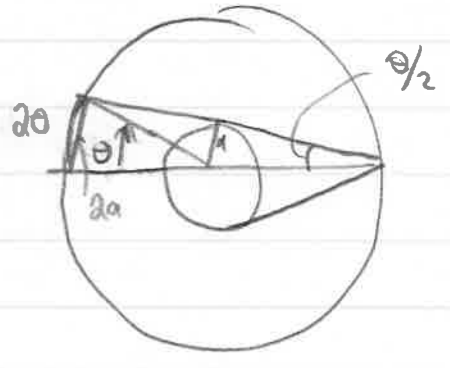
$f(z) \geq P($ $z+B$ hits A_j before other part of ∂D $)$ where $z \in V_j$

$\geq P($ $z+B$ hits A_j^z " " " " $)$

$\geq P($ $z+B$ hits A_j^z before $D \cap \partial D_j^z$ $)$

$\geq P($ $z+B$ hits $D^c \cap \partial D_j^z$ before $D \cap \partial D_j^z$ $)$

$= \theta/2\pi > \frac{2a}{2\pi} = \frac{a}{\pi}$



12/11 MARTINGALES

LEMMA: Let X be Ito integral, $X_t = \int_0^t \varphi \, dB$. If $\|\mathcal{S}(X)\|_1 < \infty$ then X is a uniformly integrable martingale

Proof. UI follows from

$$\|X^*\|_1 \leq c \|\mathcal{S}(X)\|_1$$

To show: $E(X_t \mid \overline{\mathcal{B}}_s) = X_s$ a.s. for $0 < s < t$. Let

$$Y_t = \int_0^t \psi \, dB$$

$\uparrow$ elementary $\int_0^t \psi^2 dt \leq \int_0^t \varphi^2 dt$

We have already seen that this is a martingale

$$\|E(X_t - X_s \mid \overline{\mathcal{B}}_s)\|_1 = \|E((X_t - X_s) - (Y_t - Y_s) \mid \overline{\mathcal{B}}_s\|_1$$

$$\leq \|X_t - Y_t\| + \|X_s - Y_s\|_1$$

$$\leq 2 \|(X-Y)^*\|_1$$

$$\leq c \|\mathcal{S}(X-Y)\|_1 \longrightarrow 0 \text{ as } Y \to X$$

▨

Let $F = u + iv$ be analytic with $F(0) = 0$. Then

$$\int_0^{2\pi} |u(e^{i\theta})|^p \, d\theta \approx \int_0^{2\pi} |v(e^{i\theta})|^p \, d\theta \qquad \text{(M. RIESZ)}$$

$(1 < p < \infty)$

Proof.

$$\frac{1}{2\pi}\int_0^{2\pi} |v(e^{i\theta})|^p \, d\theta = E|v(B_\tau)|^p$$

↖ 1st time you hit boundary

$$\leq E(v^*)^p \qquad v^* = \sup_{0 \leq t \leq \tau} |v(B_t)|$$

$p > 1$ →

$$\leq c_p E(u^*)^p$$

[Now $u(B_{\tau\wedge t}) = \int_0^{\tau \wedge t} \nabla u(B) \cdot dB$ is a martingale with

$$\|S(u)\|_1 \leq \sup_{z \in D} |\nabla u(s)| \, E\tau^{1/2} < \infty$$]

$$\leq C_p \|u(B_\tau)\|_p^p$$

Doob's ineq. →
$\|X^*\|_p \leq q \|X\|_p$

$$= C_p \frac{1}{2\pi} \int_0^{2\pi} |u(e^{i\theta})|^p \, d\theta$$

(Have equality for $p = 2$ by looking at $F^2 = u^2 - v^2 + 2i\,uv$ ↖ analytic

$$0 = \frac{1}{2\pi}\int_0^{2\pi} F^2(e^{i\theta}) \, d\theta \,)$$

Let $f(\theta) = u(e^{i\theta})$, $\tilde{f}(\theta) = v(e^{i\theta})$. Under the same hypotheses as above, Kolmogorov showed that

$$\lambda\, m(|\tilde{f}|>\lambda) \leq c \int_0^{2\pi} |f(e^{i\theta})|\, d\theta$$

Burgess Davis showed that the best constant is

$$c = \frac{1+1/3^2+1/5^2+1/7^2+\dots}{1-1/3^2+1/5^2-1/7^2+\dots}$$

LEMMA: For $f, g \geq 0$, if $P(g>\beta\lambda, f \leq \delta\lambda) \leq \varepsilon P(g>\lambda)$, $\forall \lambda > 0$ $\Phi(\beta\lambda) \leq \gamma\,\Phi(\lambda)$ and $\gamma\varepsilon<1$, then

$$\sup_{\lambda>0} \Phi(\lambda)\, P(g>\lambda) \leq \frac{\gamma\eta}{1-\gamma\varepsilon} \sup_{\lambda>0} \Phi(\lambda)\, P(f>\lambda)$$

Application: Let $\Phi(\lambda) = \lambda$. If $F = u+iv$ is analytic we know

$$P(v^* > \beta\lambda,\ u^* < \delta\lambda) \leq \frac{\delta^2}{(\beta-1)^2} P(v^*>\lambda) \quad \forall\lambda$$

By the lemma

$$\sup_{\lambda>0} \lambda P(v^*>\lambda) \leq c \sup_{\lambda>0} \lambda P(u^*>\lambda)$$

Then

$$\frac{1}{2\pi}\lambda m\left(|v(e^{i\theta})|>\lambda\right) = \lambda P(|v(B_\tau|>\lambda) \leq \sup_{\lambda>0} \lambda P(u^*>\lambda)$$

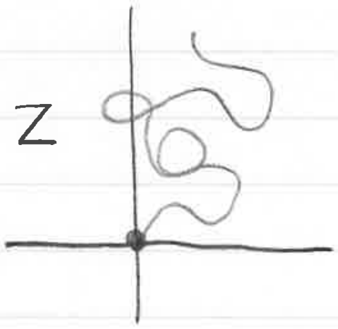
$\leq L'$-norm of martingale $u(B_{\tau\wedge t})$

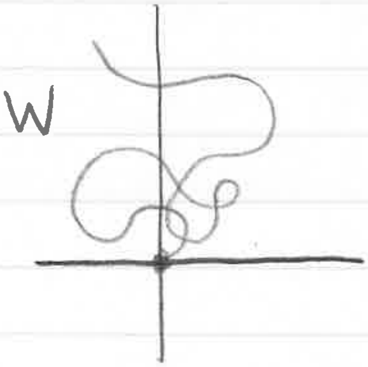
$$= E|u(B_\tau)| = \frac{1}{2\pi}\int_0^{2\pi} |u(e^{i\theta})|\, d\theta$$

This gives Kolmogorov's result.

Davis's result uses

LEVY'S THEOREM: " An analytic function maps a Brownian path onto a Brownian path "





Let F be entire, $F(0)=0$. The function $t \mapsto F(Z_t)$ is continuous has independent increments, and the image of 0 is 0. However it is not $N(0,t)$. Let $\alpha(t,w)$ be random variable. If we choose $\alpha(t,w)$ right, then $W_t = F(Z_{\alpha(t)})$ is complex Brownian motion

$\alpha(t,w)$ is the inverse of $\beta(t,w) := \int_0^t |F'(B)\omega|^2 ds$

12/13 MARTINGALE

Let $Z = X + iY$ be Brownian motion in $\mathbb{C}$ (starting at 0)

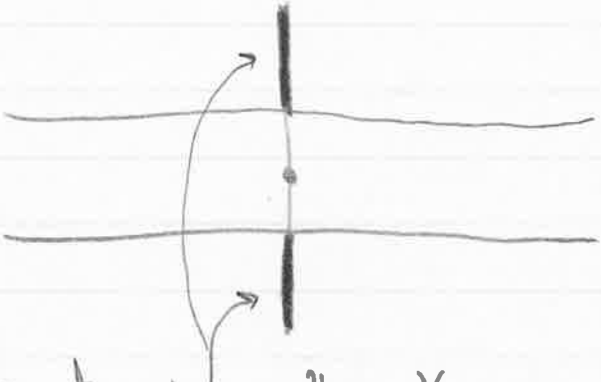
LEMMA: (B. Davis) Let $\tau = \inf\{t : |Y_t| = 1\}$. Let ν be any other stopping time of Z such that Z^ν is uniformly integrable (or just uniformly bounded). Then

$$(*) \qquad P(|Y_\nu| \geq 1) \leq \frac{E|X_\nu|}{E|X_\tau|} \leq KE|X_\nu|$$

[$E|X_\tau|$ = expected absolute displacement in the x-direction.

$$K = \frac{1}{E|X_\tau|} = \frac{1 + 1/3^2 + 1/5^2 + 1/7^2 + \dots}{1 - 1/3^2 + 1/5^2 - 1/7^2 + \dots}$$]

Remarks: (i) UI can not be entirely eliminated. Otherwise, consider



ν to be the hitting time of . Here $X_\nu = 0$, $|Y_\nu| \geq 1$, and we would get $1 \leq 0$ in $(*)$

(ii) ν can be more general

(iii) Equality holds if $\nu = \tau$

KOLMOGOROV INEQUALITY: $F = u+iv$ entire, $F(0)=0$. Then

$$\lambda m(\{\theta : |v(e^{i\theta})| \geq \lambda\}) \leq K \int_0^{2\pi} |u(e^{i\theta})|\, d\theta$$

Proof. Suffices to show this for $\lambda = 1$. Define

$$W_t := F(Z_{\alpha(t)}), \quad t \geq 0$$

depends on ω, too

Then W is a complex Brownian motion, starting at 0.

$$\alpha(t) = \beta^{-1}(t) \qquad \beta(t) = \int_0^t |F'(z_s)|^2 ds$$

Write $W = U + iV$.
Let $\mu = \inf\{t : |Z_t| = 1\}$

$$P(|v(Z_\mu)| \geq 1) = \frac{1}{2\pi} m(|v(e^{i\theta})| \geq 1)$$

$$E|u(Z_\mu)| = \frac{1}{2\pi}\int_0^{2\pi} |u(e^{i\theta})|\, d\theta$$

Let

$$\nu = \beta(\mu) = \int_0^\mu |F'(z_s)|^2 ds$$

Then

$$W_\nu = W_{\beta(\mu)} = u(Z_\mu) + iv(Z_\mu) = U_\nu + iV_\nu$$

Hence we want to show

$$P(|V_\nu| \geq 1) \leq K E|U_\nu|$$

But this follows from the lemma.

▨

Let B be BM in ℂ, starting at 0. Then

$$P(B_t \neq 0, \ t > 0) = 1$$

(Lévy, Kakutani)

Take $F(z) = e^z$, $F(0) = 0$. $F(z) = 1 - e^z$,

$$P(W_t = 1 - e^{Z_\alpha(t)} \neq 1, \ t \geq 0) = 1 \text{ a.s.}$$

Doob TAMS 1954

Browman motion without using Ito integrals

Books:

Meyer
Garsia
Neveu
Petersen

LEMMA 1: Suppose $f = (f_n : n \in \mathbb{N})$ is a (real-valued) martingale. Then $f^+ := (f_n^+ : n \in \mathbb{N})$ and $f^- := (f_n^- : n \in \mathbb{N})$ are submartingales.

Proof. For each n, f_n^+ and f_n^- are $\mathcal{A}_n$-measurable and integrable. Also

$$f_n = E(f_{n+1} | \mathcal{A}_n) = E(f_{n+1}^+ - f_{n+1}^- | \mathcal{A}_n)$$

$$= E(f_{n+1}^+ | \mathcal{A}_n) - E(f_{n+1}^- | \mathcal{A}_n)$$

Since $E(f_{n+1}^+ | \mathcal{A}_n) \geqslant 0$ and $E(f_{n+1}^- | \mathcal{A}_n) \geqslant 0$, we must have

$$f_n^+ \leqslant E(f_{n+1}^+ | \mathcal{A}_n)$$
$$f_n^- \leqslant E(f_{n+1}^- | \mathcal{A}_n)$$

Therefore f^+ and f^- are submartingales.

▨

LEMMA 2: If $f = (f_n : n \in \mathbb{N})$ is a (real-valued) martingale, then $|f| := (|f_n| : n \in \mathbb{N})$ is a submartingale.

Proof. Since $|f| = f^+ + f^-$, this is a direct consequence of lemma 1.

▨

LEMMA 3: Let B be a Banach space. If $X \in L^1(\Omega, \Sigma, P; B)$, and if $\mathcal{A}$ is a sub-σ-algebra of Σ, then

$$E(f(X) \mid \mathcal{A}) = f(E(X \mid \mathcal{A})) \quad \text{a.e.}$$

for every $f \in B^*$.

Proof. Let $f \in B^*$. Since $X \in L^1(\Omega, \Sigma, P; B)$, X is weakly measurable, and so $f(X)$ is Σ-measurable. Moreover, $E(X \mid \mathcal{A})$ is also weakly measurable with respect to $\mathcal{A}$, and so $f(E(X \mid \mathcal{A}))$ is $\mathcal{A}$-measurable.

Now

$$\int_A X \, dP = \int_A E(X \mid \mathcal{A}) \, dP$$

for every $A \in \mathcal{A}$ (where we may consider these as Pettis integrals). Then

$$\int_A f(X) \, dP = f\left(\int_A X \, dP \right) = f\left(\int_A E(X \mid \mathcal{A}) \, dP \right) = \int_A f(E(X \mid \mathcal{A})) \, dP$$

for every $A \in \mathcal{A}$, and so $E(f(X) \mid \mathcal{A}) = f(E(X \mid \mathcal{A}))$ a.e.

▨

THEOREM: If $(X_n : n \in \mathbb{N})$ is a martingale in $L^1(\Omega, \Sigma, P; B)$, then $(\|X_n\| : n \in \mathbb{N})$ is a real-valued submartingale.

Proof. Let $f \in B^*$ with $\|f\| \leqslant 1$. By lemma 3,

$$E(f(X_{n+1}) | a_n) = f(E(X_{n+1} | a_n)) = f(X_n)$$

and so $(f(X_n) : n \in \mathbb{N})$ is a (real-valued) martingale. Then by lemma 2, $(|f(X_n)| : n \in \mathbb{N})$ is a submartingale. Therefore

$$|f(X_n)| \leqslant E\Big(|f(X_{n+1})| \Big| a_n\Big) \underset{\substack{\uparrow \\ |f(X_{n+1})| \leqslant \|X_{n+1}\|}}{\leqslant} E(\|X_{n+1}\| | a_n)$$

Hence

$$\forall n \in \mathbb{N} \quad \|X_n\| = \sup_{\|f\| \leqslant 1} |f(X_n)| \leqslant E\Big(\|X_{n+1}\| \Big| a_n\Big)$$

and so $(\|X_n\| : n \in \mathbb{N})$ is a submartingale.

▨

1. Let P be Lebesgue measure on the σ-field $\mathcal{A}$ of Borel subsets of $\Omega = [0, 1)$. Let n be a positive integer. A set $B \subset \Omega$ is periodic with period $\frac{1}{n}$ if

$$x \in B \iff x + \frac{1}{n} \in B$$

(addition mod 1). Show that

(i) $\mathcal{B} = \{ B \in \mathcal{A} : B$ is periodic with period $\frac{1}{n} \}$ is a σ-field,

(ii) if F is integrable, then g defined by

$$g(x) = \frac{1}{n} \sum_{k=0}^{n-1} F\left(x + \frac{k}{n}\right)$$

is the conditional expectation of F given $\mathcal{B}$.

2. If f is integrable or nonnegative $\mathcal{A}$-measurable and $\mathcal{C} \subset \mathcal{B}$ are sub-σ-fields of $\mathcal{A}$, then

$$E\left[E(f \mid \mathcal{B}) \,\middle|\, \mathcal{C}\right] = E(f \mid \mathcal{C}) \quad \text{a.e.}$$

Write out the proof.

3. (Double-or-nothing) Let $\Omega = \{1, 2, \ldots\}$ and $\mathcal{A}$ be the σ-field of all subsets of Ω. For each positive integer n, let $\mathcal{A}_n$ be the sub-σ-field generated by the partition

$$\{\{1\}, \ldots, \{n\}, \{n+1, n+2, \ldots\}\},$$

and let

$$\begin{aligned} f_n(\omega) &= 2^n \quad \text{if} \quad \omega > n, \\ &= 0 \quad \text{if} \quad \omega \leq n. \end{aligned}$$

Define P so that $E(f_{n+1} \mid \mathcal{A}_n) = f_n$, $n \geq 1$.

4. Let $(\Omega, \mathcal{A}, P)$ be a probability space and G a finite group of transformations from Ω to Ω ($\varphi \in G \Rightarrow \varphi$ is 1-1 onto and $\varphi^{-1} \in G$; the composition of two functions in G is in G). In addition, suppose that each φ in G is measure-preserving (if $A \in \mathcal{A}$, then $\varphi^{-1}(A) \in \mathcal{A}$ and $P(\varphi^{-1}(A)) = P(A)$). Let $\mathcal{B}$ be the class of invariant sets in $\mathcal{A}$:

$$\mathcal{B} = \{A \in \mathcal{A} : \varphi^{-1}(A) = A, \ \varphi \in G\}$$

Show that

(i) $\mathcal{B}$ is a σ-field,

(ii) if f is integrable or nonnegative $\mathcal{A}$-measurable then

$$E(f \mid \mathcal{B}) = \frac{\sum_{\varphi \in G} f(\varphi)}{|G|} \quad \text{a.e.}$$

Here $|G|$ denotes the number of elements in G.

A

① Let P be Lebesgue measure on the σ-field $\mathcal{A}$ of Borel subsets of $\Omega = [0,1)$.

DEFINITION: Let n be a positive integer. A set $B \subset \Omega$ is periodic with period $1/n$ if

$$(*) \qquad x \in B \iff x + 1/n \in B$$

(addition mod 1)

PROPOSITION: $\mathcal{B}_n := \{ B \in \mathcal{A} : B \text{ is periodic with period } 1/n \}$ is a σ-field.

Proof. Clearly Ω is periodic, so $\Omega \in \mathcal{B}_n$. Now suppose $B \in \mathcal{B}_n$. Then $B^c \in \mathcal{A}$. The negation of $(*)$ says

$$x \in B^c \iff x + 1/n \in B^c$$

and so B^c is periodic with period $1/n$. Hence $B^c \in \mathcal{B}_n$. Next suppose $(B_k) \subset \mathcal{B}_n$. Then $\bigcup_{k=1}^{\infty} B_k \in \mathcal{A}$. Also

$$x \in \bigcup_{k=1}^{\infty} B_k \iff x \in B_{k_0} \text{ for some } k_0 \iff x + 1/n \in B_{k_0} \text{ for some } k_0$$

$$\iff x + 1/n \in \bigcup_{k=1}^{\infty} B_k ,$$

and so $\bigcup_{k=1}^{\infty} B_k \in \mathcal{B}_n$. Hence $\mathcal{B}_n$ is a σ-algebra. ▨

PROPOSITION: If f is integrable, then g defined by

$$g(x) := \frac{1}{n} \sum_{k=0}^{n-1} f(x + \tfrac{k}{n})$$

is the conditional expectation of f given $\mathcal{B}_n$.

Proof. First note that if $h(x) := f(x+\alpha)$, then

$$h^{-1}(U) = \{x \in \Omega : h(x) \in U\} = \{x \in \Omega : f(x+\alpha) \in U\}$$

$$= \{y - \alpha : f(y) \in U\} = f^{-1}(U) - \alpha$$

Therefore g is Borel measurable (since translates of Borel measurable sets are Borel measurable). Note that

$$g(x + 1/n) = \frac{1}{n} \sum_{k=0}^{n-1} f(x + \tfrac{k+1}{n})$$

$$= \frac{1}{n}\left[\sum_{k=1}^{n-1} f(x + k/n) + f(x+1)\right]$$

$$= \frac{1}{n}\left[\sum_{k=1}^{n-1} f(x + k/n) + f(x)\right]$$

$$= \frac{1}{n}\left[\sum_{k=0}^{n-1} f(x + k/n)\right]$$

$$= g(x)$$

Hence, if U is open, $x \in g^{-1}(U)$ if and only if $x + 1/n \in g^{-1}(U)$. Therefore $g^{-1}(U) \in \mathcal{B}_n$, and so g is $\mathcal{B}_n$-measurable.

Suppose $B \in \mathcal{B}_n$. Then

$$\int_B g = \frac{1}{n} \sum_{k=0}^{n-1} \int_B f(x+\tfrac{k}{n})\,dx$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \int_{B_k} f(y)\,dy$$

where $B_k := \{ x + k/n : x \in B \}$. But B is $1/n$-periodic, and so $B_k = B$. Therefore

$$\int_B g = \frac{1}{n} \sum_{k=0}^{n-1} \int_B f(y)\,dy = \int_B f .$$

Therefore we conclude that $E(f|\mathcal{B}_n) = g$.

✓

▨

② **PROPOSITION:** If f is integrable or non-negative $\mathcal{A}$-measurable and $\mathcal{C} \subset \mathcal{B}$ are sub-σ-fields of $\mathcal{A}$, then

$$E[E(f|\mathcal{B})\,|\,\mathcal{C}] = E[f|\mathcal{C}] \text{ a.e.}$$

Proof. First we note that $E[f|\mathcal{C}]$ is $\mathcal{C}$-measurable. Now if $C \in \mathcal{C}$, then $C \in \mathcal{B}$, and so

$$\int_C E[f|\mathcal{C}] = \int_C f = \int_C E[f|\mathcal{B}]$$

By the uniqueness of conditional expectation, we see that

$$E[f|\mathcal{C}] = E[E[f|\mathcal{B}]\,|\,\mathcal{C}] \text{ a.e.}$$

▨ ✓

③ Let $\Omega = \mathbb{N}$ and $\mathcal{A} = \mathcal{P}(\mathbb{N})$. For each $n \in \mathbb{N}$, let $\mathcal{A}_n$ be the sub-σ-field generated by the partition

$$\{\{1\}, \{2\}, \dots, \{n\}, \{n+1, n+2, \dots\}\}.$$

Also for each $n \in \mathbb{N}$ let

$$f_n(\omega) := \begin{cases} 2^n & \text{if } \omega > n \\ 0 & \text{if } \omega \leq n \end{cases}$$

for every $\omega \in \Omega$. It is immediate that $\mathcal{A}_n \subset \mathcal{A}_{n+1}$.

CLAIM - f_n is $\mathcal{A}_n$ measurable.

Proof. Let U be an open set in $\mathbb{R}$. Then

$$f_n^{-1}(U) = \begin{cases} \phi & \text{if } 0 \notin U, 2^n \notin U \\ \{n+1, n+2, \dots\} & \text{if } 0 \notin U, 2^n \in U \\ \bigcup_{k=1}^{n} \{k\} & \text{if } 0 \in U, 2^n \notin U \\ \Omega & \text{if } 0 \in U, 2^n \in U \end{cases}$$

Hence $f_n^{-1}(U) \in \mathcal{A}_n$, and so f_n is $\mathcal{A}_n$-measurable.

Choose $\alpha \in [0,1]$. Define P on $\mathcal{A}$ by

$$P(\{1\}) := \alpha$$

$$P(\{k\}) := \frac{1-\alpha}{2^{k-1}}, \quad k \geq 2$$

and for $A \in \mathcal{A}$

$$P(A) := \bigcup_{k \in A} P(\{k\}).$$

CLAIM: $E(f_{n+1} \mid \mathcal{A}_n) = f_n$

Proof. Each f_n is a simple function and therefore integrable. Now suppose $1 \leq k \leq n$. Then $f_{n+1}(k) = 0 = f_n(k)$, and so

$$\int_{\{k\}} f_{n+1} = 0 = \int_{\{k\}} f_n.$$

Now for $\{n+1, n+2, \ldots\} \in \mathcal{A}_n$,

(since $f_{n+1}(n+1) = 0$)

$$\int_{\{n+1,n+2,\ldots\}} f_{n+1} = 2^{n+1} P(\{n+2, n+3, \ldots\})$$

$$= 2^{n+1} \sum_{k=n+2}^{\infty} \frac{1-\alpha}{2^{k-1}}$$

$$= 2^{n+1}(1-\alpha)\frac{1}{2^n} = 2(1-\alpha)$$

Moreover,

$$\int_{\{n+1,n+2,\dots\}} f_n = 2^n P(\{n+1,n+2,\dots\})$$

$$= 2^n \sum_{k=n+1}^{\infty} \frac{1-\alpha}{2^{k-1}}$$

$$= 2^n (1-\alpha) \frac{1}{2^{n-1}}$$

$$= 2(1-\alpha)$$

Hence $\int_A f_{n+1} = \int_A f_n$ on the generating set of $\mathcal{A}_n$, and so

$$\int_A f_{n+1} = \int_A f_n$$

for all $A \in \mathcal{A}_n$. Hence $E(f_{n+1} | \mathcal{A}_n) = f_n$ a.e.

✓

④ Let $(\Omega, \mathcal{a}, P)$ be a probability space and G a finite group of transformations from Ω to Ω. Suppose that each φ in G is measure-preserving

Define

$$\mathcal{B} := \{ A \in \mathcal{a} : \varphi^{-1}(A) = A \text{ for all } \varphi \in G \}.$$

PROPOSITION: $\mathcal{B}$ is a σ-field.

Proof. Since $\varphi^{-1}(\Omega) = \Omega$, we have $\Omega \in \mathcal{B}$. Suppose $A \in \mathcal{B}$. Then $A^c \in \mathcal{a}$. Moreover, for all $\varphi \in G$

$$\varphi^{-1}(A^c) = [\varphi^{-1}(A)]^c = A^c$$

and so $A^c \in \mathcal{B}$. Now suppose $(A_n) \subset \mathcal{B}$. Then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{a}$. Also

$$\varphi^{-1}\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \bigcup_{n=1}^{\infty} \varphi^{-1}(A_n) = \bigcup_{n=1}^{\infty} A_n$$

and so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$. Hence $\mathcal{B}$ is a σ-field.

▨

PROPOSITION: If f is integrable or non-negative $\mathcal{a}$-measurable, then

$$E(f|\mathcal{B}) = \frac{1}{|G|} \sum_{\varphi \in G} f(\varphi) \quad \text{a.e.}$$

Proof. Let

$$h := \frac{1}{|G|} \sum_{\varphi \in G} f(\varphi)$$

Let U be an open set. Then for $\varphi \in G$,

$$(f \circ \varphi)^{-1}(U) = \varphi^{-1}(f^{-1}(U)) \in \mathcal{A}$$

since $f^{-1}(U) \in \mathcal{A}$ and G is measure-preserving. Therefore we may conclude that $h^{-1}(U) \in \mathcal{A}$.

Suppose $x \in \Omega$ and $\psi \in G$. Then

$$h(\psi(x)) = \frac{1}{|G|} \sum_{\varphi \in G} f(\varphi \circ \psi(x)) = \frac{1}{|G|} \sum_{\xi \in G} f(\xi(x))$$

$$= h(x)$$

since as φ roams over G, $\varphi \circ \psi$ also roams once over all elements of G, i.e.

$$\{\varphi \circ \psi : \varphi \in G\} = G.$$

Therefore $x \in h^{-1}(U)$ if and only if $x \in \psi^{-1}(h^{-1}(U))$, and so $\psi^{-1}(h^{-1}(U)) = h^{-1}(U)$ for every $\psi \in G$. Therefore $h^{-1}(U) \in \mathcal{B}$. Hence h is $\mathcal{B}$-measurable.

Suppose $B \in \mathcal{B}$. Then

$$\int_B h = \frac{1}{|G|}\sum_{\varphi\in G}\int_B f(\varphi(x))\,dP \underset{\substack{\uparrow\\ P(\varphi(B))=P(B)}}{=} \frac{1}{|G|}\sum_{\varphi\in G}\int_{\varphi(B)} f(y)\,dP$$

$$\underset{\substack{\uparrow\\ \varphi(B)=B\ \forall\varphi\in G}}{=} \frac{1}{|G|}\sum_{\varphi\in G}\int_B f(y)\,dP = \int_B f$$

Hence we conclude that $h = E(f|\mathcal{B})$ a.e.

✓

▨

(Due 9/29)

5. Let $f = (f_1, f_2, \dots)$ be an L^1-bounded nonnegative submartingale. Let $F_n = \sup_{k \geq n} E(f_k \mid \mathcal{A}_n)$. Show that $F = (F_1, F_2, \dots)$ is a martingale satisfying

(i) $f_n \leq F_n$ a.e., $n \geq 1$,

(ii) $\|f\|_1 = \|F\|_1$,

(iii) F is the smallest martingale with property (i), in the sense that a martingale G satisfying $f \leq G$ a.e. must also satisfy $F \leq G$ a.e.

HINT. $E(f_k \mid \mathcal{A}_n) \leq E(f_{k+1} \mid \mathcal{A}_n)$ a.e. so that $F_n = \lim_{k \to \infty} E(f_k \mid \mathcal{A}_n)$ a.e.

6. Let f be an L^1-bounded martingale. Show that $f = g - h$ where g and h are nonnegative martingales. HINT. Use 5.

A

⑤ Let $f = (f_n : n \in \mathbb{N})$ be an L^1-bounded non-negative submartingale. Let

$$F_n := \sup_{k \geq n} E(f_k \mid \mathcal{A}_n).$$

CLAIM: $F = (F_n : n \in \mathbb{N})$ is a martingale.

Proof. First note that since f is a submartingale,

$$f_k \leq E(f_{k+1} \mid \mathcal{A}_k) \quad \text{a.e.}$$

and so if $k \geq n$,

$$E(f_k \mid \mathcal{A}_n) \leq E(E(f_{k+1} \mid \mathcal{A}_k) \mid \mathcal{A}_n) \underset{\substack{\uparrow \\ \mathcal{A}_n \subset \mathcal{A}_k}}{=} E(f_{k+1} \mid \mathcal{A}_n) \quad \text{a.e.}$$

Hence

$$(1) \qquad F_n = \sup_{k \geq n} E(f_k \mid \mathcal{A}_n) = \lim_{k \to \infty} E(f_k \mid \mathcal{A}_n) \quad \text{a.e.}$$

Now, for each $k \geq n$, $E(f_k \mid \mathcal{A}_n)$ is $\mathcal{A}_n$-measurable, and so F_n is $\mathcal{A}_n$-measurable. Moreover, because the limit in (1) is actually monotone increasing (at least for $k \geq n$), we have by the Monotone Convergence theorem, $(E(F_k \mid \mathcal{A}_n) = F_k,\ k < n)$

(a) $$EF_n = \lim_{k\to\infty} E(E(f_k|\mathcal{a}_n))$$

$$= \lim_{k\to\infty} Ef_k$$

$$\leqslant \|f\|_1 < \infty$$

since f is L^1-bounded and non-negative.

Now $E(f_k|\mathcal{a}_n)$ converges monotonically to F_n, and so by the Monotone Convergence theorem for conditional expectations

$$\lim_{k\to\infty} E(E(f_k|\mathcal{a}_n)|\mathcal{a}_{n-1}) = E(F_n|\mathcal{a}_{n-1}).$$

But we also have

$$\lim_{k\to\infty} E(E(f_k|\mathcal{a}_n)|\mathcal{a}_{n-1})$$

$$= \lim_{k\to\infty} E(f_k|\mathcal{a}_{n-1}) = F_{n-1}.$$

Hence $E(F_n|\mathcal{a}_{n-1}) = F_{n-1}$, and therefore we may conclude that $F = (F_n : n \in \mathbb{N})$ is a martingale.

✓

▨

PROPOSITION: The martingale $F = (F_n : n \in \mathbb{N})$ defined above has the following properties:

(i) $f_n \leq F_n$ a.e. $\forall n$

(ii) $\|f\|_1 = \|F\|_1$

(iii) If G is a martingale satisfying $f \leq G$ a.e, then $F \leq G$ a.e.

Proof. (i) For each $k \geq n$, $E(f_k | a_n) \geq f_n$ a.e. since f is a submartingale, and so

$$F_n = \sup_{k \geq n} E(f_k | a_n) \geq f_n \text{ a.e.}$$

(ii) We have already seen by (2) that

$$EF_n \leq \|f\|_1 \quad \forall n$$

and so

$$\|F\|_1 = \sup_n EF_n \leq \|f\|_1$$

But by (i), $Ef_n \leq EF_n$ for every n, and so $\|f\|_1 \leq \|F\|_1$. Hence we actually have $\|f\|_1 = \|F\|_1$.

(iii) Suppose G is a martingale satisfying $f \leq G$ a.e. Then for each n,

$$F_n = \sup_{k \geq n} E(f_k | a_n) \leq \sup_{k \geq n} E(G_k | a_n) = \sup_{k \geq n} G_n = G_n$$

Hence $F \leqslant G$.

▨

<u>PROPOSITION</u>: Let f be an L^1-bounded martingale. Then $f = G - H$, where G and H are non-negative martingales.

Proof. For each n, let

$$g_n := f_n^+$$
$$h_n := f_n^-$$

Then $g = (g_n : n \in \mathbb{N})$ and $h = (h_n : n \in \mathbb{N})$ are L^1-bounded non-negative submartingales. For $f_n \leq g_n$ implies

$$f_{n-1} = E(f_n \,|\, \mathcal{A}_{n-1}) \leq E(g_n \,|\, \mathcal{A}_{n-1})$$

and so $g_{n-1} = f_{n-1}^+ \leq E(g_n \,|\, \mathcal{A}_{n-1})$. Similarly, $-h_n \leq f_n$ implies

$$-E(h_n \,|\, \mathcal{A}_{n-1}) = E(-h_n \,|\, \mathcal{A}_{n-1}) \leq E(f_n \,|\, \mathcal{A}_{n-1}) = f_{n-1}$$

and so $-E(h_n \,|\, \mathcal{A}_{n-1}) \leq -f_{n-1}^- = -h_{n-1}$, i.e. $h_{n-1} \leq E(h_n \,|\, \mathcal{A}_{n-1})$.
The L^1-boundedness follows from the observation that

$$\forall n \qquad \|f_n^+\|_1 + \|f_n^-\|_1 = \|f_n\|_1 < K < \infty$$

Now let

$$G_n := \sup_{k \geq n} E(g_k | \mathcal{A}_n) = \lim_{k \to \infty} E(g_k | \mathcal{A}_n)$$

$$H_n := \sup_{k \geq n} E(h_k | \mathcal{A}_n) = \lim_{k \to \infty} E(h_k | \mathcal{A}_n)$$

By the previous proposition $G := (G_n : n \in \mathbb{N})$ and $H := (H_n : n \in \mathbb{N})$ are non-negative martingales.

Now for each $n \in \mathbb{N}$

$E(f_k | \mathcal{A}_n) = f_n$ $\forall k \geq n$

$$\begin{aligned} f_n &= \lim_{k \to \infty} E(f_k | \mathcal{A}_n) \\ &= \lim_{k \to \infty} E(g_k - h_k | \mathcal{A}_n) \\ &= \lim_{k \to \infty} \left[E(g_k | \mathcal{A}_n) - E(h_k | \mathcal{A}_n) \right] \\ &= \lim_{k \to \infty} E(g_k | \mathcal{A}_n) - \lim_{k \to \infty} E(h_k | \mathcal{A}_n) \\ &= G_n - H_n \end{aligned}$$

Hence $f = G - H$.

✓

▨

7. Let f be a martingale and $\lambda > 0$. Show that

$$E\left[(f^*)^2 \underset{\overset{\uparrow}{\text{min}}}{\wedge} \lambda\right] \le 5\, E\left[s^2(f) \wedge \lambda\right].$$

8. Let f be a martingale and $\beta > 1$, $0 < \delta < \beta - 1$, $\lambda > 0$. Show that

$$P\left(f^* > \beta\lambda,\ s(f) \overset{\overset{\text{max}}{\downarrow}}{\vee} d^* \le \delta\lambda\right)$$

$$\le \frac{\delta^2}{(\beta - \delta - 1)^2}\, P(f^* > \lambda).$$

A-

⑦ **PROPOSITION**: Let f be a martingale and $\lambda > 0$. Then

$$E\left[(f^*)^2 \wedge \lambda \right] \leq 5\, E\left[s^2(f) \wedge \lambda \right]$$

Proof. Let a stopping time τ be given by

$$\tau := \inf \{ n \geq 0 : s_{n+1}^2(f) > \lambda \}$$

We claim that

$$(f^*)^2 \wedge \lambda \leq (f_\tau^*)^2 + \lambda\, I\,[\tau < \infty]$$

(where $f_\infty^* = f^*$). For if $\tau = \infty$, then

$$(f^*)^2 \wedge \lambda \leq (f^*)^2 = (f_\infty^*)^2$$

and if $\tau < \infty$, then

$$(f^*)^2 \wedge \lambda \leq \lambda \leq (f_\tau^*)^2 + \lambda$$

Let f^τ be the martingale stopped at τ. Then

$$f_n^\tau = \sum_{k=1}^{n} d_k\, I(\tau \geq k)$$

Since $I(\tau \geq k)$ is a_{k-1} measurable, we have for each n

$$s_n^2(f^\tau) = \sum_{k=1}^{n} E\left(d_k^2\, I(\tau \geq k) \,|\, a_{k-1} \right)$$

$$= \sum_{k=1}^{n} I(\tau \geq k) E(d_k^2 | a_{k-1})$$

$$\leq s_\tau^2(f) \leq \lambda$$

(Hence each $d_k I(\tau \geq k)$ is square-integrable). We see that

$$E(s^2(f^\tau)) \leq E(s_\tau^2(f)) \leq E(s_\tau^2(f) \wedge \lambda)$$

Now

$$f_n^\tau = f_{\tau \wedge n} = \begin{cases} f_n & n \leq \tau \\ f_\tau & n > \tau \end{cases}$$

so that

$$(f^\tau)^* = \sup_{n \in \mathbb{N}} |f_n^\tau| = \sup_{n \leq \tau} |f_n| = (f_\tau)^*$$

Therefore

$$\| f_\tau^* \|_2 = \|(f^\tau)^*\|_2 \underset{\substack{\uparrow \\ L^2\text{-inequality}}}{\leq} 2 \| f^\tau \|_2 = 2 \| s(f^\tau) \|_2$$

and so

$$E((f_\tau^*)^2) \leq 4 E(s^2(f^\tau)) \leq 4 E(s_\tau^2(f) \wedge \lambda)$$

We also have

$$E(\lambda I(\tau<\infty)) = \int_{\tau<\infty} \lambda \leq \int_{\Delta^2(f)>\lambda} \lambda = \int_{\Delta^2(f)>\lambda} \Delta^2(f)\wedge\lambda$$

$$\leq \int_{\Omega} \Delta^2(f)\wedge\lambda = E(\Delta^2(f)\wedge\lambda)$$

Hence

$$E((f^*)^2\wedge\lambda) \leq E((f^*_\tau)^2) + E(\lambda I(\tau<\infty))$$

$$\leq 4E(\Delta^2(f)\wedge\lambda) + E(\Delta^2(f)\wedge\lambda)$$

$$= 5E(\Delta^2(f)\wedge\lambda)$$

▨ ✓

⑧ PROPOSITION: Let f be a martingale and $\beta > 1$, $0 < \delta < \beta - 1$, $\lambda > 0$. Then

$$P\left(f^* > \beta\lambda,\ s(f) \vee d^* \leq \delta\lambda\right) \leqslant \frac{2\delta^2}{(\beta-\delta-1)^2} P\left(f^* > \lambda\right)$$

Proof. Define stopping times as follows:

$$\mu := \inf\{n : |f_n| > \lambda\}$$

$$\nu := \inf\{n : |f_n| > \beta\lambda\}$$

$$\sigma := \inf\{n \geq 0 : \underset{n+1}{s_n}(f) > \delta\lambda \text{ or } \underset{\underset{n}{\uparrow}}{|d_{n+1}|} > \delta\lambda\}.$$

← Actually not a stopping time as defined, since $|d_{n+1}|$ may be $\mathcal{A}_{n+1}$-meas but not $\mathcal{A}_n$-meas.

Let $g := {}^{\mu}f^{\nu \wedge \sigma}$, so that

$$g_n = \sum_{k=1}^{n} I(\mu < k \leq \nu \wedge \sigma)\, d_k$$

Note that if $\mu = \infty$, then $g = 0$. If $\sigma = \infty$, then

$$s^2(g) \leqslant s^2(f)\, I(\mu < \infty) \leqslant \delta^2\lambda^2\, I(\mu < \infty)$$

If $\sigma < \infty$, then

$$s^2(g) \leqslant \overbrace{s^2_\sigma(f)}^{\not\leq \delta\lambda} I(\mu < \infty) = \left[s^2_{\sigma-1}(f) + E(d^2_\sigma \mid \mathcal{A}_{\sigma-1})\right] I(\mu < \infty)$$

$$\leq (\delta^2\lambda^2 + \delta^2\lambda^2)\, I(\mu < \infty)$$

$$= 2\delta^2\lambda^2 \, I(\mu < \infty)$$

So in either case we have

$$s^2(g) \le 2\delta^2\lambda^2 \, I(\mu < \infty).$$

Now suppose $\mu \le \nu < \infty$ and $\sigma = \infty$. Then for $n > \nu$

$$g_n = \sum_{k=1}^{n} I(\mu < k \le \nu)\, d_k$$

$$= f_\nu - f_\mu$$

But

$$\lambda < |f_\mu| \le \lambda + \delta\lambda$$

$$\beta\lambda < |f_\nu| \le \beta\lambda + \delta\lambda$$

and so for $n > \nu$

$$|g_n| = |f_\nu - f_\mu| \ge |f_\nu| - |f_\mu| > \beta\lambda - (\delta\lambda + \lambda)$$

Hence

$$P(f^* > \beta\lambda,\, s(f) \vee d^* \le \delta\lambda) \le P(\mu \le \nu < \infty,\, \sigma = \infty)$$

$$\leq P\left(g^* > (\beta-\delta-1)\lambda\right) \overset{\substack{\text{weak } L^2 \text{ inequality}\\ \downarrow}}{\leq} \frac{\|g\|_2^2}{(\beta-\delta-1)^2\lambda^2}$$

$$= \frac{\|s(g)\|_2^2}{(\beta-\delta-1)^2\lambda^2} \leq \frac{2\delta^2\lambda^2 P(\mu<\infty)}{(\beta-\delta-1)^2\lambda^2}$$

$$\leq \frac{2\delta^2}{(\beta-\delta-1)^2} P\left(f^* > \lambda\right).$$

▨

9. Let f be a martingale and $a > 0$, $b > 0$. Derive the inequality

$$P(f^* > b,\ s(f) \leq a) \leq \frac{a^2}{b^2}.$$

Use this to show that

$$\{s(f) < \infty\} \subset_{a.e.} \{f \text{ converges}\}.$$

10. Suppose that f is a martingale with $|d_k|$ $\mathcal{A}_{k-1}$-measurable: For example, f might be given by $f_n = E(F \mid \mathcal{D}_n)$ where F is integrable on $[0,1)$ and $\mathcal{D}_n$ is generated by the n-th dyadic partition. Show (using inequalities derived during the course) that

$$\{f \text{ converges}\} =_{a.e.} \{S(f) < \infty\} =_{a.e.} \{f^* < \infty\}.$$

11. Suppose that u is harmonic in $\mathbb{R}^{n+1}_+$ and $a > 0$, $h > 0$, $0 < p < \infty$. Show that if

$$N_{a,h}(u) \in L^p(\mathbb{R}^n),$$

then there exists an $f \in L^p(\mathbb{R}^n)$ such that, for almost all x, u converges nontangentially to $f(x)$ at x and

$$\|u(\cdot, y) - f\|_p \to 0$$

as $y \to 0$.

A

⑨ **LEMMA:** Let f be a martingale and $a > 0$, $b > 0$. Then

$$P\left(f^* > b,\ s(f) \leq a\right) \leq \frac{4a^2}{b^2}$$

Proof. Let

$$\tau := \inf\{n : |f_n| > b\}$$
$$\sigma := \inf\{n \geq 0 : s_{n+1}(f) > a\}$$

Then τ and σ are stopping times:

$$\{\tau > n\} = \{|f_k| \leq b\ \forall k \leq n\} \in \mathcal{A}_n$$
$$\{\sigma \leq n\} = \{\sum_{k=1}^{n+1} E(d_k^2 \mid \mathcal{A}_{k-1}) > a^2\} \in \mathcal{A}_n$$

(since $E(d_{n+1}^2 \mid \mathcal{A}_n)$ is $\mathcal{A}_n$-measurable). Let $g = f^{\tau \wedge \sigma}$. Then g is also a martingale since $\tau \wedge \sigma$ is a stopping time. Note that

$$s^2(g) \leq s_\sigma^2(f) \leq a^2$$

and that if $\tau < \infty$ and $\sigma = \infty$, then

$$g^* = (f^\tau)^* \geq |f_\tau| > b.$$

Hence

$$P\left(f^* > b,\ s(f) \leq a \right) \leq P\left(\tau < \infty, \sigma = \infty \right)$$

$$\leq P\left(g^* > b \right)$$

$$\leq \frac{1}{b^2} \| g^* \|_2^2 \quad \text{(Weak } L_p\text{-inequality)}$$

$$\leq \frac{4}{b^2} \| g \|_2^2 \quad (L_2\text{-inequality})$$

Without 4 is possible by earlier mart. or submartingale ineq.

$$= \frac{4}{b^2} \| s(g) \|_2^2$$

$$\leq \frac{4a^2}{b^2}$$

✓

▨

PROPOSITION: If f is a martingale, then

$$\{ s(f) < \infty \} \subset_{a.e.} \{ f \text{ converges} \}$$

Proof. Let

$$LS(\omega) := \lim_{n \to \infty} \sup_{k,j \geq n} | f_k(\omega) - f_j(\omega) |$$

Note that if $LS(\omega) = 0$, then for every $\varepsilon > 0$, there is an $n_0 \in \mathbb{N}$ such that

$$\sup_{k,j \geq n} | f_k(\omega) - f_j(\omega) | < \varepsilon \quad \forall n > n_0$$

and so $(f_n(\omega))$ is Cauchy. Therefore if $LS(\omega)=0$, then $f(\omega)$ converges.

For each $n \in \mathbb{N}$ let

$$^n f := (f_n, f_{n+1}, f_{n+2}, \ldots, f_{n+k}, \ldots)$$

Then $^n f$ is a martingale with respect to the σ-algebras $(\mathcal{a}_n, \mathcal{a}_{n+1}, \ldots)$. Note that if $n>m$, then

$$(^n f)^* \leq (^m f)^* \quad \forall n>m$$

But if we fix $m \in \mathbb{N}$, then for each $n>m$,

$$\sup_{k,j \geq n} |f_j(\omega) - f_k(\omega)| \leq 2(^n f)^*(\omega) \leq 2(^m f)^*(\omega)$$

and so

$$LS(\omega) \leq 2(^m f)^*(\omega) \quad \forall m \in \mathbb{N}$$

Suppose $s(f)(\omega) < \infty$. Then

$$\sum_{k=1}^{\infty} E(d_k^2 \,|\, \mathcal{a}_{k-1})(\omega) < \infty$$

whence

$$s(^n f)(\omega) = \sum_{k=n+1}^{\infty} E(d_k^2 \,|\, \mathcal{a}_{k-1})(\omega) \longrightarrow 0 \quad \text{as } n \to \infty$$

Therefore, if $a>0$, we have

$$(s(f)<\infty) \subset \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} (s({}^{k}f)<a)$$

and so for $a>0, b>0$

$$P(LS>2b,\ s(f)<\infty)$$

$$\leq P\Big(LS>2b,\ \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}(s({}^{k}f)<a)\Big)$$

$$= \lim_{n\to\infty} P\Big(LS>2b,\ \bigcap_{k=n}^{\infty}(s({}^{k}f)<a)\Big)$$

$$\leq \lim_{n\to\infty} P(LS>2b,\ s({}^{n}f)<a)$$

$$\leq \lim_{n\to\infty} P(({}^{n}f)^{*}>b,\ s({}^{n}f)<a)$$

lemma applied to ${}^{n}f$

$$\leq \lim_{n\to\infty}\left(\frac{4a^2}{b^2}\right) = \frac{4a^2}{b^2}$$

So if we let $a\to 0$, we see that

$$P(LS>2b,\ s(f)<\infty)=0$$

Now if we let $b\to 0$, we have

$$P(LS>0,\ s(f)<\infty)=0$$

Therefore

$$\{\Delta(f) < \infty\} \subset_{a.e.} \{LS = 0\} \subset \{f \text{ converges}\}.$$ ✓

▨

⑩ **PROPOSITION**: Suppose that f is a martingale with $|d_k|$ $\mathcal{A}_{k-1}$-measurable. Then

$$\{f \text{ converges}\} =_{a.e} \{S(f)<\infty\} =_{a.e.} \{f^*<\infty\}$$

Proof. Since $|d_k|$ is $\mathcal{A}_{k-1}$-measurable, we have

$$(1) \qquad P(S(f)>\beta\lambda,\ f^*\vee d^*\le\delta\lambda)\le\varepsilon P(S(f)>\lambda)$$

where $\beta>1$, $\delta<\sqrt{\beta^2-1}$, λ and $\varepsilon=\varepsilon(\beta,\delta)\longrightarrow 0$ as $\beta\to\infty$ a as $\delta\to 0$. Now

$$|d_n|=|f_n-f_{n-1}|\le 2f^*$$

so that $f^*\vee d^*\le 2f^*$. Hence (1) says that

$$P(S(f)>\beta\lambda,\ f^*\le\delta\lambda/2)\le\varepsilon P\{S(f)>\lambda\}$$

If we first let $\beta\to\infty$ and then $\delta\to 0$ we see that

$$P(S(f)=\infty,\ f^*<\infty)=0$$

and so $\{f^*<\infty\}\subset\{S(f)<\infty\}$ a.e. ✓

We also have the dual inequality

$$P(f^*>\beta\lambda,\ S(f)\vee d^*\le\delta\lambda)\le\varepsilon P(f^*>\lambda)$$

In this case $S(f) \vee d^* = S(f)$ since

$$S(f) = \sup_n \left(\sum_{k=1}^{n} |d_k|^2 \right)^{1/2} \geq \sup_n \left(\max_{k \leq l \leq n} |d_k| \right)$$

$$= \sup_n |d_n| = d^*$$

and so

$$P(f^* > \beta\lambda, S(f) \leq \delta\lambda) \leq \varepsilon P(f^* > \lambda)$$

Again we let $\beta \to \infty$ and then $\delta \to 0$ to obtain

$$P(f^* = \infty, S(f) < \infty) = 0$$

Therefore $\{S(f) < \infty\} \subset \{f^* < \infty\}$ a.e., whence we have

$$\{S(f) < \infty\} = \{f^* < \infty\} \text{ a.e.} \qquad \checkmark$$

Now suppose $f(\omega)$ converges. Then $\{f_n(\omega) : n \in \mathbb{N}\}$ is bounded, and so $f^*(\omega) = \sup |f_n(\omega)| < \infty$. Therefore

$$\{f \text{ converges}\} \subset \{f^* < \infty\} \subset_{a.e.} \{S(f) < \infty\}$$

Now each $|d_k|$ is $\mathcal{A}_{k-1}$-measurable, and so $d_k^2 = |d_k|^2$ is also $\mathcal{A}_{k-1}$-measurable. Hence $E(d_k^2 | \mathcal{A}_{k-1}) = d_k^2$, so that

$$s^2(f) = \sum_{k=1}^{\infty} E(d_k^2 \mid \mathcal{A}_{k-1}) = \sum_{k=1}^{\infty} d_k^2 = S^2(f)$$

Therefore

$$\{S(f) < \infty\} = \{s(f) < \infty\} \overset{\text{problem 9}}{\underset{a.e.}{\subset}} \{f \text{ converges}\} \underset{a.e.}{\subset} \{S(f) < \infty\}$$

whence

$$\{f \text{ converges}\} =_{a.e.} \{S(f) < \infty\} \underset{a.e.}{=} \{f^* < \infty\}$$ ✓

▨

(11) **THEOREM**: Suppose that u is harmonic in $\mathbb{R}^{n+1}_+$ and that $a>0$, $h>0$, $0<p<\infty$. If

$$N_{a,h}(u) \in L^p(\mathbb{R}^n)$$

then there exists an $f \in L^p(\mathbb{R}^n)$ such that, for almost all x, u converges nontangentially to $f(x)$ at x and

$$\| u(\cdot, y) - f \|_p \to 0$$

as $y \to 0$.

Proof. Since $N_{a,h}(u)^p$ is integrable, $N_{a,h}(u)(x) < \infty$ for almost all x. But

$$\{x \in \mathbb{R}^n : u \text{ converges nontangentially at } x\} =_{\text{a.e.}} \{x \in \mathbb{R}^n : N_{a,h}(u)x < \infty\}$$

and so for almost all x, u converges nontangentially at x, to $f(x)$ say. On the set of measure 0 where u does not converge nontangentially, say N, define $f(x) := 0$.

Now

$$N_{a,h}(u)x = \sup\{|u(s,y)| : (s,y) \in \Gamma_{a,h}(x)\}$$

and so

$$|u(x, 1/n)| \le N_{a,h}(u)x \quad \forall x, \forall n > 1/h$$

For $x \notin N$,

$$u(x, {}^1/_n) \longrightarrow f(x)$$

as $n \to \infty$ since $(x, {}^1/_n) \in \Gamma_{a,h}(x)$ and $(x, {}^1/_n) \to x$. Therefore

$$|f(x)| \leq N_{a,h}(u)x \quad \forall x \in \mathbb{R}^n$$

and so

$$\int_{\mathbb{R}^n} |f(x)|^p \, dx \leq \int_{\mathbb{R}^n} |N_{a,h}(u)x|^p \, dx < \infty$$

since $N_{a,h}(u) \in L^p(\mathbb{R}^n)$. Therefore $f \in L^p(\mathbb{R}^n)$.
Now for each x

$$u(x,y) \longrightarrow f(x)$$

as $y \to 0$, and so $u(\cdot,y) - f \longrightarrow 0$ pointwise. Moreover,

$$|u(\cdot,y) - f|^p \leq (2\max(|u(\cdot,y)|, |f|))^p \leq 2^p N_{a,h}(u)^p \in L^1(\mathbb{R}^n)$$

and so by the Dominated Convergence theorem

$$\|u(\cdot,y) - f\|_p \longrightarrow 0$$

✓

as $y \to 0$ ▨

1/19 MARTINGALES

$(\Omega, \mathcal{A}, P)$ will always be a probability space

Example: $\Omega = \{\omega : [0,\infty) \longrightarrow \mathbb{R}^n$ continuous, $\omega(0) = 0\}$

$\mathcal{A}$ = smallest σ-field containing all sets of the form $\{\omega : \omega(t) \in \mathcal{O}\}$ for $0 \le t \le 1$, $\mathcal{O}$ open

P = Wiener measure

$\mathcal{B} \subset \mathcal{A}$ sub-σ-field

Example: $\Omega = [0,1)$ $\quad 0 \le \omega < 1$, write

$$\omega = .\omega_1 \omega_2 \omega_3 \cdots = \frac{\omega_1}{10} + \frac{\omega_2}{10^2} + \cdots$$

Consider map $\omega \mapsto \omega_1$. Let $\mathcal{B}_1$ be the smallest σ-field that makes the transformation measurable.

$\omega \mapsto (\omega_1, \omega_2)$ $\quad \mathcal{B}_2$ smallest σ-field making this measurable

$$\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \subset \mathcal{A}$$

CONDITIONAL EXPECTATION

f integrable or non-negative $\mathcal{A}$-measurable.

$$Ef := \int_\Omega f \, dP$$

If f is square-integrable, then

$$\min_a E(f-a)^2 = E(f - Ef)^2$$

DEFINITION: f integrable or non-negative $\mathcal{A}$-measurable. $\mathcal{B}$ sub-σ-field. Suppose that there is a $\mathcal{B}$-measurable function g s.t.

$$\int_B f \, dP = \int_B g \, dP$$

for all $B \in \mathcal{B}$. Then g is the conditional expectation of f (unique up to sets of measure zero)

Note: $$\min_{\substack{h \in L_2 \\ \mathcal{B}\text{-meas}}} E(f-h)^2 = E(f - \underset{\substack{\uparrow \\ \text{conditional expectation}}}{g})^2$$

Note: f integrable $\Rightarrow$ g integrable

THEOREM: (i) Conditional expectations always exist

(ii) g is essentially unique

(iii) $f_1 \le f_2$ a.e. $\Rightarrow g_1 \le g_2$ a.e. where $\int_B f_i = \int_B g_i \quad \forall B \in \mathcal{B} \quad i=1,2$

Examples

(1) $\mathcal{B} = \mathcal{A}$ Then $g = f$

(2) $\mathcal{B} = \{\emptyset, \Omega\}$ Then $g = Ef$

(3) $\mathcal{B}$ generated by a finite partition $\{B_1, \dots, B_n\}$ (disjoint)

$$g = \sum_{i=1}^{n} \frac{\int_{B_i} f}{\mu(B_i)} \chi_{B_i}$$

$$= \sum_{i=1}^{n} E(f \mid \mathcal{B}_i) \chi_{B_i}$$

Proof of th^m: Assume f non-negative $\mathcal{B}$-measurable

(iii) Suppose otherwise. Let $B = \{\omega : g_1(\omega) > g_2(\omega)\}$

Then $P(B) > 0$. Note

$$B = \bigcup_{r \in \mathbb{Q}} [g_2 < r < g_1]$$

Then for some r, $P(\overset{\overset{B'}{\parallel}}{g_2 < r < g_1}) > 0$, so

$$\int\limits_{B'} (g_2 - r) < 0 < \int\limits_{B'} (g_1 - r)$$

$$\Rightarrow \int\limits_{B'} (f_2 - r) < 0 < \int\limits_{B'} (f_1 - r) \leq \int\limits_{B'} (f_2 - r) \quad ↯$$

(iii) follows immediately by taking $f_1 = f_2$.

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Easy to show that $\|g\|_1 \leq \|f\|_1$ (where g is the conditional expectation of f)

Note: Even if f is finite a.e., g may be infinite

e.g. $f(x) = \frac{1}{x}$ on $(0,\infty)$ $\mathcal{B} = \{\phi, \Omega\}$
Then $g(x) = +\infty$ for all x

Proof of existence of g: May assume $f \geq 0$ and integrable

(if $f \geq 0$, not integrable, work with $f \wedge n$ - if

$$\int_B f \wedge n = \int_B g_n \quad \forall B$$

by monotonicity, $g_n \uparrow g$ a.e. and $\int_B f \wedge n \longrightarrow \int_B f \quad \forall B$

$$\Rightarrow \int_B f = \int_B g \quad \forall B \quad)$$

Let $\varphi(B) := \int_B f \, dP$ for $B \in \mathcal{B}$. Then φ is a countably additive positive measure which is P-continuous. By Radon-Nikodym theorem there exists $g : \Omega \longrightarrow \mathbb{R}$ $\mathcal{B}$-measurable such that

$$\int_{B} f\,dP = \varphi(B) = \int_{B} g\,dP \quad \forall B \in \mathcal{B}$$

$\blacksquare$

More intuitive proof of existence: Assume $0 \le f$ square integrable

Define

$$\delta^2 = \inf_{\substack{h \in L^2 \\ \mathcal{B}\text{-meas.}}} E(f-h)^2$$

Let $M = L^2(\Omega, \mathcal{B}, P)$ (subspace of $L^2(\Omega, \mathcal{a}, P)$). Choose $g_n \in M$ s.t.

$$\delta^2 \le E(f-g_n)^2 \le \delta^2 + \frac{1}{4^n}$$

Then

$$E(g_{n+1}-g_n)^2 = 2E(f-g_n)^2 + 2E(f-g_{n+1})^2 - 4E\Big(f - \underset{\uparrow\, \in M}{\frac{g_{n+1}+g_n}{2}}\Big)^2$$

$$\le 2\left(\delta^2 + \frac{1}{4^n}\right) + 2\left(\delta^2 + \frac{1}{4^{n+1}}\right) - 4\delta^2$$

$$= 2 \cdot \frac{1}{4^{n+1}} \cdot 5$$

$$\therefore \quad E|g_{n+1} - g_n| \le \frac{K}{2^{n+1}}$$

$$\therefore \quad E \sum_{n=0}^{\infty} |g_{n+1} - g_n| \le E|g_1| + \sum_{n=1}^{\infty} \frac{K}{2^{n+1}} < \infty$$

Hence $g_n = \sum_{k=0}^{n-1} (g_{k+1} - g_k) \xrightarrow[a.e.]{} g$ and g is $\mathcal{B}$-measurable

Also

$$E(f-g)^2 = E \liminf (f-g_n)^2 \le \liminf E(f-g_n)^2 = \delta^2$$

$$\Rightarrow \quad g \in M$$

$$\Rightarrow \quad \delta^2 \le E(f-g)^2 \le \delta^2$$

$$\Rightarrow \quad E(f-g)^2 = \delta^2$$

Clear that $\int fh = \int gh$ for all $h \in M$ (show $(f-g,h) \le 0 \ \forall h$
$\Rightarrow (f-g,h) = 0$)

and thus $\int_B f = \int_B g \quad \forall B \in M$

▨

NOTATION: f integrable or non-negative $\mathcal{A}$-measurable.
Denote the conditional expectation of f given $\mathcal{B}$ by

$$E(f|\mathcal{B})$$

Properties of conditional expectations

① Monotonicity: If $f_1 \le f_2$ a.e., then $E(f_1|\mathcal{B}) \le E(f_2|\mathcal{B})$ a.e.

② $\| E(f|\mathcal{B}) \|_1 \le \|f\|_1$

③ $E(f_1 + f_2 | \mathcal{B}) = E(f_1|\mathcal{B}) + E(f_2|\mathcal{B})$ a.e.
$E(\alpha f | \mathcal{B}) = \alpha E(f|\mathcal{B})$ a.e.

④ If f is $\mathcal{A}$-measurable and h is $\mathcal{B}$-measurable and both f and fh are integrable, then

$$E(fh|\mathcal{B}) = h\, E(f|\mathcal{B}) \text{ a.e.}$$

(If $f \ge 0, h \ge 0$, then can drop integrability assumption on f, fh)

Proof. Let $g = E(f|\mathcal{B})$. Want to show

$$\int_B fh = \int_B gh \quad \text{for all } B \in \mathcal{B}$$

(Assume $f, h \ge 0$)
Special case $h = \chi_C$, where $C \in \mathcal{B}$. Then clearly true since $g = E(f|\mathcal{B})$. Thus true for simple functions, etc.

⑤ $\| E(f|\mathcal{B}) \|_\infty \le \|f\|_\infty$

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① $\Omega = (-\frac{1}{2}, \frac{1}{2})$

$\mathcal{A}$ = Borel

P = Lebesgue

$G = \{x \to x,\ x \to -x\}$

$\mathcal{B} = \{A \in \mathcal{A} : A = -A\}$

$$E(f|\mathcal{B}) = \frac{1}{2}\left(f(x) + f(-x)\right)$$

② $\Omega = [0,1] \times [0,1] \times [0,1]$

$\mathcal{A}$ = Borel

P = Lebesgue

$G = \{(x_1, x_2, x_3) \longrightarrow (x_{\sigma_1}, x_{\sigma_2}, x_{\sigma_3})\}$

$= \sigma x$

σ permutation

$\mathcal{B}$ = class of invariant sets

$$E(f|\mathcal{B}) = \frac{1}{3!} \sum_{\sigma} f(\sigma x)$$

③ $\Omega = [0,1)$

$\mathcal{A}, P$ as in ①

$\varphi(x) = x + \frac{1}{n} \mod 1$

$G = \{\varphi, \varphi^2, \varphi^3, \ldots, \varphi^n = \text{identity}\}$

$$E(f|\mathcal{B}) = \frac{\sum_{k=0}^{n-1} f(x + \frac{k}{n})}{n}$$

Proof of ⑤ $\| E(f|\mathcal{B}) \|_\infty \leq \|f\|_\infty$

Assume $\|f\|_\infty < \infty$. Then

$$- \|f\|_\infty \leq f \leq \|f\|_\infty$$

$$\Rightarrow E(-\|f\|_\infty, \mathcal{B}) \leq E(f|\mathcal{B}) \leq E(\|f\|_\infty | \mathcal{B})$$

$$\Rightarrow - \|f\|_\infty \leq E(f|\mathcal{B}) \leq \|f\|_\infty$$

It follows from Jensen's inequality or Riesz-Thorin that

⑥ $\| E(f|\mathcal{B}) \|_p \leq \|f\|_p$ for $1 \leq p \leq \infty$

(Proof later)

⑦ $| E(f|\mathcal{B}) | \leq E(|f| \, | \mathcal{B})$ (since $-|f| \leq f \leq |f|$)

⑧ $E[E(f|\mathcal{B})] = Ef$

⑨ $E[E(f|\mathcal{B}) | \mathcal{C}] = E(f|\mathcal{C})$ a.e. $(\mathcal{C} \subset \mathcal{B} \subset \mathcal{A})$

Proof. If $C \in \mathcal{C}$, then

$$\int_C f = \int_C E(f|\mathcal{B}) \quad \text{since } C \in \mathcal{B}$$

but $\displaystyle\int_C f = \int_C E(f|\mathcal{C})$ by definition

⑩ $E\left[E(f|\mathcal{C})\,|\,\mathcal{B}\right] = E(f|\mathcal{C})$ $\quad(\mathcal{C} < \mathcal{B} < \mathcal{A})$

⑪ $E\left[E(f|\mathcal{B})\,\middle|\,\mathcal{B}\right] = E(f|\mathcal{B})$ $\quad$ (from ⑩)

(idempotent)

⑫ Let $f_1 \in L^p$, $f_2 \in L^q$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$

$$E\big(f_2 E(f_1|\mathcal{B})\big) = E\big(f_1 E(f_2|\mathcal{B})\big)$$

Proof.

$$E\big(f_2 E(f_1|\mathcal{B})\big) = E\Big(E\big(f_2 E(f_1|\mathcal{B})\,\big|\,\mathcal{B}\big)\Big)$$

$$= E\big(E(f_2|\mathcal{B})\,E(f_1|\mathcal{B})\big) \quad (\text{since } f_1 f_2 \in L_1)$$

$$= E\Big(E\big(f_1 E(f_2|\mathcal{B})\,\big|\,\mathcal{B}\big)\Big)$$

$$= E\big(f_1 E(f_2|\mathcal{B})\big)$$

(Thus E "self-adjoint")

<u>Conditional Expectation Operator</u>

Define for $f \in L^p(\Omega,\mathcal{A},P)$, $Tf := E(f|\mathcal{B})$ (equivalence class)

$$T : L^p(\Omega,\mathcal{A},P) \to L^p(\Omega,\mathcal{B},P)$$

Assume that $\mathcal{B}$ contains all sets $A \in \mathcal{a}$ with $P(A) = 0$. Then $L^p(\Omega, \mathcal{B}, P) \subset L^p(\Omega, \mathcal{a}, P)$. Then

T is linear contraction, idempotent (ie. contraction)
$T(1) = 1$
$Tf \geq 0$ if $f \geq 0$
T self-adjoint in L_2

positive
In L_2, if T is any linear operator that is idempotent, self-adjoint and $T1 = 1$, then T is a conditional expectation operator.

$$\mathcal{B} = \{A \in \mathcal{a} : T\chi_A = \chi_A\}$$

$$\therefore L_2(\Omega, \mathcal{B}, P) = \{f \in L_2(\Omega, \mathcal{a}, P) : Tf = f\}$$

(May 1954 Pacific J. of Math ; Bahadur 1955 PAMS ; Douglas 1965 PJM)

$T1 = 1$
In L_1, if T is linear projection and contraction ($\|Tf\|_1 \leq \|f\|_1$) then T is cond. expectation operator.

Ando 1966 (PJM) In L_p ($p \neq 2$) T linear contraction in L_p, projection, $T1 = 1$, then T is cond. expectation operator

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$$\left.\begin{array}{l} T \text{ linear} \\ T^2 = T \\ T\mathbb{1} = \mathbb{1} \\ \|Tf\|_1 \le \|f\|_1 \end{array}\right\} \Leftrightarrow T \text{ conditional exp. operator}$$

Example: $\Omega = \{1,2,\dots,n\}$

$\mathcal{A}$ all subsets

P unif measure $\quad P(\{i\}) = \frac{1}{n}$

$$T = (a_{jk})_{n\times n}$$

$$(*) \quad T\mathbb{1} = \mathbb{1} \Rightarrow \sum_{k=1}^{n} a_{jk} = 1$$

$$\frac{1}{n}\sum_{j=1}^{n}\left|\sum_{k=1}^{n} a_{jk} f(k)\right| \le \frac{1}{n}\sum_{k=1}^{n} |f(k)|$$

Letting $f = e_i \Rightarrow \sum_{j=1}^{n}\sum_{k=1}^{n} |a_{jk}| \le n$. Coupled with $(*)$ this implies

$a_{jk} \ge 0$

additional information provided by $T^2 = T$

JENSEN'S INEQUALITY: $(\Omega, \mathcal{A}, P)$ $\mathcal{B} \subset \mathcal{A}$

Let f be integrable and φ convex on a convex set $S \subset \mathbb{R}$
Then

$$\varphi(E(f|\mathcal{B})) \leq E(\varphi(f)|\mathcal{B}) \quad \text{a.e.}$$

Example: Let $\varphi(x) = |x|^p$ $1 \leq p < \infty$. Then

$$|E(f|\mathcal{B})|^p \leq E(|f|^p|\mathcal{B}) \quad \text{a.e.}$$

$$\Rightarrow E|E(f|\mathcal{B})|^p \leq E(E(|f|^p|\mathcal{B})) = E|f|^p$$

$$\Rightarrow \quad \|E(f|\mathcal{B})\|_p \leq \|f\|_p$$

Proof. There is a sequence $\psi_1, \psi_2, \ldots$ of affine functions ($\psi_n(x) = a_n x + b_n$) s.t.

$$\psi_n(x) \leq \varphi(x) \quad \forall x \in S$$
$$\sup_n \psi_n(x) = \varphi(x) \quad \forall x \in \operatorname{int} S$$

Let $B = \{\omega : E(f|\mathcal{B})(\omega) \in \operatorname{int} S\}$. Then on B

$$\varphi(E(f|\mathcal{B})) = \sup_n \psi_n(E(f|\mathcal{B})) \underset{\substack{\uparrow \\ \text{since } \psi_n \text{ affine}}}{=} \sup_n E(\psi_n(f)|\mathcal{B})$$

$$\le \sup_n E(\varphi(f) \mid \mathcal{B}) \quad \text{a.e.}$$

$$= E(\varphi(f) \mid \mathcal{B}) \quad \text{a.e.}$$

Now suppose a is the left endpoint of S and is in S. Let $B_a = \{\omega : E(f|\mathcal{B})(\omega) = a\}$. Then $f = a$ a.e. on B_a, since

$$f - a \ge 0 \Rightarrow 0 \le \int_{B_a} (f-a) = \underset{\substack{\uparrow \\ \text{since } B_a \in \mathcal{B}}}{\int_{B_a}} E(f|\mathcal{B}) - a = 0$$

Then

$$\chi_{B_a} \cdot (\varphi(E(f|\mathcal{B}))) = \chi_{B_a} \varphi(a)$$

$$= E(\chi_{B_a} \varphi(a) \mid \mathcal{B})$$

$$= E(\chi_{B_a} \varphi(f) \mid \mathcal{B}) \quad \text{a.e.}$$

$$= \chi_{B_a} E(\varphi(f) \mid \mathcal{B}) \quad \text{a.e.}$$

$$\Rightarrow \varphi(E(f|\mathcal{B})) = E(\varphi(f)|\mathcal{B}) \quad \text{a.e. on } B_a$$

Similarly for other endpoint.

▨

Conditional Expectation and independence

$$E(f \mid \mathcal{B} \vee \mathcal{D}) = E(f \mid \mathcal{B}) \quad \text{a.e.}$$

under the following conditions:

f is $\mathcal{C}$-measurable, $\mathcal{B} \subset \mathcal{C}$
$\mathcal{C}$ and $\mathcal{D}$ are independent sub-σ-fields of $\mathcal{A}$, i.e.
$P(C \cap D) = P(C)P(D) \quad \forall C \in \mathcal{C}, D \in \mathcal{D}$

where $\mathcal{B} \vee \mathcal{D}$ = smallest σ-field containing both $\mathcal{D}$ and $\mathcal{B}$
= smallest σ-field containing the field of finite disjoint union of sets of the form $B \cap D$

Thus it suffices to show

$$\int_{B \cap D} f = \int_{B \cap D} g \qquad \forall B \in \mathcal{B}, D \in \mathcal{D}$$

where $g = E(f \mid \mathcal{B})$

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Corollary: If f is independent of $\mathcal{D}$, then $E(f|\mathcal{D}) = Ef$ a.e.

Proof. Take $\mathcal{B} = \{\phi, \Omega\}$

Proof of previous theorem (cont.) $E(f|\mathcal{B} \vee \mathcal{D}) = E(f|\mathcal{B})$ a.e.

If g is $\mathcal{C}$-measurable, h $\mathcal{D}$-measurable, with $\mathcal{C}$ and $\mathcal{D}$ independent, then $Egh = EgEh$. This is obvious for characteristic functions, ∴ simple functions, etc.

Must show

$$\int_{B\cap D} f = \int_{B\cap D} g \qquad g = E(f|\mathcal{B})$$

But

$$\int_{B\cap D} f = E f \chi_B \chi_D = E f \chi_B E \chi_D$$

$$\underset{\uparrow}{=} E g \chi_B E \chi_D = E g \chi_B \chi_D = \int_{B\cap D} g$$

since $g = E(f|\mathcal{B})$

▨

Limits under conditional expectation

Monotone convergence: $0 \le f_n \uparrow f$ a.e. $\Rightarrow E(f_n|\mathcal{B}) \uparrow E(f|\mathcal{B})$ a.e.

Fatou's lemma: $0 \le f_n \Rightarrow E(\liminf_n f_n|\mathcal{B}) \le \liminf_n E(f_n|\mathcal{B})$ a.e.

Dominated convergence: $f^* = \sup_n |f_n|$ integrable, $f_n \to f$ a.e.

$$\Rightarrow E(f_n|\mathcal{B}) \to E(f|\mathcal{B}) \text{ a.e.}$$

Proof. Note $E|f| \le \liminf E|f_n| \le Ef^* < \infty$.

$$|E(f_n|\mathcal{B}) - E(f|\mathcal{B})| \le E(\underbrace{|f_n - f|}_{F_n}\,|\,\mathcal{B}) = E(F_n|\mathcal{B})$$

where $F_n \to 0$ a.e. F^* int. Then

$$\begin{aligned} E(F^*|\mathcal{B}) &= E(F^* - \limsup F_n\,|\,\mathcal{B}) \\ &= E(\liminf (F^* - F_n)\,|\,\mathcal{B}) \\ &\le \liminf E(F^* - F_n\,|\,\mathcal{B}) \\ &= E(F^*|\mathcal{B}) - \limsup E(F_n|\mathcal{B}) \end{aligned}$$

$$\therefore \limsup E(F_n|\mathcal{B}) \le 0 \Rightarrow \lim_n E(F_n|\mathcal{B}) = 0$$

▨

Conditional form of dominated convergence

If f_n, f integrable, $f_n \to f$ a.e., then

$$E(f_n|\mathcal{B}) \to E(f|\mathcal{B})$$

a.e. on the set $\{E(f^*|\mathcal{B}) < \infty\}$.

Proof. Let $B_j = \{E(f^*|\mathcal{B}) \le j\}$, $j \in \mathbb{N}$. Fix j and let $g_n = f_n \chi_{B_j}$. Then

$$E g^* = E f^* \chi_{B_j} = E(E(f^*|\mathcal{B}) \underset{\uparrow\ \mathcal{B}\text{-measurable}}{\chi_{B_j}}) \le j$$

so ordinary DCT applies. Now true on $\bigcup B_j$

▨

Uniformly integrable families of measurable functions

$$\lim_{b \to \infty} \sup_{f \in \mathcal{F}} \int_{|f|>b} |f| = 0$$

Example: Let f be integrable. Then the family

$$\{E(f|\mathcal{B}) : \mathcal{B} \text{ sub-}\sigma\text{-field of } \mathcal{A}\}$$

is uniformly integrable. Since $|E(f|\mathcal{B})| \le E(|f|\,|\mathcal{B})$, we may assume WLOG that $f \geq 0$. Let $g = E(f|\mathcal{B})$. Then

$$\int_{g>b} g = \int_{g>b} f = \int_{g>b,\, f\leq a} f + \int_{g>b,\, f>a} f$$

$$\leq a P(g>b) + \int_{f>a} f$$

$$\leq a \frac{Eg}{b} + \int_{f>a} f \qquad \text{(Chebyschov's ineq)}$$

$$= a \frac{Ef}{b} + \int_{f>a} f$$

Thus

$$\lim_{b\to\infty} \sup_{g} \int_{g>b} g \leq \int_{f>a} f \to 0 \text{ as } a \to \infty$$

Lemma If (f_n) is uniformly integrable, then

$$E(\liminf_n f_n) \le \liminf_n Ef_n \le \limsup_n Ef_n \le E(\limsup_n f_n)$$

Proof. Let $f_n^a = \begin{cases} a & \text{if } f_n < a \\ f_n & \text{if } f_n \ge a \end{cases}$, so $a \le f_n^a$.

Apply Fatou's lemma

$$E(\liminf f_n) \le E(\liminf f_n^a) \le \liminf Ef_n^a$$

Error from Ef_n in last term is unif. small by uniform integrability.

NOTE: In general, this is not true for conditional expectations.

(For counterexample see Zheng, Z. Wahrschein 53 (1980) p291
Burkholder, Ann. Math. Stat. 1962)

$X_1, \dots X_n, \dots$ i.i.d. integrable

$$\Rightarrow \frac{X_1 + X_2 + \dots + X_n}{n} \longrightarrow EX_1 \text{ a.e.}$$

$$\exists \mathcal{B} \text{ s.t. } \limsup_n E\left(\frac{X_1 + \dots + X_n}{n} \middle| \mathcal{B}\right) = +\infty \text{ a.e.}$$

MARTINGALES

Given $(\Omega, \mathcal{A}, P)$, T partially ordered set
$(\mathcal{A}_t : t \in T)$ family of sub-σ-field satisfying

(*) $\quad s \leq t \Rightarrow \mathcal{A}_s \subset \mathcal{A}_t$

$(f_t : t \in T)$ family of integrable functions satisfying

① f_t is $\mathcal{A}_t$-measurable

② $E(f_t | \mathcal{A}_s) = f_s$ a.e. if $s \leq t$

Example: let F be integrable. Suppose $(\mathcal{A}_t : t \in T)$ satisfies (*). Then if

$$f_t := E(F | \mathcal{A}_t)$$

$(f_t, \mathcal{A}_t : t \in T)$ is a martingale.

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Submartingale $\quad E(f_t|\mathcal{A}_s) \geq f_s \quad \forall s \leq t$

Supermartingale $\quad E(f_t|\mathcal{A}_s) \leq f_s \quad \forall s \leq t$

Suppose φ is convex on a convex $S \subset \mathbb{R}$ and $f = (f_t)_{t\in T}$ is a martingale with values in S. If $s \leq t$, then

$$\varphi(f_s) = \varphi(E(f_t|\mathcal{A}_s)) \underset{\uparrow \text{Jensen's ineq.}}{\leq} E(\varphi(f_t)|\mathcal{A}_s)$$

Thus $F_t = \varphi(f_t)$ defines a submartingale (if integrability holds)

Examples: ① F integrable $\quad f_t = E(F|\mathcal{A}_t)$

$$s \leq t \Rightarrow E(f_t|\mathcal{A}_s) = E\left(E(f|\mathcal{A}_t)\,\middle|\,\mathcal{A}_s\right)$$

$$= E(F|\mathcal{A}_s) = f_s$$

② Let T be the set of all sub-σ-fields $\mathcal{B}$ of $\mathcal{A}$ and let F be integrable $\qquad (\mathcal{B} \leq \mathcal{C}$ if $\mathcal{B} \subset \mathcal{C})$

$$f_{\mathcal{B}} = E(F|\mathcal{B})$$

Then $(f_{\mathcal{B}})$ is a uniformly integrable martingale

Let $T = \mathbb{N}$. A reversed martingale satisfies

$$\mathcal{A} \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \mathcal{A}_3 \supset \dots$$

$$E(f_n | \mathcal{A}_{n+1}) = f_{n+1} \quad \text{a.e.}$$

$$\left(\Rightarrow E(f_n | \mathcal{A}_m) = f_m \quad \text{a.e.} \quad \forall n < m \right)$$

Note: If $(f_t)_{t \in T}$ is a martingale

$$E f_t = E(E(f_t | \mathcal{A}_s)) = E f_s \quad (s \leq t)$$

and so martingales are expectation preserving. Similarly, submartingales are expectation increasing.

Example $\Omega = [0,1)$ $\mathcal{A}$ Borel sets P Lebesgue measure

Let F be integrable

$$g_n = \frac{1}{n} \sum_{k=0}^{n-1} F\left(x + \frac{k}{n}\right)$$

↑ addition mod 1

$$\mathcal{B}_n = \{ A \in \mathcal{A} : \mathbb{1}_A \text{ periodic with period } 1/n \}$$

Then $g_n = E(F | \mathcal{B}_n)$. This neither a martingale nor a reversed martingale.

Let $\mathfrak{f}_n = g_{2^n} = \frac{1}{2^n} \sum_{k=0}^{2^n-1} F(x + \frac{k}{2^n})$, $\mathcal{A}_n = \mathcal{B}_{2^n}$

Then $\mathcal{A}_n \supset \mathcal{A}_{n+1}$ and $(\mathfrak{f}_n)$ is a reversed martingale. Note, if F is continuous, then

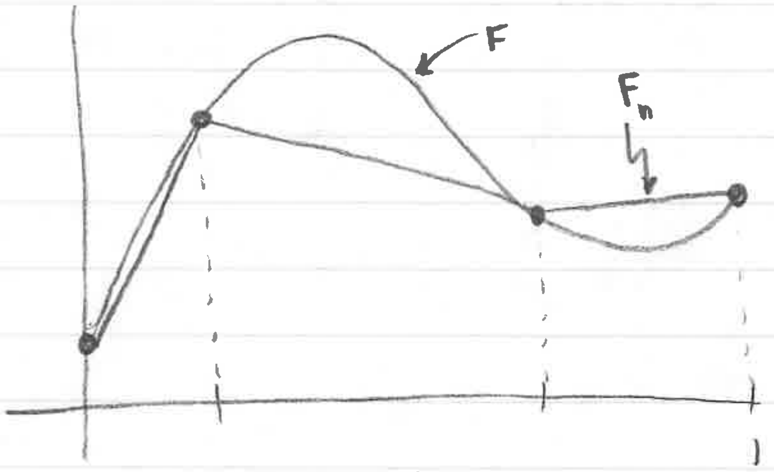
$$g_n \longrightarrow \int_0^1 F(x)\,dx$$

However, if F is an arbitrary element of L_1, then the g_n's don't necessarily converge a.e. (Rudin PAMS, 1960's). However, by the reversed martingale convergence theorem, if $F \in L_1$, then

$$\mathfrak{f}_n \longrightarrow \int_0^1 F(x)\,dx \quad \text{a.e.}$$

(Jessen 1934)

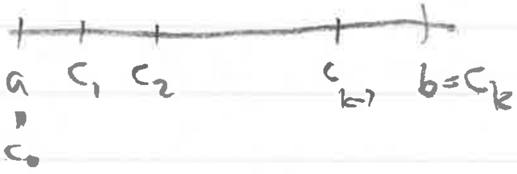
Example: $F : [0,1] \longrightarrow \mathbb{R}$ any function



n^{th} partition of $[0,1]$

$\mathfrak{f}_n = F_n'$ (right-hand derivative), $\mathcal{A}_n$ generated by n^{th}-partition

Check $E(f_{n+1} | a_n) = f_n$. Let $[a,b)$ be an interval in the n^{th} partition



$$\int_{[a,b)} f_{n+1} = \sum_{j=1}^{k} \int_{[c_{j-1}, c_j)} f_{n+1}$$

$$= \sum_{j=1}^{k} \frac{F(c_j) - F(c_{j-1})}{c_j - c_{j-1}} \cdot (c_j - c_{j-1})$$

$$= F(c_k) - F(c_0)$$

$$= F(b) - F(a)$$

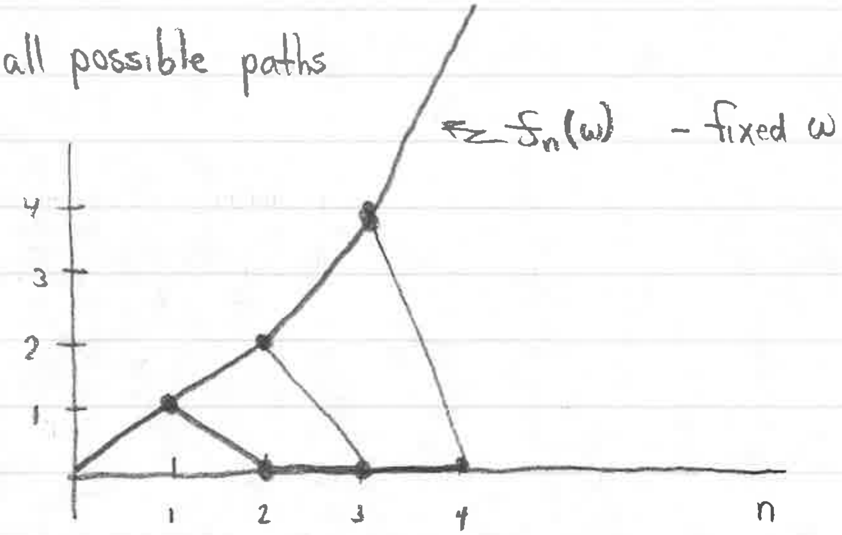
$$= \int_{[a,b)} \frac{F(b) - F(a)}{b-a} \, dx$$

$$= \int_{[a,b)} f_n$$

Example: $F(\omega) = \begin{cases} 0 & 0 \leq \omega < 1 \\ 1 & \omega = 1 \end{cases}$ with dyadic partitions

$$f_n = 2^{n-1} \chi_{\left(1 - \frac{1}{2^{n-1}}, 1\right)}$$

Graph of all possible paths



2/2 MARTINGALES

Note: Any non-negative martingale is L_1-bounded, i.e. if $f=(f_1,f_2,\dots)$

$$\|f\|_1 := \sup_n \|f_n\|_1 < \infty$$

Double-or-nothing martingale satisfies

(i) L_1-bounded
(ii) $f_n \to 0$ a.e.
(iii) not uniformly bounded
(iv) $f^* \notin L_1$ (otherwise $1 = Ef_n \longrightarrow E0 = 0$ $\lightning$)

Thus there is no F (integrable) such that $f_n = E(F|\mathcal{A}_n)$

Model - Let d_n = net amount won by gambler playing n^{th} game
Let $f_n = \sum_{k=1}^{n} d_k$ = fortune after game n. Fair game $\Rightarrow$

$$E(d_n \mid \text{past}) = 0, \quad n \geq 2$$

This gives $E(f_{n+1} \mid \mathcal{A}_n) = f_n$
$\uparrow$ past & present information

Any sequence $f=(f_1,f_2,\dots)$ of integrable functions is a martingale if for some non-decreasing seq. of σ-fields $\mathcal{A}_1,\mathcal{A}_2,\dots$ we have

$$E(d_n | \mathcal{a}_{n-1}) = 0, \; n \geq 2$$

where $d_n = f_n - f_{n-1}$ $(f_0 = 0)$ is $\mathcal{a}_n$-measurable. If we define $\mathcal{B}_n$ to be the smallest σ-field with respect to which $d_1, \dots, d_n$ are measurable.

<u>DEFINITION</u>: $d = (d_1, d_2, \dots)$ a sequence of int. functions on $(\Omega, \mathcal{a}, P)$ is a martingale difference sequence iff

$$\varphi_n(d_1, \dots, d_n) \perp d_{n+1} \quad \forall n \geq 1$$

for all $\varphi_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ bounded Borel measurable functions

$$(\, h \perp f \iff E\, hf = 0 \,)$$

Note: This is equivalent to requirement $E(d_{n+1} | \underset{\uparrow \, \sigma(d_1, \dots, d_n)}{\mathcal{B}_n}) = 0 \;\forall n$

A sequence of measurable functions on $(\Omega, \mathcal{a}, P)$ is orthogonal if $E d_n^2 < \infty$ $\forall n$ and $E d_n d_m = 0$ for $n \neq m$.

An L_2-bdd martingale has a difference seq d that is orthogonal in the above sense, but also

$$\varphi(d_1, \dots, d_n) \perp d_{n+1}$$

Note: $\|f_n\|_2^2 = \sum_{k=1}^{n} Ed_k^2$ $\quad (d_k = f_k - f_{k-1})$

Menchoff 1923 showed $\exists$ a seq. of orthogonal functions $d=(d_1, d_2, \ldots)$ with $\sum Ed_k^2 < \infty$ s.t.

$$f_n = d_1 + \ldots + d_n$$

does not converge a.e. as $n \to \infty$ $\quad$ ($P(f_n \text{ converges}) = 0$)

Weak L_1-inequality for the martingale maximal function

$f = (f_1, \ldots)$ martingale or non-negative submartingale

$$f_n^* = \sup_{1 \le k \le n} |f_k(\cdot)|$$

$$f^* = \sup_k |f_k(\cdot)|$$

Then

(also with $\geq$)

(a) $$\lambda P(f^* > \lambda) \le \|f\|_1 \quad \forall \lambda > 0$$

(b) $$\lambda P(f_n^* > \lambda) \le \int_{(f_n^* > \lambda)} |f_n| \le \|f_n\|_1 \le \|f\|_1$$

Note (b) $\Rightarrow$ (a) since $f_n^* \uparrow f^*$

(Not true that $\|f\|_1 < \infty \Rightarrow \|f^*\|_1 < \infty$ as double-or-nothing martingale shows)

Proof: WLOG f is non-neg. submartingale (replace f_n by $|f_n|$)

Let

$$A_1 = (f_1 > \lambda)$$

$$A_2 = (f_1 \le \lambda, f_2 > \lambda) \qquad \text{(disjoint)}$$

$$A_3 = (f_1 \le \lambda, f_2 \le \lambda, f_3 > \lambda)$$

$$\vdots$$

Then $A_k \in \mathcal{A}_k$.

$$\lambda P(f_n^* > \lambda) = \lambda P\Big(\bigcup_{k=1}^n A_k\Big) = \lambda \sum_{k=1}^n P(A_k)$$

$$= \sum_{k=1}^n \int_{A_k} \lambda \, dP \le \sum_{k=1}^n \int_{A_k} f_k \, dP$$

$$\underset{\uparrow \text{ submart.}}{\le} \sum_{k=1}^n \int_{A_k} f_n = \int_{\cup A_k} f_n = \int_{(f_n^* > \lambda)} f_n$$

▨

2/4 MARTINGALES

Let $f = (f_1, f_2, \ldots)$ be a nonnegative martingale. Note that

$$P(f^* = \infty) \le P(f^* > \lambda) \le \frac{\|f\|_1}{\lambda} = \frac{Ef_1}{\lambda} \to 0$$

$$\therefore \quad f^* < \infty \text{ a.e.}$$

$$\therefore \quad \limsup_{n\to\infty} f_n \le f^* < \infty \quad \text{a.e.}$$

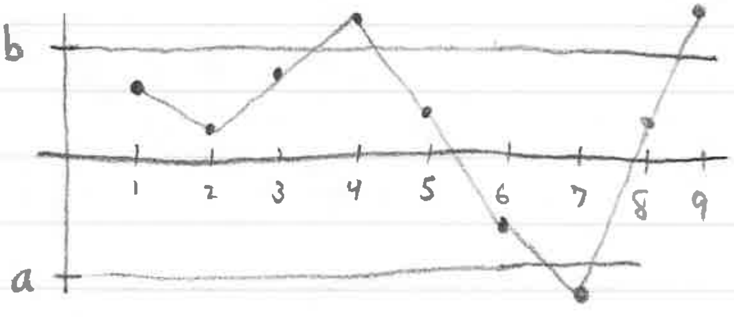
Now non-convergence $\Rightarrow$ $\exists\, a, b$ with $\liminf f_n < a < b < \limsup f_n$

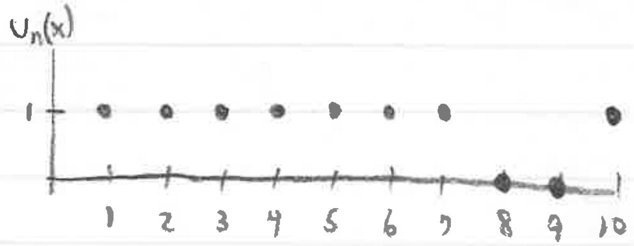
Upcrossing inequality

Let $x = (x_1, x_2, \ldots)$ be a sequence of real numbers. Let $U_1(x) = 1$ and by induction let

$$U_{n+1}(x) = \begin{cases} 1 & \text{if } x_n \ge b \\ U_n(x) & \text{if } a < x_n < b \\ 0 & x_n \le a \end{cases}$$

($a < b$ prescribed numbers)





Note: there is an upcrossing at time n $\Leftrightarrow$ $u_{n+1}(x) - u_n(x) = 1$.
Let $n \geq 2$ and define

$$U_n^{ab}(x) = \underset{\text{by time } n}{\text{total number of upcrossings}} = \sum_{k=2}^{n} \left(u_{k+1}(x) - u_k(x)\right)^+$$

Assume that $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2, \ldots)$ is a submartingale.

Remark: $$(b-a)\, U_n^{ab}(x) \leq \sum_{k=2}^{n} (x_k - a)(u_{k+1}(x) - u_k(x))$$

since $(b-a)(u_{k+1}(x) - u_k(x))^+ \leq (x_k - a)(u_{k+1}(x) - u_k(x))$. Why?

(i) if $u_{k+1}(x) - u_k(x) > 0$, then $x_k > b$ so ok

(ii) if $u_{k+1}(x) - u_k(x) = 0$, then both sides 0

(iii) if $u_{k+1}(x) - u_k(x) < 0$, then $x_k \leq a$ and left side is 0
and right side is $(-)(-) > 0$

Back to submartingale, let

$$v_n(\cdot) = u_n(\mathfrak{f}(\cdot)) = \text{function of } \mathfrak{f}_1, \ldots, \mathfrak{f}_{n-1}$$

$$\Rightarrow v_n \text{ is } \mathfrak{a}_{n-1} \text{ measurable}$$

Then

$$(b-a)\, E\, U_n^{ab}(f) \leq E\left(\sum_{k=2}^{n} (f_k - a)(V_{k+1} - V_k) \right)$$

$$= \sum_{k=2}^{n} \left[E\left(V_{k+1}(f_k - a)\right) - E\left(V_k (f_k - a)\right) \right]$$

Observe that $E\left(V_k (f_k - a)\right) \underset{\uparrow}{=} E\left(V_k\, E(f_k - a \,|\, \mathfrak{a}_{k-1})\right)$

since V_k is $\mathfrak{a}_{k-1}$-measurable

$$\geq E\left(V_k (f_{k-1} - a)\right) \quad \text{(by submartingale property)}$$

$$\therefore\ (b-a)\, E\, U_n^{ab}(f) \leq \sum_{k=2}^{n} \left[E\left(V_{k+1}(f_{k-a})\right) - E\left(V_k (f_{k-1} - a)\right)\right]$$

$$= E\left(V_{n+1}(f_n - a)\right) - E\left(V_2 (f_1 - a)\right)$$

(telescoping sum)

$$\leq E\left(V_{n+1}(f_n - a)\right) \quad \left(\begin{matrix} \text{ok if } f_1 > a \ ✓ \\ f_1 \leq a \Rightarrow V_2 = 0 \ ✓ \end{matrix} \right)$$

$$\leq E|f_n| + |a|$$

$$\leq \|f\|_1 + |a|$$

Upcrossing Inequality (Doob, Snell). Let $U^{ab} = \lim_{n \to \infty} U_n^{ab}$. Then

$$E\, U^{ab}(\mathfrak{f}) \leq \frac{\|\mathfrak{f}\|_1 + |a|}{b-a}$$

where $\mathfrak{f}$ is a submartingale

MARTINGALE CONVERGENCE THEOREM (Doob) If $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2, \dots)$ is either a martingale, submartingale, or supermartingale and $\mathfrak{f}$ is L_1-bounded, then $\mathfrak{f}$ converges a.e. to a finite valued measurable function $\mathfrak{f}_\infty$.

Proof. WLOG $\mathfrak{f}$ is a submartingale. To show first that

$$P\Big(\underbrace{\omega : \liminf_n \mathfrak{f}_n(\omega) < \limsup_n \mathfrak{f}_n(\omega)}_{A}\Big) = 0$$

Note that

$$P(A) \leq \sum_{\substack{a<b \\ a,b \in \mathbb{Q}}} P(\liminf \mathfrak{f}_n < a < b < \limsup \mathfrak{f}_n)$$

For a fixed pair $a<b$

$$P(\liminf \mathfrak{f}_n < a < b < \limsup \mathfrak{f}_n) \leq P(U^{ab}(\mathfrak{f}) = \infty) = 0$$

$$\hookrightarrow \leq P(U^{ab}(\mathfrak{f}) > \lambda) \leq \frac{E U^{ab}(\mathfrak{f})}{\lambda}$$

(since $E\, U^{ab}(\mathfrak{f}) = 0$)

▨

Corollary: F integrable $a_1 \subset a_2 \subset \dots$ $f_n := E(F|a_n)$
Then $f = (f_1, f_2, \dots)$ is an L_1-bdd martingale and hence converges a.e.

2/6 MARTINGALES

$\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2, \dots)$ satisfies $E(\mathfrak{f}_{n+1} | \mathfrak{f}_n) = \mathfrak{f}_n$ a.e. and $\|\mathfrak{f}\|_1 < \infty$

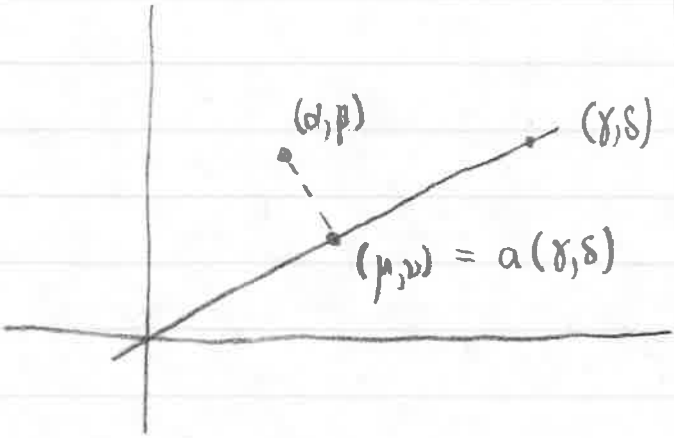
$\not\Rightarrow$ $\mathfrak{f}$ converges a.e.

Example: $X, Y \sim \mathcal{N}(0,1)$ indep $EX = EY = 0$

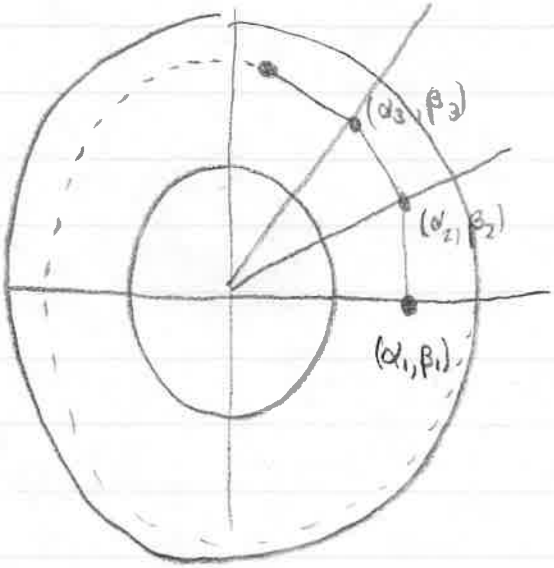
$EX^2 = EY^2 = 1$

$$E(\alpha X + \beta Y \mid \gamma X + \delta Y) = ? = \mu X + \nu Y = a(\gamma X + \delta Y)$$

$\uparrow$ because of normality



Let $\mathfrak{f}_n = \alpha_n X + \beta_n Y$



$E(\mathfrak{f}_{n+1} | \mathfrak{f}_n) = \mathfrak{f}_n$

but (α_n, β_n) spiral around infinitely often so $\mathfrak{f}_n$ does not converge

Theorem: If $f = (f_1, f_2, \dots)$ is a reversed martingale, then f converges a.e.

(Note: f is automatically L_1-bdd since $E|f_1| \geq E|f_2| \geq \cdots$)

Proof. Consider the martingale $(f_n, f_{n-1}, f_{n-2}, \dots, f_1, f_1, f_1, \dots)$
Apply the upcrossing to this to get

$$\underbrace{E\, V^{ab}(f_n, f_{n-1}, f_{n-2}, \dots f_1, f_1, \dots)}_{\text{non-decreasing in } n \ \uparrow\ V^{ab}(f)} \leq \frac{\|f\|_1 + a}{b-a} < \infty$$

$$\therefore\ E\, V^{ab}(f) \leq \frac{\|f\|_1 + a}{b-a}$$

Note $\quad \liminf f_n < a < b < \limsup f_n \Rightarrow V^{ab}(f) = \infty$

$$\therefore P(\liminf f_n < a < b < \limsup f_n) \leq P(V^{ab}(f) = \infty) = 0$$

$\blacksquare$

CONTINUITY THEOREM FOR CONDITIONAL EXPECTATION

① If F is integrable and $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$ then

$$E(F|\mathcal{A}_n) \longrightarrow E(F| \bigvee_{k=1}^{\infty} \mathcal{A}_k)$$

a.e. and in L_1 $\quad\uparrow$ smallest σ-field containing $\bigcup \mathcal{A}_k$

② If F is integrable and $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \dots$ then

$$E(F|\mathcal{A}_n) \rightarrow E(F| \bigcap_{k=1}^{\infty} \mathcal{A}_k)$$

a.e. and in L_1

Proof. Let $f_n = E(F|\mathcal{A}_n)$. In ① $f = (f_1, f_2, \dots)$ is a martingale and in ② f is a reversed martingale. Thus

$$f_n \longrightarrow f_\infty \text{ a.e.}$$

where $f_\infty \in L_1$ and $E|f_\infty| \leq \|f\|_1$. To show $f_\infty = E(F| \bigvee_{k=1}^{\infty} \mathcal{A}_k)$ a.e.

$A_n \in \mathcal{A}_n$

$$\int_{A_n} F \underset{\substack{\uparrow \\ \forall k}}{=} \int_{A_n} f_{n+k} \xrightarrow[k \to \infty]{} \int_{A_n} f_\infty \quad (\text{since } f = (f_1, f_2, \dots) \text{ is UI}$$

$$\therefore \int_B F = \int_B f_\infty \quad \forall B \in \bigvee_{k=1}^{\infty} \mathcal{A}_k$$

Note each f_n is $\bigvee_{k=1}^{\infty} \mathcal{A}_k$ measurable, so f_∞ is also $\bigvee_{k=1}^{\infty} \mathcal{A}_k$ measurable

Now to show $\|f_n - f_\infty\|_1 \to 0$. Follows because $(f_1 - f_\infty, f_2 - f_\infty, \dots)$ is UI and converges to 0 a.e.

$$\limsup E|f_n - f_\infty| \underset{\substack{\uparrow \\ \text{U.I.}}}{\le} E(\limsup |f_n - f_\infty|) = 0$$

For ②, things are similar except f_∞ is $\bigcap_{k=1}^{\infty} \mathcal{A}_k$-measurable because

$$\underset{\substack{\uparrow \\ \mathcal{A}_n \text{ measurable}}}{f_{n+k}} \longrightarrow f_\infty \text{ a.e. } (k \to \infty)$$

$\therefore$ f_∞ $\mathcal{A}_n$-measurable $\forall n \Rightarrow f_\infty$ $\bigcap_{n=1}^{\infty} \mathcal{A}_n$ measurable.

▨

2/9 MARTINGALES

<u>Proposition</u>: Suppose $\mathcal{A}_n \uparrow \mathcal{A}$ (i.e. $\bigvee_{k=1}^{\infty} \mathcal{A}_k = \mathcal{A}$). Then $\bigcup_{n=1}^{\infty} L_1(\Omega, \mathcal{A}_n, P)$ is dense in $L_1(\Omega, \mathcal{A}, P)$

example: $\mathcal{A}_n$ = n^{th} dyadic partition of $[0,1]$. Then dyadic functions are dense in L_1

Proof of prop. Let $f_n = E(F|\mathcal{A}_n)$, $F \in L_1$. Then

$$f_n \underset{\uparrow\,\text{continuity th}^{m}}{\longrightarrow} E\Big(F \,\Big|\, \bigvee_{k=1}^{\infty} \mathcal{A}_k\Big) = E(F|\mathcal{A}) = F$$

a.e. and in L_1. ▨

Let F be integrable on $[0,1)$. Then

$$\underbrace{\frac{1}{2^n}\sum_{k=0}^{2^n-1} F\Big(\underset{\uparrow\,\text{addition mod } 1}{x+\frac{k}{2^n}}\Big)}_{= E(F|\mathcal{A}_n)} \longrightarrow \int_0^1 F(x)\,dx \text{ a.e.}$$

$= E(F|\mathcal{A}_n)$ where $\mathcal{A}_n$ = σ-field generated by measurable sets of period $1/2^n$

$\mathcal{A}_n \downarrow \mathcal{A}_\infty$

Know $E(F|\mathcal{A}_n) \longrightarrow E(F|\mathcal{A}_\infty)$.

But if G is continuous, then $E(G|\mathcal{A}_n) \rightarrow EG$ by Riemann integration.

$$\searrow E(G|\mathcal{A}_\infty) = EG$$

$$\|E(F|\mathcal{A}_\infty) - EF\|_1 \leq \|E(G|\mathcal{A}_\infty) - EG\|_1$$

$$+ \|E(F-G|\mathcal{A}_\infty) - E(F-G)\|_1$$

$$\leq \|E(F-G|\mathcal{A}_\infty)\|_1 + |E(F-G)|$$

$$\leq 2\|F-G\|_1 < \varepsilon \text{ for appropriate } G$$

▨

Corollary: Then only integrable functions that are periodic with period $1/2^n$ for all n are the constant functions

Theorem: $F: [0,1] \longrightarrow \mathbb{R}$ nondecreasing, abs. continuous. Then there exists an integrable function f such that

$$\int_0^x f(t)\,dt = F(x) - F(0) \quad \text{a.e.}$$

Proof. Let $f_n = F_n' =$ right hand derivative on $[0,1)$

$\uparrow$ linear approx. over n^{th} partition.

(norm n^{th} part. $\rightarrow 0$)

$f = (f_1, f_2, \ldots)$ nonnegative martingale

$\therefore f$ converges a.e. to f_∞ , $\|f_\infty\|_1 \le \|f\|_1$

To show abs. cont. of F implies UI of f. Write

$$(f_n > \lambda) = \bigcup_{k=1}^{m} [a_k , b_k)$$

↑ sets in n^{th} partition

Then

$$\int_{(f_n > \lambda)} f_n = \sum_{k=1}^{m} \int_{[a_k, b_k)} f_n = \sum_{k=1}^{m} \int_{[a_k, b_k)} \frac{F(b_k) - F(a_k)}{b_k - a_k}$$

$$= \sum_{k=1}^{m} \left(F(b_k) - F(a_k) \right) < \varepsilon$$

if $P(f_n > \lambda)$ is small. But

$$P(f_n > \lambda) \le \frac{1}{\lambda} E f_n = \frac{E f_1}{\lambda} \to 0$$

↑ indep of n

Now let x be a point in one of the partitions, say n^{th} partition. As above

$$\int_0^x f_{n+h} = F(x) - F(0) \qquad \text{(telescoping sum)}$$

But UI $\Rightarrow \int_0^x f_{n+h} \xrightarrow[h \to \infty]{} \int_0^\infty f_\infty$ ∎

Radon-Nikodym Theorem : $(\Omega, \overset{\swarrow \text{ Separable}}{\mathcal{A}}, P)$, $\varphi \ll P$ non-negative ∧ finite measure on $(\Omega, \mathcal{A})$. Then there is an integrable f s.t.

$$\varphi(A) = \int_A f \, dP \qquad \forall A \in \mathcal{A}$$

Proof. Let Π_n be finite partition of Ω, $\mathcal{A}_n = \sigma(\Pi_n)$, $\mathcal{A}_n \uparrow \mathcal{A}$
let

$$f_n = \sum_{A \in \Pi_n} \frac{\varphi(A)}{P(A)} \chi_A$$

Then $(f_1, f_2, \dots)$ is a non-negative $\underset{\uparrow \text{ show as above}}{UI}$ integrable, ∴ $f_n \to f_\infty$ a.e. and in L_1

If $A \in \mathcal{A}_n$

$$\varphi(A) = \int_A f_{n+k} \xrightarrow{k} \int_A f_\infty$$

∴ true for $\cup \mathcal{A}_n$ $\Rightarrow$ true for $\sigma(\cup \mathcal{A}_n) = \mathcal{A}$

2/11 MARTINGALE

Martingale Analogue of Hardy-Littlewood Ineq.

$f = (f_1, f_2, \dots)$ martingale $\qquad f^*(\omega) := \sup_n |f_n(\omega)|$

$$(*) \qquad \|f^*\|_p \leq q \|f\|_p \qquad 1 < p \leq \infty$$

where $1/p + 1/q = 1$.

For $p=1$, best you can do is weak L_1 inequality $\lambda P(f^* > \lambda) \leq \|f\|_1$
This inequality implies $\quad \uparrow$ mart. or non-neg sub.

$$\lambda^p P(f^* > \lambda) \leq \|f\|_p^p \qquad 1 \leq p < \infty$$

since $|f_n|^p$ is a non-neg. submartingale.

Lemma: f, g non-negative $\mathcal{a}$-measurable functions. Suppose

$$\lambda P(g > \lambda) \leq \int_{\{g > \lambda\}} f \qquad \forall \lambda > 0$$

Then

$$\|g\|_p \leq q \|f\|_p \qquad 1 < p < \infty$$

((*) follows from this lemma and $\lambda P(f_n^* > \lambda) \leq \int_{f_n^* > \lambda} |f_n|$ by taking sups)

Proof. May assume $f \in L_p$, $g \in L_\infty$. Otherwise replace g by $g \wedge n$

Now

$$\|g\|_p^p = E g^p = \int_0^\infty p \lambda^{p-1} P(g > \lambda)\, d\lambda$$

$$\leq \int_0^\infty p \lambda^{p-1} \frac{1}{\lambda} \int_{\{g>\lambda\}} f \, dP \, d\lambda$$

$$= \int_\Omega p \int_0^g \lambda^{p-2} f \, d\lambda \, dP$$

$$= \frac{p}{p-1} \int_\Omega g^{p-1} f \, dP$$

$$\leq q \left(\int (g^{p-1})^q \right)^{1/q} \left(\int f^p \right)^{1/p}$$

$$= q \left(\int g^p \right)^{1/q} \|f\|_p$$

$$= q \, \|g\|^{p/q} \|f\|_p$$

$$\therefore \|g\|_p^{p-p/q} \leq q \|f\|_p \Rightarrow \|g\|_p \leq q \|f\|_p$$

▨

__Corollary__: If f is an L^p-bounded martingale then f converges a.e. and in L^p $(1<p<\infty)$

Proof. f is also L_1-bdd $\Rightarrow$ converges a.e. to f_∞. By Fatou $\|f_\infty\|_p \le \|f\|_p$. Then

$$|f_n - f_\infty| \le f^* + |f_\infty|$$

$|f_\infty|$ $\in L_p$

f^* $\in L_p$ by above result

$\therefore$ by DCT $E(|f_n - f_\infty|^p) \longrightarrow 0$

▨

Let (Ω, a, P) prob. space $\quad a_n \uparrow a_\infty = \bigvee_{k=1}^{\infty} a_k$

<u>Spaces of functions</u>		<u>Spaces of martingales</u>
$1<p<\infty \quad L^p(\Omega, a_\infty, P)$	$\simeq$ isomorphic isometric	$L^p(a_1, a_2, \dots) = \{f \ L^p\text{-bdd martingale}\}$ $f+g = (f_1+g_1, f_2+g_2, \dots)$ $\alpha f = (\alpha f_1, \alpha f_2, \dots)$ $\|f\|_p = \sup \|f_n\|_p$
f_∞	$\longrightarrow$	$E(f_\infty \mid a_n) = f_n$
$f_\infty = \lim f_n$	$\longleftarrow$	$f = (f_1, f_2, \dots)$

$L_1(\Omega, a_\infty, P) \quad \simeq \quad UI\,(a_1, a_2, \ldots) = \{$ unif. int. martingales $\}$

$H^p\,(a_1, a_2, \ldots) = \{$ martingales f s.t. $f^* \in L_p \}$

$0 < p < \infty$

isomorphic to L_p

if $1 < p < \infty$

$\|f\|_p \le \|f^*\|_p \le q\,\|f\|_p$

Hardy's Ineq $\quad f$ int. $\quad F(x) = \frac{1}{x}\int_0^x f(t)\,dt$

$$\int_0^\infty |F(x)|^p\,dx \le q \int_0^\infty |f(x)|^p\,dx$$

2/13 MARTINGALES

Theorem: $f = (f_1, f_2, \dots)$ non-negative submart. Then

$$\|f^*\|_1 \leq 2 + 2 \sup_n E f_n \log^+(f_n)$$

Proof. To show: $\lambda P(f_n^* > 2\lambda) \leq \int_{(f_n > \lambda)} f_n$. Let n be fixed, define

$$h_k = E\left[f_n \chi_{(f_n > \lambda)} \mid a_k\right]$$

Then $(h_1, h_2, \dots)$ is a martingale. If $1 \leq k \leq n$, then

$$f_k \leq E(f_n \mid a_k) = E(f_n \chi_{(f_n \leq \lambda)} \mid a_k) + h_k$$

$$\leq \lambda + h_k$$

$$\therefore f_n^* = \sup_{1 \leq k \leq n} f_k > 2\lambda \Longrightarrow h_n^* > \lambda$$

$$\therefore \lambda P(f_n^* > 2\lambda) \leq \lambda P(h_n^* > \lambda) \leq E h_n = E f_n \chi_{(f_n > \lambda)}$$

Thus

$$\|f_n^*\|_1 = \int_0^\infty P(f_n^* > \lambda)\, d\lambda = 2\int_0^\infty P(f_n^* > 2\lambda)\, d\lambda$$

$$= 2\int_0^1 P(f_n^* > 2\lambda)\, d\lambda + 2\int_1^\infty P(f_n^* > 2\lambda)\, d\lambda$$

$$\leq 2 + 2\int_1^\infty \frac{1}{\lambda} \int_{f_n > \lambda} f_n \, dP \, d\lambda$$

$$= 2 + 2 \int_{f_n > 1} f_n \int_1^{f_n} \frac{1}{\lambda} \, d\lambda$$

$$= 2 + 2 \, E f_n \log^+ f_n \qquad (\text{since integrating over } f_n > 1)$$

Now take $\sup_n$.

▨

Stopping times

$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots \subset \mathcal{A}$. A stopping time τ is a function from Ω to $\mathbb{N} \cup \{\infty\}$ s.t. $\{\tau \leq n\}$ is $\mathcal{A}_n$-measurable $\forall n \geq 1$

Example: Suppose $f = (f_1, f_2, \dots)$ is any seq. of functions adapted to $\mathcal{A}_1, \mathcal{A}_2, \dots$, i.e. f_n is $\mathcal{A}_n$-measurable. Then

$$\tau(\omega) = \inf \left(n : f_n(\omega) > \lambda \right)$$

$(\inf \emptyset = \infty)$ is a stopping time since

$$\{\tau \leq n\} = \bigcup_{k=1}^{n} \underbrace{\{f_k > \lambda\}}_{\mathcal{A}_k \text{ meas} \; \therefore \; \mathcal{A}_n \text{ measurable}} \in \mathcal{A}_n$$

Properties

① τ_1, τ_2 stopping times $\Rightarrow$ $\tau_1 \vee \tau_2$, $\tau_1 \wedge \tau_2$ stopping times

$$\{\tau_1 \vee \tau_2 \leq n\} = \{\tau_1 \leq n\} \cap \{\tau_2 \leq n\} \in \mathcal{A}_n$$
$$\{\tau_1 \wedge \tau_2 > n\} = \{\tau_1 > n\} \cap \{\tau_2 > n\} \in \mathcal{A}_n$$

If $\tau < \infty$ a.e. or if f_∞ is $\mathcal{A}$-measurable, define

$$f_\tau(\omega) = f_{\tau(\omega)}(\omega)$$

② f_τ is $\mathcal{A}$-measurable.

$$\{f_\tau \in \underset{\nearrow \\ \text{Borel}}{B}\} = \bigcup_{n=1}^{\infty} \{f_\tau \in B\} \cap \{\tau = n\} \cup \{f_\tau \in B\} \cap \{\tau = \infty\}$$

$$= \bigcup_{n=1}^{\infty} \underbrace{\{f_n \in B\} \cap \{\tau = n\}}_{\mathcal{A}_n\text{-meas}} \cup \underbrace{\{f_\infty \in B\} \cap \{\tau = \infty\}}_{\mathcal{A}\text{-meas}}$$

$$\in \mathcal{A}$$

③ $f_{\tau \wedge n}$ is $\mathcal{A}_n$-measurable

Definition: $f = (f_1, f_2, \dots)$ any seq. of functions, τ stopping time

$$f^{\tau} := (f_{\tau\wedge 1}, f_{\tau \wedge 2}, f_{\tau\wedge 3}, \dots)$$

(called f stopped at τ

Note $$(f^{\tau})_n = \begin{cases} f_n & n < \tau \\ f_{\tau} & n > \tau \end{cases}$$

Note $$f_{\tau\wedge n} = \sum_{k=1}^{n} I(\tau \geq k)\, d_k$$

$\uparrow$ diff. seq $f_n = \sum_{k=1}^{n} d_k$

Lemma: If f is a martingale (submart.), then f^{τ} is a martingale (submart.)

In the martingale case $Ef_1 = Ef_{\tau\wedge n} = Ef_n$

In the submartingale case $Ef_1 \leq Ef_{\tau \wedge n} \leq Ef_n$. Furthermore, if f is an L_1-bdd martingale or non-neg. submartingale, then

(*) $$\|f_{\tau}\|_1 \leq \|f^{\tau}\|_1 \leq \|f\|_1$$

Application: Let $\tau = \inf\{n : |f_n| > \lambda\}$. Then

$$\lambda P(f^* > \lambda) \underset{\substack{\uparrow \\ \text{since } \{f^* > \lambda\} = \{\tau < \infty\}}}{\le} \lambda P(f_\tau > \lambda) \underset{\substack{\uparrow \\ \text{Chebychev}}}{\le} \|f_\tau\|_1 \underset{\substack{\uparrow \\ \text{lemma}}}{\le} \|f\|_1$$

Proof of lemma: Note $I(\tau \ge k) d_k$ is $\mathcal{A}_k$-measurable and integrable

$$\therefore\ E(I(\tau \ge k) d_k \mid \mathcal{A}_{k-1}) = \underset{\substack{\uparrow \\ \mathcal{A}_{k-1} \text{ meas.}}}{I(\tau \ge k)} E(d_k \mid \mathcal{A}_{k-1})$$

$$= 0 \quad \text{mart.}$$
$$\ge 0 \quad \text{submart.}$$

$\therefore$ $I(\tau \ge k) d_k$ mart. diff. seq. $\Rightarrow$ $f_{\tau \wedge n} = \sum_{k=1}^{n} I(\tau \ge k) d_k$ is martingale (resp. submart.)

$$Ef_n - Ef_{\tau \wedge n} = \sum_{k=1}^{n} E[I(\tau < k) d_k]$$

$$= \sum_{k=1}^{n} E\Big[\underset{\overset{\shortparallel}{0}\ k=1}{I(\tau < k)}\ \underset{\overset{\shortparallel}{0}\ k\ge 2}{E(d_k \mid \mathcal{A}_{k-1})}\Big]$$

$$= 0 \quad (\ge 0 \text{ submart})$$

To get (*) use fact that $\xi_\tau = \lim_{n\to\infty} \xi_{\tau\wedge n}$, so

$$E|\xi_\tau| \le \liminf E|\xi_{\tau\wedge n}| \le \liminf E|\xi_n| \le \|\xi\|_1$$

$$\le \|\xi^\tau\|_1$$

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Example: $f_1 = 10$ $\quad \tau = \inf\{n : f_n > 10\}$ $\quad f = (f_1, f_2, \dots)$ nonneg. mart. or supermart.

Note: Here $P(\tau = \infty) > 0$ for otherwise $f_\tau > 10$ a.e.

$$\Rightarrow \quad 10 = Ef_1 \ge Ef_\tau > 10 \quad ↯$$

THEOREM: $f = (f_1, f_2, \dots)$ martingale with difference seq. d where $Ed^* < \infty$. Then the following sets are equal a.e.

(1) $\{f \text{ converges}\}$

(2) $\{\sup_n f_n < \infty\}$

(3) $\{\inf_n f_n > -\infty\}$

Proof. Clearly (1) $\subset$ (2) a.e. Now to show (2) $\subset$ (1) a.e. Let $\lambda > 0$ and let $\tau = \inf\{n : f_n > \lambda\}$

Claim: f^τ is an L_1-bdd martingale.

If this holds, then f^τ converges a.e. by the martingale convergence theorem

$$\{f \text{ converges}\} \supset \{f^\tau \text{ converges},\ \overset{\Rightarrow \tau = \infty \text{ so } f^\tau = f}{\sup f_n \le \lambda}\}$$

$$\therefore \{ f \text{ converges} \} \underset{a.e.}{\supseteq} \{ \sup f_n \leq \lambda \} \quad \text{since } f^{\tau} \text{ converges a.e.}$$

Now let $\lambda \to \infty$

To show claim: $$f_{\tau \wedge n} \leq \begin{cases} \lambda & \tau > n \\ \lambda + d^{*} & \tau \leq n \end{cases}$$

$$\therefore \quad E f_{\tau \wedge n} \leq \lambda + E d^{*} < \infty$$

$$\therefore E|f_{\tau \wedge n}| = E f^{+}_{\tau \wedge n} + E f^{-}_{\tau \wedge n} = 2 E f^{+}_{\tau \wedge n} - E f_1$$
$$\leq 2(\lambda + E d^{*}) - E f_1 \leq M$$

$$\left(E f_1 = E f_{\tau \wedge n} = E f^{+}_{\tau \wedge n} - E f^{-}_{\tau \wedge n} \right)$$

▨

Martingale Transforms

Suppose d_k dollars are to be won by a gambler playing kth game (fair game). His fortune is

$$f_n = \sum_{k=1}^{n} d_k$$

Let

$$g_n = \sum_{k=1}^{n} v_k d_k$$

$\swarrow$ "value of k^{th} game"

where V_k is a function depending only on the past (e.g. V_k is $\mathcal{a}_{k-1}$ measurable)

(e.g. $V_k = I(\underset{\uparrow \text{stopping time}}{\tau} \geq k)$, then $g = f^{\tau}$)

<u>Definition:</u> f martingale $\mathcal{a}_0 \subset \mathcal{a}_1 \subset \mathcal{a}_2 \subset \cdots \subset \mathcal{a}$
d difference sequence. $v = (v_1, v_2, \cdots)$ where V_k is $\mathcal{a}_{k-1}$-measurable
Then $g = (g_1, g_2, \cdots)$ where $\nwarrow$ predictable seq.

$$g_n = \sum_{k=1}^{n} v_k d_k$$

is called a martingale transform.

<u>Remarks</u> (1) g need not be a martingale (for example, $v_k d_k$ might not be integrable)

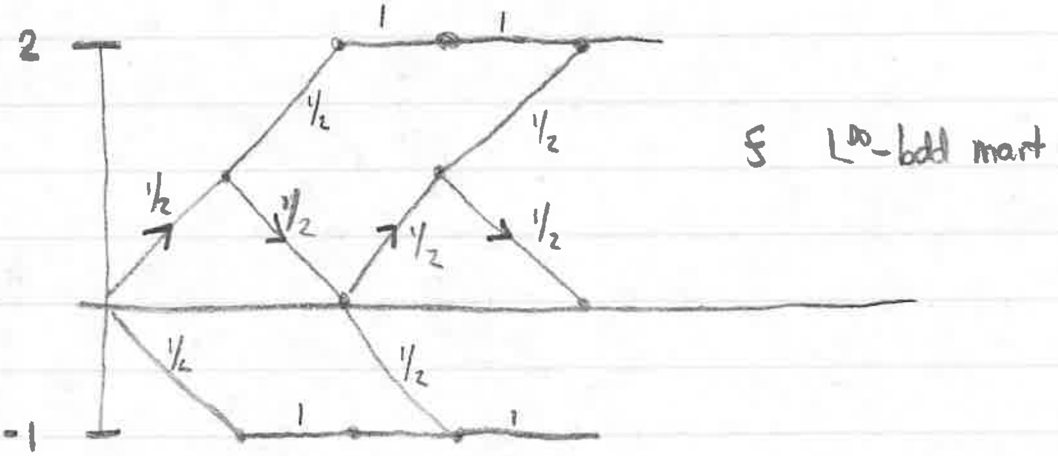
(2) If each V_k is bounded, then g is a martingale

$$V_k d_k \quad \mathcal{a}_k\text{-measurable}$$
integrable
$$E(V_k d_k | \mathcal{a}_{k-1}) = V_k E(d_k | \mathcal{a}_{k-1}) = 0 \text{ for } k \geq 2$$

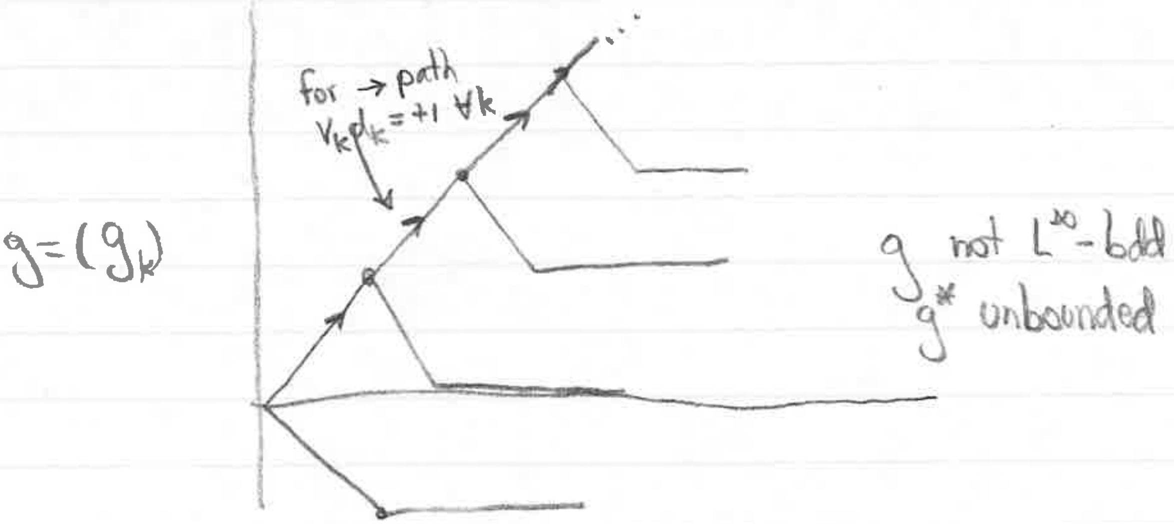
(3) Even if the V_k's are unif. bdd by 1 (i.e. $v^* \leq 1$) g can be worse than f: it can happen that $\|f\|_\infty < \infty$ but $\|g\|_\infty = \infty$ or even $\|f\|_1 < \infty$ but $\|g\|_1 = \infty$

THEOREM: If f is an L^1-bdd martingale and g is the transform of f by a predictable sequence $v=(v_1,v_2,\dots)$, then g converges a.e. on the set where $v^* < \infty$.

In particular, $\|f\|_1 < \infty \Rightarrow g$ converges a.e. if $v^* \le 1$.



$$V_1 \equiv 1 \quad V_2 \equiv -1 \;, \dots, \; V_k \equiv (-1)^{k+1}$$



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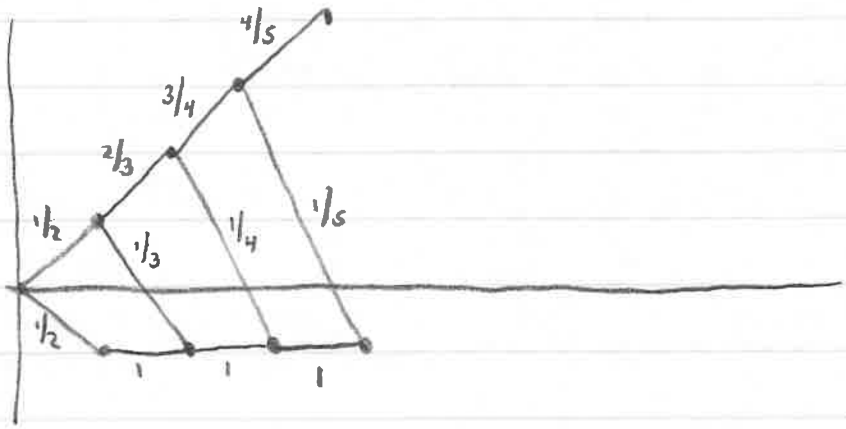
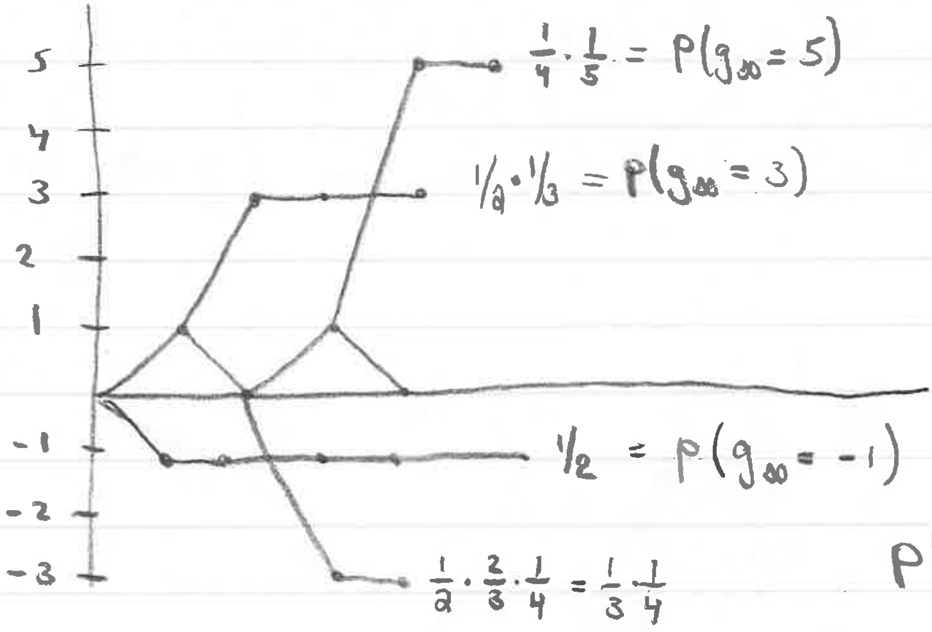


f is L_1-bdd since $f_n = (f_n + 1) - 1$

↑ non-negative mart

Let $v = (1, -1, 1, -1, \dots)$



$\frac{1}{4} \cdot \frac{1}{5} = P(g_\infty = 5)$

$1/2 \cdot 1/3 = P(g_\infty = 3)$

$1/2 = P(g_\infty = -1)$

$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{3} \frac{1}{4}$

$P(|g_\infty| \geq k) \geq \frac{1}{k}$

$E|g_\infty| = 1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} \cdot \frac{1}{4} + 5 \cdot \frac{1}{4} \cdot \frac{1}{5} + \dots$ diverges

THEOREM: f L_1-bdd $g_n = \sum_{k=1}^{n} \underset{\substack{\uparrow \\ \mathcal{A}_{k-1} \text{ measurable}}}{v_k} d_k$

Then $g = (g_1, g_2, \dots)$ converges a.e. on the set where $\{v^* < \infty\}$

Proof. (i) $\|f\|_2 < \infty$, $v^* \leq 1 \Rightarrow g$ converges a.e.
Let $e_k = v_k d_k$. Then $|e_k| \leq |d_k|$ and so

$$\|g_n\|_2^2 = \sum_{k=1}^{n} \|e_k\|_2^2 \leq \sum_{k=1}^{n} \|d_k\|_2^2 = \|f_n\|_2^2$$

Thus $\|g\|_1 \leq \|g\|_2 < \infty$ so g is L_1-bdd hence converges a.e. by martingale convergence theorem.

(ii) f uniformly bounded submartingale, $v^* \leq 1 \Rightarrow g$ converges a.e.
Let $k \geq 2$

$$d_k = d_k - E(d_k | \mathcal{A}_{k-1}) + E(d_k | \mathcal{A}_{k-1})$$

$$= \hat{d}_k + E(d_k | \mathcal{A}_{k-1})$$

Let $\hat{d}_1 = d_1$. Note that $\hat{d}_k$ is a martingale difference seq. and $E(d_k | \mathcal{A}_{k-1}) \geq 0$ by submart. property. Also, $\hat{f}$ is L_2-bdd since

$$E\hat{d}_k^2 = \|d_k - E(d_k|\mathcal{A}_{k-1})\|_2^2 \leq \|d_k\|_2^2$$

↑ since $E(d_k|\mathcal{A}_{k-1})$ is best predictor in L_2

$$\therefore \ \|\hat{f}\|_2 \leq \|f\|_2 < \infty$$

$\therefore$ $\hat{f}$ converges a.e.

Then by (i), $\hat{g}_n = \sum_{k=1}^{n} v_k \hat{d}_k$ converges a.e. Now

$$\begin{tikzcd} g_n = & \hat{g}_n \arrow[d] & + \sum_{k=2}^{n} v_k E(d_k | a_{k-1}) \arrow[d] \\ & \hat{g}_\infty & \sum_{k=2}^{n} E(d_k | a_{k-1}) = f_n - \hat{f}_n \end{tikzcd}$$

both L_2 bdd

$\Rightarrow$ converges a.e.

$\therefore$ given series converges absolutely a.e.

Hence g_n converges a.e.

(iii) f non-negative martingale, $v^* \leq 1 \Rightarrow g$ converges a.e.

Let

$$F_n = f_n \wedge \lambda \qquad (\lambda > 0)$$

This is a unif. bdd supermartingale since

$$E(F_{n+1} | a_n) \leq \lambda$$
$$E(F_{n+1} | a_n) \leq E(f_{n+1} | a_n) = f_n$$

By (ii), $G_n = \sum_{k=1}^{n} v_k D_k$ converges a.e. Thus

$$\{g \text{ converges}\} \underset{a.e.}{\supseteq} \{g = G\} \supseteq \{f^* \le \lambda\} \uparrow \{f^* < \infty\} \underset{a.e.}{=} \Omega$$

(iv) f L^1-bdd martingale, $v^* \le 1 \Rightarrow g$ converges a.e.

By HW #3, $f = f' - f''$ where f', f'' are non-negative martingales. Then

$$\begin{tikzcd} g = & g' \arrow[d] & - & g'' \arrow[d] \\ & g'_\infty & - & g''_\infty \text{ by (iii)} \end{tikzcd}$$

To complete the proof of the theorem, let $u_k = \begin{cases} v_k & |v_k| \le \lambda \\ 0 & \text{otherwise} \end{cases}$

Then u_k is $\mathcal{A}_{k-1}$-measurable and $\frac{u_k}{\lambda}$ is unif. bdd by 1. By (iv)

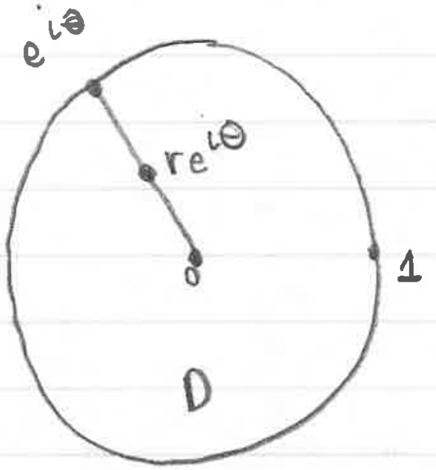
$$\sum_{k=1}^{n} u_k d_k$$

converges a.e. Then

$$\{g \text{ converges}\} \underset{a.e.}{\supseteq} \{\underset{\substack{\uparrow \\ \text{here } u_k = v_k}}{v^* \le \lambda}\} \uparrow \{v^* < \infty\}$$

▨

Harmonic function analogue (Privalov 1919)



u harmonic in D

v conjugate of u

Fatou (1906) showed $u \geq 0 \Rightarrow \lim_{r\to 1-} u(re^{i\theta})$ exists and is finite a.e.

$$\|u\|_p = \sup_{r<1} \int_0^1 |u(re^{i\theta})|^p \frac{d\theta}{2\pi}$$

$\|u\|_1 < \infty \Rightarrow \lim_{r\to 1-} u(re^{i\theta})$ exists and is finite a.e

Privalov showed $\|u\|_1 < \infty \Rightarrow \lim_{r\to 1-} v(re^{i\theta})$ exists and is finite a.e.

M. Riesz: $\|v\|_p \leq C_p \|u\|_p \qquad 1 < p < \infty$

Kolmogorov: $u = \mathrm{Re}\,\frac{1+z}{1-z}$, $v = \mathrm{Im}\,\frac{1+z}{1-z}$ then $\|u\|_1 < \infty$ but $\|v\|_1 = \infty$

but is true that $\lambda\, m\{\theta : |\lim_{r\to 1-} v(re^{i\theta})| > \lambda\} \leq c\,\|u\|_1$

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0-1 LAWS

Kolmogorov's 0-1 Law: $X_1, X_2, \dots$ indep. r.v.. If A is a tail event, then $P(A) = 0$ or $P(A) = 1$.

$\uparrow$ Let $A_n = \sigma(X_{n+1}, X_{n+2}, \dots)$ and $A_\infty = \bigcap_{n=1}^{\infty} \mathcal{A}_n$. Sets in A_∞ = tail events

Proof. Let $\mathcal{B}_n = \sigma(X_1, \dots, X_n)$. Note that $\mathcal{A}_n$ and $\mathcal{B}_n$ are independent σ-fields.

$$\mathcal{B}_n \uparrow \mathcal{B}_\infty = \sigma(X_1, \dots, X_n, \dots)$$

Since $\mathcal{A}_n \subset \mathcal{B}_\infty \; \forall n$, $\mathcal{A}_\infty \subset \mathcal{B}_\infty$. Let $A \in \mathcal{A}_\infty$

$$P(A) = E(\chi_A) \underset{\uparrow \text{ since } A \in \mathcal{A}_n \text{ and } \mathcal{B}_n \text{ ind. of } \mathcal{A}_n}{\overset{\text{a.e.}}{=}} E(\chi_A \mid \mathcal{B}_n) \xrightarrow{\text{a.e.}} E(\chi_A \mid \mathcal{B}_\infty) \overset{\text{a.e.}}{=} \chi_A$$

$$\therefore \; P(A) = \chi_A \text{ a.e.}$$

so either $P(A) = 0$ or $P(A) = 1$

Applications: (1) $$P\Big(\underbrace{\sum_{k=1}^{\infty} a_k \underset{\uparrow \text{ indep.}}{X_k} \text{ converges}}_{\mathcal{A}_n\text{-meas. } \forall n \Rightarrow \text{ tail event}}\Big) = 0 \text{ or } 1$$

(2) $$\limsup \frac{X_1+\dots+X_n}{n} = \text{constant (possibly } \pm\infty)$$

$$\left(= \limsup_{n\to\infty} \frac{X_{m+}\dots+X_n}{n} \Rightarrow a_m\text{-meas } \forall m \Rightarrow a_\infty\text{-meas} \right)$$

Strong Law of Large Numbers $X_1, X_2, \dots$ i.i.d random variables

$\uparrow$ $P(X_1\in B) = P(X_2 \in B) = \dots$

$$E|X_1| < \infty \Rightarrow \frac{X_1+\dots+X_n}{n} \longrightarrow EX_1 \text{ a.e.}$$

If $X_1 \geq 0$ but not integrable, then

$$\frac{X_1+\dots+X_n}{n} \longrightarrow +\infty \text{ a.e.}$$

(Apply 1st part to $X_n \wedge \lambda$ and let $\lambda \to \infty$)

Proof. $$E(X_k \mid X_1+X_2+\dots+X_n) = Y_n \quad \forall 1 \leq k \leq n$$

where $Y_n := E(X_1 \mid X_1+\dots+X_n)$

Claim: $$\int_{X_1+X_2\in B} X_1 = \int_{X_1+X_2 \in B} X_2$$

Follows from observation that for bdd meas φ, $E\varphi(X_1,X_2) = E\varphi(X_2,X_1)$

Since

$$\iint_{\mathbb{R}^2} \varphi(x_1,x_2)\, d\mu(x_1)\, d\mu(x_2) = \iint_{\mathbb{R}^2} \varphi(x_2,x_1)\, d\mu(x_1)\, d\mu(x_2)$$

(works for $\chi_{B_1 \times B_2}$ by indep) Let $\varphi(x_1,x_2) = x_1 \chi_B(x_1+x_2)$

Thus

$$E(\underset{\overset{\|}{X_1+\cdots+X_n}}{X_1+X_2+\cdots+X_n} \mid X_1+\cdots+X_n) = nY_n$$

$$\therefore \quad \frac{X_1+\cdots+X_n}{n} = Y_n$$

Note $Y_n = E(X_1 \mid X_1+\cdots+X_n) = E(X_1 \mid \underbrace{X_1+\cdots+X_n}_{S_n}, \underbrace{X_1+\cdots+X_{n+1}}_{S_{n+1}}, \ldots)$

$$= E(X_1 \mid X_1+\cdots+X_n, X_{n+1}, X_{n+2}, \ldots)$$

since $X_1, \ldots, X_n$ indep of $X_{n+1}, X_{n+2}, \ldots$

This last term converges to $E(X_1 \mid \text{tail } \sigma\text{-alg of } S_n\text{'s})$

$\therefore \dfrac{X_1+\cdots+X_n}{n} \longrightarrow$ a.e. to a finite constant $\alpha \Rightarrow EX_1 = E\left(\dfrac{X_1+\cdots+X_n}{n}\right) \rightarrow E\alpha = \alpha$

▨

Borel-Contelli lemma $A_1, A_2, \ldots \in \mathcal{A}$. If $\sum P(A_i) < \infty$, then

$$P(A_n \text{ occurs i.o.}) = P\Big(\bigcap_n \bigcup_{k \geq n} A_k\Big) = 0$$

If the $A_1, A_2, \ldots$ are independent, then

$$\sum P(A_i) = \infty \Rightarrow P(A_n \text{ occurs i.o.}) = 1$$

Proof. Let $v_k = \chi_{A_k}$. $E v_k = P(A_k)$

$$\therefore E\Big(\sum_{k=1}^{\infty} v_k\Big) = \sum_{k=1}^{\infty} P(A_k) < \infty$$

$$\therefore \sum_{k=1}^{\infty} v_k < \infty$$

Alt: $P\Big(\bigcap_n \bigcup_{k \geq n} A_k\Big) \leq P\Big(\bigcup_{k \geq n} A_k\Big) \leq \sum_{k=n}^{\infty} P(A_k) \xrightarrow[n \to \infty]{} 0$

For other statement, let $f_n = \sum_{k=1}^{n} \big(v_k - E(v_k \mid \underset{\uparrow \sigma(v_1, \ldots, v_{k-1})}{\mathcal{A}_{k-1}})\big)$

$$= \sum_{k=1}^{n} d_k$$

$|d_k| \leq 2 \quad E d^* < \infty$

so $\sum \chi_{A_k} < \infty$

$\Leftrightarrow \sum P(A_k) < \infty$

Lemma: $v_1, v_2, \ldots$ any $\mathcal{A}$-meas functions $v^* \leq 1$,

$$\sum_{k=1}^{\infty} v_k \text{ converges a.e.} \Leftrightarrow \sum_{k=1}^{\infty} E(v_k \mid \mathcal{A}_{k-1}) \text{ converges a.e.}$$

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$0 \le u_k \le 1$ $\qquad \mathcal{A}_k = \sigma(u_1, \dots, u_k)$

$\mathcal{A}$-meas.

$$\sum_{k=1}^{\infty} u_k < \infty \overset{a.e.}{\Longleftrightarrow} \sum_{k=1}^{\infty} E(u_k \mid \mathcal{A}_{k-1}) < \infty$$

Proof. $\xi_n = \sum_{k=1}^{n} \left(u_k - E(u_k \mid \mathcal{A}_{k-1}) \right)$ martingale with $Ed^* < \infty$

$$\left\{ \sum_{k=1}^{\infty} u_k = \infty, \ \sum_{k=1}^{\infty} E(u_k \mid \mathcal{A}_{k-1}) < \infty \right\}$$

$$\subset \{ \xi_n \longrightarrow \infty \}$$

$$\subset \{ \inf \xi_n > -\infty \}$$

$$\underset{a.e.}{\subset} \{ \xi_n \text{ converges} \} \quad ↯$$

Banach space valued martingales

$\mathcal{X}$ Banach space as usual
$(\Omega, \mathcal{A}, P)$ probability space

$$f = \sum_{k=1}^{n} x_k \chi_{A_k} \quad \text{simple function}$$

$\uparrow$ disjoint

$$\|f\|_1 = E\|f\| = \sum_{k=1}^{n} \|x_k\| P(A_k)$$

$$\geq \|Ef\| = \|\sum_{k=1}^{n} x_k P(A_k)\|$$

Definition: $f : \Omega \to \mathcal{X}$ is measurable if it is the a.e. pointwise limit of a sequence of simple functions (f_n)

If, furthermore, $\|f_n - f\|_1 \to 0$ then f is Bochner integrable

$$Ef := \lim_{n \to \infty} Ef_n$$

$\uparrow$ exists by completeness

$E(f|\mathcal{B})$ defined as before

THEOREM: $E(f|\mathcal{B})$ exists and is unique a.e.

Proof. If $f = \sum_{k=1}^{n} x_k \chi_{A_k}$, then $E(f|\mathcal{B}) = \sum_{k=1}^{n} x_k E(\chi_{A_k}|\mathcal{B})$

For arbitrary f use approximation by simple functions.

▨

As before $\| E(f|\mathcal{B}) \|_1 \leq \| f \|_1$

Martingales defined as in real-valued case

Doob's results

$f: \Omega \to \mathbb{R}$ martingale $f = (f_1, f_2, \ldots)$

(1) $\|f\|_1 < \infty \Rightarrow f$ converges a.e.

(2) $\lambda P(f^* > \lambda) \leq \|f\|_1$

(3) $\|f^*\|_p \leq q \|f\|_q \quad 1 < p < \infty, \; 1/p + 1/q = 1$

Only (2) and (3) remain true for $\mathfrak{X}$-valued martingales ($f^* = \sup_n \|f_n(\cdot)\|$

THEOREM: (Chatterji) Condition (1) holds for all $\mathfrak{X}$-valued martingales iff $\mathfrak{X}$ has RNP.

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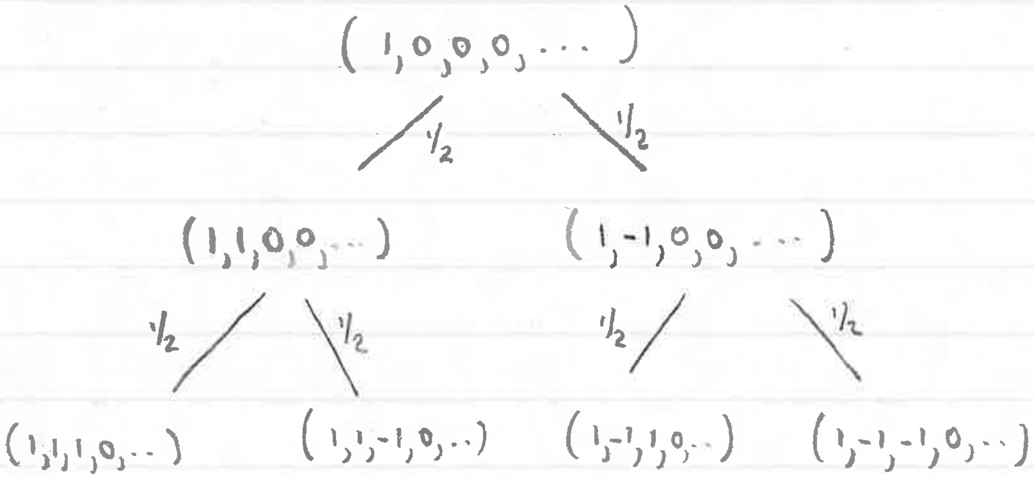
Example: Let $\mathcal{X} = \ell_\infty$



etc.

L_∞-bdd but f does not converge a.e. $(|d_n(\omega)| = 1)$

Example: Let $\mathcal{X} = \ell_1$

Let f be L_1-bdd ℓ_1-valued. Then f converges a.e.

Proof. $f = (f_1, f_2, \dots)$ where

$$f_n = \begin{pmatrix} f_{1n} \\ f_{2n} \\ f_{3n} \\ \vdots \end{pmatrix} \in \ell_1 \quad \therefore f = (f_{jn})$$ where rows are real-valued martingales

$$\|f_n\|_1 = E|f_n| = \sum_{j=1}^{\infty} E|f_{jn}|$$

$$\therefore\ \infty > \|f\|_1 = \sup \|f_n\|_1 \geq \sup_n E|f_{jn}| \quad \text{for each } j$$

Hence (f_{jn}) is L_1-bdd $\mathbb{R}$-valued martingale so $f_{jn} \longrightarrow f_{j\infty}$ a.e.

$$f_\infty = \begin{pmatrix} f_{1\infty} \\ f_{2\infty} \\ \vdots \end{pmatrix}$$

increasing seq in n since $E|f_{jn}| \uparrow_n$

$$\|f\|_1 = \sup_n \|f_n\| = \sup_n \sum_{j=1}^{\infty} E|f_{jn}|$$

$$= \sum_{j=1}^{\infty} \sup_n E|f_{jn}| \Rightarrow \sum_{j>i} \sup E|f_{jn}| \longrightarrow 0$$

as $i \to \infty$

by Monotone Convergence

$$\therefore\ \|f_n(\omega) - f_\infty(\omega)\|_1 = \sum_{j=1}^{\infty} |f_{jn}(\omega) - f_{j\infty}(\omega)|$$

$$= \sum_{j\leq i} |f_{jn}(\omega) - f_{j\infty}(\omega)| + \sum_{j>i} |f_{jn}(\omega) - f_{j\infty}(\omega)|$$

$\leq 2M_i$

$$M_i = \sup_n \sum_{j>i} |f_{jn}| \longrightarrow 0 \quad i \to \infty$$

$$\limsup_{n\to\infty} \|f_n(\omega) - f_\infty(\omega)\|_1 \le 0 + 2M_i$$

Let $i \to \infty$ to see $\limsup_{n\to\infty} \|f_n(\omega) - f_\infty(\omega)\|_1 = 0$

To see why $\lim_{i\to\infty} M_i = 0$: $\forall \lambda > 0$

$$\lambda P\Big(\sup_n \underbrace{\sum_{j>i} |f_{jn}|}_{\text{submartingale in } n} > \lambda\Big) \le \sup_n \sum_{j>i} E|f_{jn}|$$

$$= \sum_{j>i} \sup_n E|f_{jn}|$$

$$\longrightarrow 0 \text{ as } i \to \infty$$

▨

Let $g_n = \sum_{k=1}^{n} \underset{\uparrow\ \mathcal{A}_{k-1}\text{-meas.}}{v_k} d_k$ Assume $v^* \le 1$

(1) $\|f\|_1 < \infty \Rightarrow g$ converges a.e.

(2) $\lambda P(g^* > \lambda) \le c\|f\|_1$

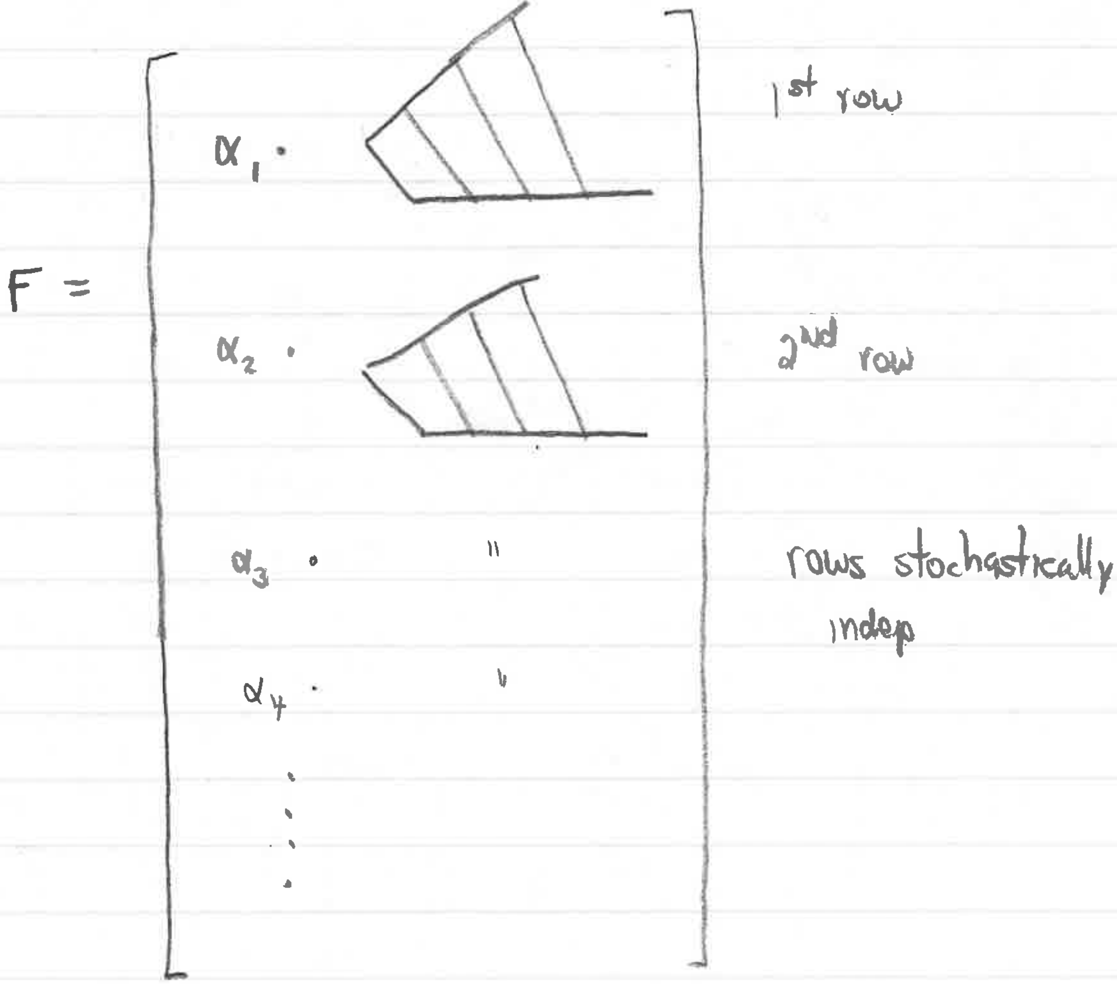
(3) $\|g\|_p \le c_p \|f\|_p$ $(1 < p < \infty)$

What about case when d_k is $\mathfrak{X}$-valued and v_k is $\mathbb{R}$-valued?

If for $\mathfrak{X}$, (1) always holds, then (2) always holds and
(3) always holds

Actually, if one always holds, then all the other holds. We say $\mathfrak{X} \in MT$ if (1) holds for $\mathfrak{X}$.

Example: $\ell_1 \notin MT$



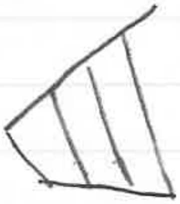
$\alpha_j > 0 \quad \sum_j \alpha_j < \infty \qquad v = (1,-1,1,-1,\dots)$

$$G = \begin{bmatrix} \alpha_1 \cdot & \\ & \\ \alpha_2 \cdot & " \\ \vdots & \end{bmatrix}$$

$\alpha_1 g_{1\infty}$ exists

$\alpha_2 g_{2\infty}$ exists

$\mathbb{R}$-valued $\begin{cases} \hat{f} = (\hat{f}_1, \hat{f}_2, \ldots) = \\ \hat{g} = (\hat{g}_1, \hat{g}_2, \ldots) \text{ transform with } (1,-1,1,-1,\ldots) \end{cases}$



$$\|\hat{f}\|_1 < \infty \qquad E|\hat{g}_\infty| = \infty$$

Let $f_j = (f_{j1}, f_{j2}, \ldots)$ $j=1,2,\ldots$ be an indep seq of martingales each having the same dist. as $\hat{f}$

$$F_{jn} = \alpha_j f_{jn} \qquad G_{jn} = \alpha_j g_{jn}$$

$F = (F_1, F_2, \ldots)$ ℓ_1-valued L_1-bdd $\qquad G^* = \infty$ a.e. G diverges a.e.

$$\|F_n\|_1 = \sum_{j=1}^{\infty} |F_{jn}| = \sum_{j=1}^{\infty} \alpha_j |\xi_{jn}|$$

$$E\|F_n\|_1 = \sum_{j=1}^{\infty} \alpha_j \underbrace{E|\xi_{jn}|}_{E|\hat{\xi}_n| \le \|\hat{\xi}\|_1} \le \|\hat{\xi}\|_1 \sum_{j=1}^{\infty} \alpha_j < \infty$$

$\therefore \|F\|_1 < \infty$

$$G^* = \sup_n \|G_n\|_1 = \sup_n \sum_{j=1}^{\infty} \alpha_j |g_{jn}|$$

$$\underset{\uparrow \text{Fatou}}{\ge} \sup_n \sum_{j=1}^{\infty} \alpha_j |g_{j\infty}|$$

$$= \sum_{j=1}^{\infty} \alpha_j \underbrace{|g_{j\infty}|}_{X_j} = +\infty \text{ a.e.}$$

Lemma: $X_1, X_2, \ldots$ non-neg i.i.d $EX_1 = \infty$. Then $\exists\, \alpha_1, \alpha_2, \ldots$ positive s.t. $\sum \alpha_j < \infty$ and

$$\sum \alpha_j X_j = \infty \text{ a.e.}$$

(Recall SLLN $\Rightarrow \dfrac{X_1 + \cdots + X_n}{n} \to \infty$ a.e.

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Lemma: $X_1, X_2, \ldots$ i.i.d non-negative $EX_1 = \infty$. Then $\exists\, \alpha_1, \alpha_2 \ldots > 0$ s.t. $\sum \alpha_j < \infty$ but $\sum \alpha_j X_j$ diverges a.e

Proof. By SLLN $\dfrac{X_1 + \ldots + X_n}{n} \to \infty$ a.e. Given $\varepsilon > 0$,

there is a $\lambda > 0$ s.t.

$$P\left(\frac{X_1 + \ldots + X_n}{n} > \lambda \right) > 1 - \varepsilon$$

Let $n_0 = 0 < n_1 < n_2 < \ldots$ satisfy

$$P\Big(\sum_{n_{k-1} < j \le n_k} X_j > k^2 (n_k - n_{k-1}) \Big) > \frac{1}{k+1}$$

event A_k

Let $\alpha_j = \dfrac{1}{k^2} \dfrac{1}{n_k - n_{k-1}}$ if $n_{k-1} < j \le n_k$

Note $\sum P(A_k) > \sum \frac{1}{k+1} = \infty$, so by Borel-Cantelli A_k occurs infinitely often

$$\sum_{j=1}^{\infty} \alpha_j X_j = \sum_k \underbrace{\sum_{n_{k-1} \le j \le n_k} \alpha_j X_j}_{>1 \text{ for } \omega \in A_k \text{ which happens inf. often}}$$

Khintchine's inequality: r_n n^{th} Rademacher function; $a_0, a_1, \ldots \in \mathbb{R}$

$$\forall n, \quad c_p\Big(\sum_{k=0}^{n} a_k^2\Big)^{1/2} \le \Big(\int_0^1 \Big|\underbrace{\sum_{k=0}^{n} a_k r_k(t)}_{f}\Big|^p dt\Big)^{1/p} \le C_p\Big(\sum_{k=0}^{n} a_k^2\Big)^{1/2}$$

$(0 < p < \infty)$

Note: $r_0, r_1, r_2, \ldots$ are independent

Proof. WLOG $\sum_{k=1}^{n} a_k^2 = 1$. Enough to prove RHS for $p = 2m$ (if $p \le 2m$, $\|f\|_p \le \|f\|_{2m} \le C_{2m}$). Now

$$\frac{f^{2m}}{(2m)!} \le \frac{e^f + e^{-f}}{2}$$

$$\frac{Ef^{2m}}{(2m)!} \le \frac{Ee^f + Ee^{-f}}{2} = Ee^f$$

$$Ee^f = E\prod_{k=0}^{n} e^{a_k r_k} \underset{\text{indep.}}{=} \prod_{k=0}^{n} E e^{a_k r_k} = \prod_{k=0}^{n} \frac{e^{a_k} + e^{-a_k}}{2}$$

$$\le \prod_{k=0}^{n} e^{a_k^2} = e^{\sum a_k^2} = e$$

$$\therefore \quad Ef^{2m} \le e(2m)!$$

For LHS: Case $p \geq 2$. Then

$$1 = \left(\sum a_k^2\right)^{1/2} = \|f\|_2 \leq \|f\|_p$$

Case $p < 2$.

$$1 = \|f\|_2^2 = E|f|^2 = E|f|^{p/2}|f|^{\frac{4-p}{2}}$$

$$\leq \left(E|f|^p\right)^{1/2}\left(E|f|^{4-p}\right)^{1/2}$$

$$\therefore \quad 1 \leq E|f|^p \underbrace{E|f|^{4-p}}_{\leq C_{4-p}^{4-p}}$$

$$\therefore \quad \frac{1}{C_{4-p}^{4-p}} \leq \|f\|_p^p$$

▨

Let $S(f) = \left(\sum_{k=1}^{\infty} d_k^2\right)^{1/2}$ $\quad$ (d_k) mart. diff. seq. of f

(1) $\|f\|_1 < \infty \Rightarrow S(f) < \infty$ a.e.

(2) $\lambda P(S(f) > \lambda) \leq c\|f\|_1$ $\quad$ (Note (2) $\Rightarrow$ (1) Let $\lambda \to \infty$)

(3) $c_p\|S(f)\|_p \leq \|f\|_p \leq C_p\|S(f)\|_p$ $\quad$ $1 < p < \infty$

Recall results for martingale transform $v^* \le 1$

(1') $\|f\|_1 < \infty \Rightarrow g$ converges a.e.

(2') $\lambda P(g^* > \lambda) \le c\|f\|_1$

(3') $\|g\|_p \le c_p \|f\|_p \quad (1<p<\infty)$

(3') $\Rightarrow$ (3). If $0 \le t \le 1$

$$(3') \Rightarrow \int_\Omega \Big| \sum_{k=1}^n \underbrace{r_k(t)}_{v_k} d_k \Big|^p dP \le C_p^p \int_\Omega \Big| \sum_{k=1}^n d_k \Big|^p dP$$

$$\therefore \int_0^1 \int_\Omega \Big| \sum_{k=1}^n r_k(t) d_k \Big|^p dP\,dt \le C_p^p \int_0^1 \int_\Omega \Big| \sum_{k=1}^n d_k \Big|^p dP = C_p^p \|f_n\|_p^p$$

$$\therefore \int_\Omega \int_0^1 \Big| \sum_{k=1}^n r_k(t) d_k \Big|^p dt\,dP \le C_p^p \|f_n\|_p^p$$

$\rotatebox{-90}{$\geq$}$ (Khintchin)

$$c_p \int_\Omega \Big(\sum_{k=1}^n d_k^2 \Big)^{1/2} dP$$

$\therefore c_p \|S_n(f)\|_p^p \le C_p^p \|f_n\|_p^p$ Let $n \to \infty$

$(3) \Rightarrow (3')$

$$S(g) = \Big(\sum_{k=1}^{\infty} v_k^2 d_k^2 \Big)^{1/2} \leq S(f)$$

$$\therefore \quad c_p \|g\|_p \leq \| S(g) \|_p \leq \| S(f) \|_p \leq C_p \|f\|_p$$

Suppose $a_0, a_1, a_2, \ldots$ belong to Banach space $\mathfrak{X}$. For what $\mathfrak{X}$'s does Khintchine's inequality still hold? If the inequality does hold (for all choices of $a_0, a_1, a_2, \ldots$) then $\mathfrak{X}$ is essentially a Hilbert space (Kwapien)

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martingale

Control problem: Given f, find a sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots)$ $\varepsilon_k = \pm 1$ such that $g^* \geq b$ a.e. $(b \in [0, \infty])$

For double-or-nothing martingale with $\varepsilon = (1, -1, -1, -1, \dots)$, one gets $g^* \geq 2$

Dual problem: $\min \|f\|_1$ subject to $g^* > 1$ a.e.

Main problem: $\varepsilon = (1, \varepsilon_2, \varepsilon_3, \dots)$
$f = (f_1, f_2, f_3, \dots)$ f_n simple function into $\mathfrak{X}$

Let $x, y \in \mathfrak{X}$. $M(x,y) :=$ all $\mathfrak{X}$-valued martingale f of simple functions starting at x (i.e. $f_1 = x$ a.e.) such that for some sequence $(1, \varepsilon_1, \varepsilon_2, \dots)$, $\varepsilon_k = \pm 1$, the transform g of f by this sequence satisfies

$$P\left(\|g_n - y\| \geq 1 \text{ for some } n \geq 1 \right) = 1$$

$$\text{i.e. } P\left(g_n \in B(y,1) \ \forall n \right) = 0$$

Let $\psi(x,y) = \inf \{ \|f\|_1 : f \in M(x,y) \}$.

THEOREM: (1) $\psi(x,y) = \psi(x, 2x-y)$

(2) $\psi(\cdot, y)$ is convex for each fixed y

(3) $\psi(x,y) \leq \|x\|$ if $\|y\| \geq 1$

(4) ψ is the largest function $\mathfrak{X} \times \mathfrak{X} \to \mathbb{R}$ satisfying (1), (2), (3)

Proof. (1) Show $M(x,y) = M(x, 2x-y)$. Let $\mathfrak{f} \in M(x,y)$ and suppose

$$g_n = x + \varepsilon_2 d_2 + \varepsilon_3 d_3 + \ldots + \varepsilon_n d_n$$

is the assoc. transform. Define

$$G_n = x - \varepsilon_2 d_2 - \varepsilon_3 d_3 - \ldots - \varepsilon_n d_n$$

so G is a transform of $\mathfrak{f}$ by $(1, -\varepsilon_2, -\varepsilon_3, \ldots)$

$$\|G_n - (2x-y)\| = \|-x + y - \varepsilon_2 d_2 - \varepsilon_3 d_3 - \ldots - \varepsilon_n d_n\|$$

$$= \|x + \varepsilon_2 d_2 + \ldots + \varepsilon_n d_n - y\|$$

$$= \|g_n - y\| \geq 1 \text{ for some } n$$

Hence $\mathfrak{f} \in M(x, 2x-y)$. Hence $M(x,y) \subset M(x, 2x-y)$

$$\therefore M(x, 2x-y) \subset M(x, 2x-(2x-y)) = M(x,y)$$

(2) Choose $x_1, x_2 \in \mathfrak{X}$, $\alpha_1, \alpha_2 > 0$ $\alpha_1 + \alpha_2 = 1$, $\alpha_1 x_1 + \alpha_2 x_2 = x$
Show $\psi(x,y) \leq \alpha_1 \psi(x_1,y) + \alpha_2 \psi(x_2,y)$

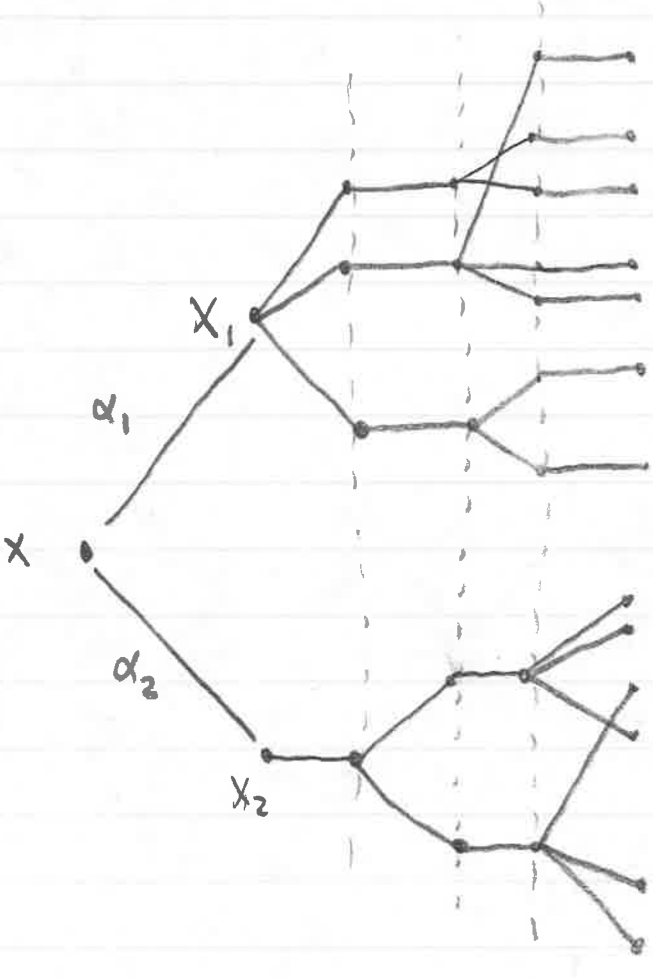
Let $\delta_1, \delta_2 > 0$. Let $\mathfrak{f}_j = (\mathfrak{f}_{j1}, \mathfrak{f}_{j2}, \dots)$ satisfy

$$\psi(x_j,y) \leq \|\mathfrak{f}_j\|_1 \leq \psi(x_j,y) + \delta_j$$

$j = 1,2$. Consider

$$\hat{\mathfrak{f}}_1 = (\mathfrak{f}_{11}, \mathfrak{f}_{11}, \mathfrak{f}_{12}, \mathfrak{f}_{12}, \mathfrak{f}_{13}, \mathfrak{f}_{13}, \dots)$$

$$\hat{\mathfrak{f}}_2 = (0, \mathfrak{f}_{21}, \mathfrak{f}_{21}, \mathfrak{f}_{22}, \mathfrak{f}_{22}, \mathfrak{f}_{23}, \mathfrak{f}_{23}, \dots)$$



F

Define G similarly as shifted version of g_1, g_2. If

$$f_j \sim \varepsilon_j = (1, \varepsilon_{j2}, \varepsilon_{j3}, \ldots)$$

the ε for G is

$$\varepsilon = (1, 1, \varepsilon_{12}, \varepsilon_{22}, \varepsilon_{13}, \varepsilon_{23}, \ldots)$$

Then $P(\|G_n - y\| \geq 1 \text{ for some } n) = 1$. Thus $F \in M(x,y)$

$$E\|F_{2n}\| \leq \alpha_1 E\|f_{1n}\| + \alpha_2 E\|f_{2n}\|$$

$$\psi(x,y) \leq \|F\|_1 \leq \alpha_1 \|f_1\|_1 + \alpha_2 \|f_2\|_1$$

$$\leq \alpha_1 \psi(x_1, y) + \alpha_1 \delta_1 + \alpha_2 \psi(x_2, y) + \alpha_2 \delta_2$$

Let $\delta_1, \delta_2 \to 0$ to see that $\psi(x,y) \leq \alpha_1 \psi(x_1,y) + \alpha_2 \psi(x_2, y)$.

(3) Case 1: $\|x-y\| \geq 1$. Let $f = (x, x, x, \ldots) = g$ (so $\varepsilon_k = +1 \; \forall k$)
Then $\|g_n - y\| = \|x-y\| \geq 1$ for all n, so $f \in M(x,y)$

$$\therefore \psi(x,y) \leq \|f\|_1 = \|x\| \qquad \left[\text{Corollary: } \|y\| \geq 1 \Rightarrow \psi(0,y) = 0\right]$$

Case 2: $\|x-y\| < 1$.

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Proof of (3) cont. Let $x \neq 0$ and choose $\lambda > 1$ so that $\|\lambda x - y\| > 1$

By convexity

$$\psi(x,y) \leq \left(1 - \frac{1}{\lambda}\right) \cancelto{0}{\psi(0,y)} + \frac{1}{\lambda}\psi(\lambda x, y)$$

$$\leq \frac{1}{\lambda}\|\lambda x\| = \|x\|$$

$\uparrow$ by the 1st part since $\|\lambda x - y\| > 1$

Proof of (4). Let $\varphi : \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathbb{R}$ satisfy (1), (2), (3). Want to show $\varphi(x,y) \leq \psi(x,y)$. Define

$$M_n^+(x,y) = \{ f \in M(x,y) : \text{for some } (1, 1, \varepsilon_3, \varepsilon_4, \dots), \; P(|g_k - y| \geq 1 \text{ for some } k \leq n) = 1 \}$$

where $g_k = x + d_2 + \varepsilon_3 d_3 + \dots + \varepsilon_k d_k$. Let

$$\psi_n^+(x,y) = \inf\{ \|f_n\|_1 : f \in M_n^+(x,y) \}$$

Define $M_n^-(x,y)$ similarly with $(1, -1, \varepsilon_3, \varepsilon_4, \dots)$ and let $\psi_n^-(x,y) = \inf\{\|f_n\|_1 : f \in M_n^-(x,y)\}$. Finally, let

$$M_n(x,y) = M_n^+(x,y) \cup M_n^-(x,y)$$

$$\psi_n(x,y) = \inf\{\|f_n\|_1 : f \in M_n(x,y)\} = \psi_n^+(x,y) \wedge \psi_n^-(x,y)$$

To show: $\varphi(x,y) \le \psi_n(x,y) \quad n \geq 2$

$n=2$: We need $\varphi(x,y) \le \|x\|$ if $\|x-y\| \geq 1$. If $\|y\| \geq 1$, then (3) applies. If $\|y\| \le 1$, then

$$\|2x-y\| = \|2x-2y+y\| \geq 2\|x-y\| - \|y\|$$

$$\geq 2-1 = 1$$

so by (1) and (3),

$$\varphi(x,y) = \varphi(x, 2x-y) \le \|x\|.$$

Now let $f \in M_2^+(x,y)$. Note $f_2 = g_2$ since $\varepsilon_1 = \varepsilon_2 = 1$, so either $\|x-y\| \geq 1$ or $\|f_2 - y\| \geq 1$ a.e.

In this case $\varphi(f_2, y) \le \|f_2(\cdot)\|$ a.e.

$$\begin{array}{c} \therefore \quad E\,\varphi(f_2, y) \le E\,\|f_2\| = \|f_2\|_1 \\ \rotatebox{90}{$\le$} \\ \varphi(E f_2, y) \\ \rotatebox{90}{$=$} \\ \varphi(x,y) \end{array}$$

Jensen's ineq. for simple fct.

$$\therefore \quad \varphi(x,y) \le \psi_2^+(x,y)$$

Note: by same proof as for (1), we have $M_n^-(x,y) = M_n^+(x, 2x-y)$

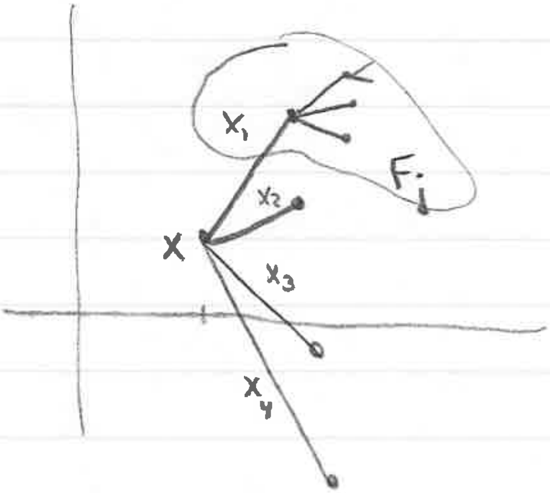
Thus $\psi_n^-(x,y) = \psi_n^+(x, 2x-y) \geq \varphi(x, 2x-y) = \varphi(x,y)$. Hence

$$\varphi(x,y) \leq \psi_n(x,y) \quad \text{for } n=2$$

Now suppose $\varphi(x,y) \leq \psi_n(x,y)$ for some $n \geq 2$. Will show it holds for n+1. Let $f \in M_n^+(x,y)$

$$\|f_{n+1}\|_1 = \int_\Omega \|x+d_2+d_3+\ldots+d_{n+1}\| dP$$

Let $P(\underbrace{f_2 = x_j}_{\Omega_j}) = \alpha_j > 0$, $\sum_{j=1}^{m} \alpha_j = 1$



$$= \sum_{j=1}^{m} \int_{\Omega_j} \frac{\overbrace{\|x_j+d_3+\ldots+d_{n+1}\|}^{F_{jn}}}{P(\Omega_j)} \alpha_j \, dP$$

$$= \sum_{j=1}^{m} \alpha_j \int_\Omega \|F_{jn}\| dP_j$$

Define $P_j(\cdot) = P(\cdot \mid \Omega_j)$, i.e. $P_j(A) = \dfrac{P(A \cap \Omega_j)}{P(\Omega_j)}$

$F_j = (F_{j1}, F_{j2}, \ldots)$ martingale on $(\Omega, \mathcal{A}, P_j)$

$F_j \in M_n(x_j, y)$

$$\therefore \sum_{j=1}^{m} \alpha_j \int_{\Omega} \|F_{jn}\| dP_j$$

$$\geq \sum_{j=1}^{m} \alpha_j \psi_m (x_j, y)$$

$$\geq \sum_{j=1}^{m} \alpha_j \varphi(x_j, y)$$

$$\geq \varphi\left(\sum_{j=1}^{m} \alpha_j x_j, y\right) = \varphi(x,y)$$

Hence $\varphi(x,y) \leq \psi^{+}_{n+1}(x,y)$

Claim: $\varphi(x,y) \leq \psi_n(x,y) \quad \forall n \geq 2 \implies \varphi(x,y) \leq \psi(x,y)$

Let $f \in M(x,y)$. To show $\varphi(x,y) \leq \|f\|_1$. Let $\delta > 0$.
For some large n,

$$P\big(\underbrace{|g_k - y| \geq 1 \text{ for some } k \leq n}_{\text{good set}}\big) > 1 - \delta$$

Let $F_k = f_k \quad 1 \leq k \leq n$ $\qquad F_{n+1} = \begin{cases} f_n & \text{on good values} \\ f_n + D_{n+1} & \text{on bad} \end{cases}$ $\quad \|D_{n+1}\| = 2$ on bad

$F \in M_{n+1}(x,y)$ $\qquad D_n = \begin{cases} +z & \text{prob } 1/2 \\ -z & \text{prob } 1/2 \end{cases}$ $\quad \|z\| = 2$

$$\varphi(x,y) \leq \psi_{n+1}(x,y) \leq \|F_{n+1}\|_1 \leq \|f_n\|_1 + \|D_{n+1}\|_1$$

$$\leq \|f\|_1 + 2\delta + (1-\delta)\cdot 0$$

Let $\delta \to 0$ to get desired result. ▨

THEOREM: $\psi(0,0) > 0 \iff \mathfrak{X} \in MT$

Proof $(\Leftarrow)$: $\mathfrak{X} \in MT$ says $\lambda P(g^* > \lambda) \leq c\|f\|_1$

Take $f \in M(0,0)$. If $\lambda < 1$,

$$g^* > \lambda \text{ a.c.} \Rightarrow \frac{\lambda}{c} \leq \|f\|_1$$

$$\Rightarrow \frac{\lambda}{c} \leq \psi(0,0) \quad (\forall \lambda < 1)$$

$$\Rightarrow \frac{1}{c} \leq \psi(0,0)$$

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THEOREM: $\mathfrak{X} \in MT \iff \exists$ symmetric biconvex function $\zeta : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}$ s.t. $\zeta(0,0) > 0$ and $\underset{(*)}{\zeta(x,y) \le \|x+y\|}$ if $\|x\| \le 1 \le \|y\|$

Example: ① $\mathfrak{X} = \mathbb{R}$ $\quad \zeta(x,y) = 1 + xy$

$$(1+xy)^2 = 1 + 2xy + x^2y^2 = (x+y)^2 + (1-x^2)\underset{\uparrow \text{negative}}{(1-y^2)}$$

$$\le (x+y)^2$$

② $\mathfrak{X} = \mathcal{H}$ (Hilbert space) $\quad \underset{\substack{\uparrow \\ \text{real inner product}}}{(x,y)} = \mathrm{Re}\underset{\uparrow \text{complex inner product}}{\langle x,y \rangle}$

$$\zeta(x,y) = 1 + (x,y)$$

$$\zeta(x,y) = \left(1 + 2(x,y) + \|x\|^2\|y\|^2\right)^{1/2}$$

May always go from a ζ satisfying (*) to $\zeta_1(x,y) \le \|x+y\|$ if $\|y\| \ge 1$

$$\zeta_1(x,y) = \left\{ \begin{array}{ll} \zeta(x,y) \vee \|x+y\| & \|y\| < 1 \\ \|x+y\| & \|y\| \ge 1 \end{array} \right\} \text{convex in } x$$

Let

$$\zeta_2(x,y) = \begin{cases} \zeta(x,y) \vee \|x+y\| & \text{if } \|x\| \vee \|y\| < 1 \\ \|x+y\| & \text{if } \|x\| \vee \|y\| \geq 1 \end{cases} \Bigg\} \text{symmetric}$$

Note: $\zeta_1 = \zeta_2$! so ζ_1 is symmetric and biconvex

Clear if $\|y\| \geq 1$ or $\|x\| < 1$, so only look at $\|y\| < 1 \leq \|x\|$

$$\zeta_1(x,y) = \zeta(x,y) \vee \|x+y\| = \zeta(y,x) \vee \|x+y\| = \|x+y\| = \zeta_2(x,y)$$

Assume from now on that $\zeta(x,y) \leq \|x+y\|$ if $\|y\| \geq 1$.

Let

$$\varphi(x,y) = \tfrac{1}{2}\zeta(2x-y, y)$$

Then φ satisfies (1), (2), (3) of penultimate theorem

(1) $\varphi(x,y) = \varphi(x, 2x-y)$

(2) $\varphi(\cdot, y)$ convex

(3) $\varphi(x,y) \leq \|x\|$ if $\|y\| \geq 1$

$$\varphi(x,2x-y) = \frac{1}{2}\left(\zeta(2x-(2x-y),2x-y)\right)$$

$$= \frac{1}{2}\zeta(y,2x-y) = \frac{1}{2}\zeta(2x-y,y) = \varphi(x,y)$$

Convexity clear

$$\|y\| \leq 1 \Rightarrow \varphi(x,y) \leq \frac{1}{2}\|2x-y+y\| = \|x\|$$

Conversely, if you have a φ, then let

$$\zeta(x,y) = 2\varphi\left(\frac{x+y}{2}, y\right)$$

$$\psi(x,y) = \inf\{\|\xi\|_1 : \xi \in M(x,y)\} = \max_{\varphi} \varphi(x,y)$$

$$= \frac{1}{2}\max_{\zeta} \zeta(2x-y,y)$$

$$\therefore \psi(0,0) = \frac{1}{2}\max_{\zeta}\zeta(0,0) > 0$$

(Recall $\psi(0,0)>0 \Leftrightarrow \mathcal{X} \in MT$)

Example: $\mathcal{X} = \mathbb{R}$ $\quad \psi(x,y) = \begin{cases} \frac{1+(2x-y)y}{2} \vee |x| & |y|<1 \\ |x| & |y| \geq 1 \end{cases}$

$$\max_{\zeta}\zeta(x,y) = \begin{cases} 1+xy & |x|\vee|y|<1 \\ |x+y| & |x|\vee|y| \geq 1 \end{cases}$$

3/9 MARTINGALES

Suppose (Ω, a, P) probability space. $X = (X_t)_{0 \le t < \infty}$ is a family of measurable functions from Ω into $\mathbb{R}^n$ s.t.

(i) $P(X_t \in B) = \int_B \frac{e^{-\|x\|^2/2t}}{(2\pi t)^{n/2}} dx$ $\forall$ Borel $B \subset \mathbb{R}^n$, $\forall t > 0$

$\|x\|^2 = x_1^2 + \dots + x_n^2$

(Gaussian dist.)

$\uparrow$ Gauss kernal

(ii) If $t_0 \le t_1 \le \dots \le t_k$ $(k \ge 2)$, then

(indep. increments) $X_{t_1} - X_{t_0}, \dots, X_{t_k} - X_{t_{k-1}}$

are independent

(iii) If $\omega \in \Omega$, the map $t \longmapsto X_t(\omega)$ on $[0, \infty)$ into $\mathbb{R}^n$ is continuous

(cont. paths)

(iv) If $\omega \in \Omega$, then $X_0(\omega) = 0$

(starts at 0)

Then X is called a standard Brownian motion in $\mathbb{R}^n$ starting at 0.

THEOREM (Wiener, 1923): Brownian motion exists

Canonical Prob. Space

Ω = all continuous $\omega : [0,\infty) \longrightarrow \mathbb{R}^n$ with $\omega(0) = 0$
$\mathcal{A}$ = the smallest σ-field containing sets of the form $\{X_t \in B\}$

$$X_t(\omega) := \omega(t)$$

P = Wiener measure

Brownian motion is rotationally invariant - Suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is linear and $\|Tx\| = \|x\|$. Let $Y_t = TX_t$, $t \geq 0$.

Claim: $Y = (Y_t)_{0 \leq t < \infty}$ is a standard Brownian motion starting at 0

(i) $P(Y_t \in B) = P(X_t \in T^{-1}B)$

$$= \int_{T^{-1}B} \frac{e^{-\|x\|^2/2t}}{(2\pi t)^{n/2}}\, dx = \int_{B} \frac{e^{-\|Tx\|^2/2t}}{(2\pi t)^{n/2}}\, dx$$

$(m(T^{-1}B) = m(B))$

$$= \int_{B} \frac{e^{-\|x\|^2/2t}}{(2\pi t)^{n/2}}\, dx$$

(ii) - (iv) done similarly

Consequence : Let $\mu(\omega) = \inf\{t : \|X_t(\omega)\| = 1\}$
$(\inf \phi = +\infty)$

$$X_\mu(\omega) := X_{\mu(\omega)}(\omega) \quad \text{if } \mu(\omega) < \infty$$

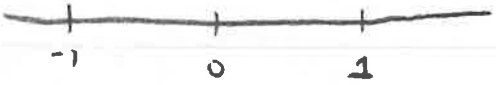
(will show later $\mu(\omega) < \infty$ a.e. and X_μ is measurable)

$$P(\mu = \infty) \leq P(\|X_t\| < 1) = \int_{\|x\|<1} \frac{e^{-\|x\|^2/2t}}{(2\pi t)^{n/2}} dx$$

$$= \frac{1}{(2\pi t)^{n/2}} \int_{\|x\|<1} 1 \, dx \longrightarrow 0 \text{ as } t \to \infty$$

THEOREM: X_μ has unif. dist. on the unit sphere $\|x\| = 1$ in $\mathbb{R}^n$

Proof : $n = 1$



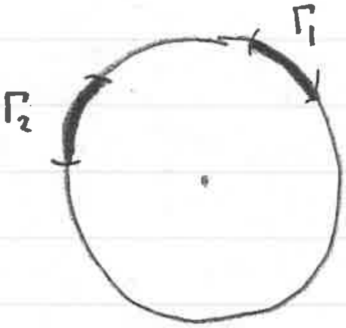
Consider X , $Y = -X$

$$P(X_\mu = 1) \overset{\uparrow}{=} P(Y_\mu = 1) = P(-X_\mu = 1) = P(X_\mu = -1)$$

invariance

$\therefore$ $P(X_\mu = 1) = 1/2 = P(X_\mu = -1)$

$n=2$:



arcs have same length

Let $Y = TX$ where T is chosen so that T maps Γ_1 onto Γ_2

$$P(Y_\mu \in \Gamma_2) = P(X_\mu \in \Gamma_1)$$

$$\parallel$$

$$P(X_\mu \in \Gamma_2) \quad \text{(by invariance)}$$

$n \geq 3$: similar proof

▨

$$X_t = (X_{t,1}, X_{t,2}, \dots, X_{t,n})$$

Claim: Each component $(X_{t,i})_{0 \leq t < \infty}$ is Brownian motion

(i) Write $B = B_1 \times \mathbb{R} \times \dots \times \mathbb{R}$

$$P(X_{t,1} \in B_1) = P(X_t \in B)$$

$$= \int_{B_1} \frac{e^{-|x_1|^2/2t}}{\sqrt{2\pi t}} dx_1 \cdot \overset{=1}{\int_{\mathbb{R}} \frac{e^{-|x_2|^2/2t}}{\sqrt{2\pi t}}} \cdot \dots \cdot \overset{=1}{\int_{\mathbb{R}} \frac{e^{-|x_n|^2/2t}}{\sqrt{2\pi t}} dx_n}$$

(ii) - (iv) similar

Note $(X_{t,1})_{0 \le t < \infty}$, $(X_{t,2})_{0 \le t < \infty}$, ... indep

Can also go backwards

Thus enough to prove Wiener's th^m for $n=1$ and for $0 \le t \le 1$

Let (Ω, a, P) be a prob. space on which is defined an indep. sequence $Z_1, Z_2, \ldots$ of real-valued measurable functions satisfying

$$P(Z_k \in B) = \int_B \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \qquad (\eta(0,1))$$

e.g. $\Omega = [0,1)$ $P =$ Lebesgue

$\omega = .b_1(\omega) b_2(\omega) \ldots$ binary expansion

3/11 MARTINGALES

$$\mathcal{U}(\omega) = \sum_{k=1}^{\infty} \frac{b_k(\omega)}{2^k} \qquad \text{unf. dist. on } [0,1)$$

Put

$$U_1(\omega) = \sum_{k=1}^{\infty} \frac{b_{2^k}(\omega)}{2^k}$$

$$U_2(\omega) = \sum_{k=1}^{\infty} \frac{b_{3^k}(\omega)}{2^k}$$

$$U_3(\omega) = \sum_{k=1}^{\infty} \frac{b_{5^k}(\omega)}{2^k}$$

$$\vdots$$

(use primes for subscript of b

Each of the U_i are independent and unf. on $[0,1]$

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy$$

$$Z_k := F^{-1}(\mathcal{U}_k)$$

$$P(Z_k \leq x) = P(U_k \leq F(x)) = F(x)$$

$\therefore$ $Z_1, Z_2, \ldots$ indep. $\eta(0,1)$

Facts: (i) If $a_1, a_2, \ldots$ are real numbers with $\sum' a_k^2 < \infty$, then the series $\sum' a_k Z_k$ converges a.e. and in L_2 to a $\eta(0, \sum a_k^2)$ random var.

Proof. $\xi_n = \sum_{k=1}^{n} a_k Z_k$ martingale since

$$E(\xi_{n+1} | Z_1, \ldots, Z_n) = \sum_{k=1}^{n} a_k Z_k + a_{n+1} E(Z_{n+1} | Z_1, \ldots, Z_n)$$

$$= \xi_n + E(Z_{n+1}) a_{n+1} \quad \text{a.e.}$$

$$= \xi_n \quad \text{a.e.}$$

$$E\xi_n^2 = E \sum_{k=1}^{n} a_k^2 Z_k^2 = \sum_{k=1}^{n} a_k^2 \quad \therefore \ \xi \ L_2\text{-bdd}$$

(ii) If $\underset{\substack{\shortparallel \\ X}}{\sum_{j=1}^{\infty} a_j Z_j}$ and $\underset{\substack{\shortparallel \\ Y}}{\sum_{k=1}^{\infty} b_k Z_k}$ are orthogonal, i.e.

$$E(XY) = 0$$

then X and Y are independent

Let $\mathcal{M} = \left\{ \sum_{k=1}^{\infty} a_k Z_k : (a_k) \in \ell_2 \right\} \subset L_2(\Omega, \mathcal{a}, P)$

$$\left\| \sum_{k=1}^{\infty} a_k Z_k \right\| = \|(a_k)\|_2$$

The map $(a_k) \longrightarrow \sum a_k Z_k$ is an isometry from ℓ_2 onto M

Let $\varphi_1, \varphi_2, \ldots$ be a complete orthonormal seq on $L_2[0,1]$

$$f \in L_2[0,1] \Rightarrow \|f\|_2^2 = \sum_{k=1}^{\infty} (f, \varphi_k)^2$$

$$f = \sum_{k=1}^{\infty} (f, \varphi_k) \varphi_k$$

LEMMA: The series $X_t = \sum_{k=1}^{\infty} Z_k(\omega) \int_0^t \varphi_k(x)\,dx$ converges uniformly in $t \in [0,1]$ for almost all ω

Construct Brownian motion: Let $\Omega_1 = \{\omega \in \Omega :$ the series converges unif on $[0,1]\}$. $P(\Omega_1) = 1$. Consider $X_t(\cdot)$ defined on Ω_1

$X = (X_t)_{0 \le t \le 1}$ is standard Brownian motion

(i) Gaussian. Let $0 \le s < t \le 1$

$$X_t - X_s = \sum_{k=1}^{\infty} Z_k(\cdot) \int_s^t \varphi_k$$

$$= \sum_{k=1}^{\infty} a_k Z_k(\cdot) \qquad a_k = \int_s^t \varphi_k = (I_{[s,t]}, \varphi_k)$$

$$\sum a_k^2 = \sum (I_{[s,t]}, \varphi_k)^2 = \| I_{[s,t]} \|_2^2$$

$$= t-s$$

Thus $X_t - X_s$ is normal $(0, t-s)$. In particular, if $s=0$, $X_t \sim \eta(0,t)$

(ii) independence of increments. $s_1 < t_1 \le s_2 < t_2 \le \dots$

$$(X_{t_1} - X_{s_1}, X_{t_2} - X_{s_2}) = (\sum a_k Z_k, \sum b_k Z_k)$$

$$= \sum a_k b_k$$

$$= (I_{[s_1,t_1]}, I_{[s_2,t_2})$$

$$= 0$$

(iii) continuous paths. Follows from uniform convergence

(iv) $X_0 = 0$ ✓

Note $X_{(\cdot)}(w)$ is a $C[0,1]$-valued martingale for each w

Define $(X_t^{(j)})_{0 \leq t \leq 1}$ indep. Brownian motion

(copies of $(X_t)_{0 \leq t \leq 1}$)

$$X_t(w) := X_t^{(1)} \quad 0 \leq t \leq 1$$

$$= X_1^{(1)} + X_{t-1}^{(2)} \quad 1 \leq t \leq 2$$

$$= X_1^{(1)} + X_1^{(2)} + X_{t-2}^{(3)} \quad 2 \leq t \leq 3$$

$$\vdots$$

Defines Brownian motion on $[0,\infty)$

3/13 MARTINGALES

Haar Orthonormal system

$$h_0(t) = 1 \qquad 0 \le t < 1$$

$$h_1(t) = \begin{cases} +1 & 0 \le t < 1/2 \\ -1 & 1/2 \le t < 1 \end{cases}$$

$$h_2(t) = \begin{cases} 2^{1/2} & 0 \le t < 1/4 \\ -2^{1/2} & 1/4 \le t < 1/2 \\ 0 & 1/2 \le t < 1 \end{cases}$$

$$h_3(t) = \begin{cases} 0 & 0 \le t < 1/2 \\ 2^{1/2} & 1/2 \le t < 3/4 \\ -2^{1/2} & 3/4 \le t < 1 \end{cases}$$

$$\vdots$$

$$h_{2^n+k}(t) = \begin{cases} 2^{n/2} & \text{approp. interval} \\ -2^{n/2} & \\ 0 & \end{cases}$$

LEMMA: This is complete in $L_2[0,1]$

Proof. Let $\mathcal{a}_n = \sigma\{h_0, h_1, \dots, h_n\}$ generated by a partition of $n+1$ sets. Suppose that f_n is $\mathcal{a}_n$-measurable

Claim: $f_n = \sum_{k=0}^{n} a_k h_k \qquad a_k = (f_k, h_k)$

Follows since only need to solve

$$f_n(t) = \sum_{k=0}^{n} a_k h_k(t)$$

for n+1 values of t ∴ n+1 equations in n+1 unknowns. $a_k = (f_k, h_k)$ by orthonormality. Note (f_n) is a martingale

Now let $f \in L_2[0,1]$. Then

$$E(f|\mathcal{A}_n) = \sum_{k=0}^{n} (E(f|\mathcal{A}_n), h_k)\, h_k$$

$$\uparrow = E(h_k E(f|\mathcal{A}_n)) = E(E(h_k f|\mathcal{A}_n)) = E\, h_k f = (h_k, f) = a_k$$

$$= \sum_{k=0}^{n} a_k h_k$$

Let $n \to \infty$. By continuity thm

$$f \underset{\text{a.e.}}{=} E(f| \bigvee_{n=1}^{\infty} \mathcal{A}_n) = \sum_{k=0}^{\infty} a_k h_k \quad (a_k = (h_k, f))$$

$$\|f\|_2^2 = \lim_n \|E(f|\mathcal{A}_n)\|_2^2 = \lim_n \sum_{k=0}^{n} a_k^2 = \sum_{k=0}^{\infty} a_k^2$$

▨

LEMMA: $Z_0, Z_1, \ldots$ ind $\eta(0,1)$. Almost surely,

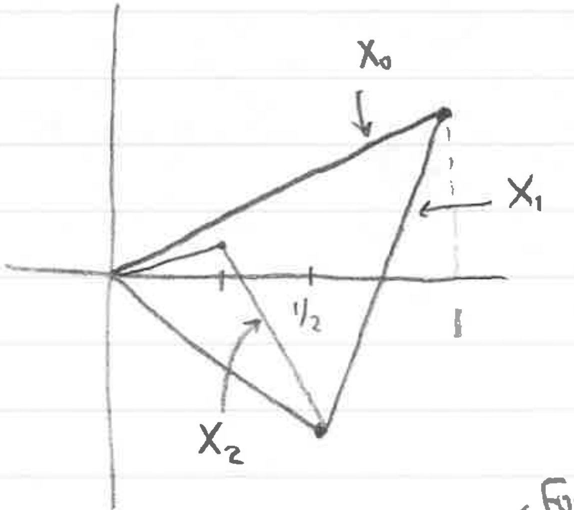
$$\sum_{k=0}^{\infty} Z_k(\omega) \int_0^t h_k(x)\,dx$$

converges uniformly on $[0,1]$.

Proof. $X_n(t,\omega) := \sum_{k=0}^{n} Z_k(\omega) \int_0^t h_k$

$$X_0(t,\omega) = Z_0(\omega)\, t \quad \text{(linear)}$$

$$X_1(t,\cdot) = \text{linear} + \text{triangular}$$



For fixed t, non-zero in only one term

$$\text{Call } Y_j = \sup_{0 \le t \le 1} \sum_{2^j \le k < 2^{j+1}} |X_k(t) - X_{k-1}(t)|$$

$$|X_k(t) - X_{k-1}(t)| \le |Z_k| \int_0^t |h_k|$$

$$\le |Z_k|\, 2^{j/2} \cdot \frac{1}{2^j} = |Z_k| \frac{1}{2^{j/2}}$$

$$\therefore Y_j \le \frac{1}{2^{j/2}} \sup_{2^j \le k < 2^{j+1}} |Z_k|$$

Claim: $\sum_{j=0}^{\infty} Y_j < \infty$ a.e.

$$\lambda > 0 \Rightarrow \int_\lambda^\infty e^{-y^2/2}\, dy \le \int_\lambda^\infty \frac{y}{\lambda} e^{-y^2/2}\, dy$$

$$= \frac{1}{\lambda} e^{-\lambda^2/2}$$

$$\therefore P(Y_j > \lambda_j) = P\Big(\sup_{2^j \le k < 2^{j+1}} |Z_k| > 2^{j/2} \lambda_j\Big)$$

$$\le \sum_{k=2^j}^{2^{j+1}-1} P(|Z_k| > 2^{j/2} \lambda_j)$$

$$= 2^j \cdot 2 \cdot \frac{1}{2^{j/2} \lambda_j} e^{-2^j \lambda_j^2/2}$$

(If $\lambda_j = 1/j^2$)

$$\le c e^{j/2} e^{-2^j/2j^4} \le c' e^{-j} \Rightarrow \sum P(Y_j > \lambda_j) < \infty$$

By Borel Cantelli
$P(Y_j > \lambda_j$ happens only finite # times$) = 1$

3/30 MARTINGALES

Interpolation

① Riesz-Thorin $\|Tf\|_{p_i} \le c_{p_i} \|f\|_{p_i}$ $\quad 1 \le p_1 < p_2 \le \infty$

$$\Rightarrow \|Tf\|_p \le c_p \|f\|_p \qquad p_1 < p < p_2$$

② Marcinkiewicz $\lambda^{p_i} P(|Tf| > \lambda) \le c_{p_i} \|f\|_{p_i}^{p_i}$ $\quad 1 \le p_1 < p_2 < \infty$

$$\Rightarrow \lambda^p P(|Tf| > \lambda) \le c_p \|f\|_p^p \qquad p_1 < p < p_2$$

Let

$$g_n = \sum_{k=1}^{n} v_k d_k \qquad v^* \le 1$$

Then

$$\|g_n\|_2^2 = \sum_{k=1}^{n} E v_k^2 d_k^2 \le \|f_n\|_2^2$$

Also $\lambda P(g^* > \lambda) \le 2\|f\|_1$. Define

$$\forall f_\infty \in L^1(\Omega, \mathcal{A}_\infty, P),\ T_n f_\infty = v_1 E(f_\infty | \mathcal{A}_1) + \sum_{k=2}^{n} v_k \left(E(f_\infty | \mathcal{A}_k) - E(f_\infty | \mathcal{A}_{k-1}) \right)$$

Note $T_n f_\infty = g_n$

↑ limit of f_n

Now we have

$$\lambda^2 P(|T_n f_\infty| > \lambda) \le \|f_\infty\|_2^2$$

$$\lambda P(|T_n f_\infty| > \lambda) \le \lambda P(g^* > \lambda) \le 2\|f\|_1 = 2\|f_\infty\|_1$$

$$\therefore \quad \|T_n f_\infty\|_p \le c_p \|f_\infty\|_p \qquad 1 < p < 2$$

$$\therefore \quad \|g_n\|_p \le c_p \|f_n\|_p$$

$$\therefore \quad \Big\|\sum_{k=1}^{n} \pm d_k\Big\|_p \le c_p \|f_n\|_p \qquad 1<p<2 \qquad (v_k = -1 \text{ or } +1)$$

(See Burkholder Annals of Math. Stat. 1966)

Extrapolation

Know $\|S(f)\|_p \approx \|f^*\|_p \qquad 1<p<\infty$

$\uparrow$ means $c_1 \|f^*\|_p \le \|S(f)\|_p \le c_2 \|f^*\|_p$

Actually holds on range $0<p<\infty$ for regular martingales, i.e.

$$f_n = \sum_{k=1}^{n} v_k d_k$$

where $E(d_k^2 \mid a_{k-1}) = 1$ a.e. and $E(|d_k| \mid a_{k-1}) \ge \alpha > 0$

Consider special martingale

$$f_n = \sum_{k=1}^{n} d_k$$

with $|d_k|$ a_{k-1}-measurable $\left(\text{e.g. } d_k = \underset{\uparrow \text{ constant}}{a_k}\, \underset{\uparrow \text{ Haar function}}{h_k} \right)$

<u>**THEOREM**</u>: For this martingale

$$\|S(f)\|_p \approx \|f^*\|_p \quad 0<p<\infty$$

Proof. Step 1. $\|S(f)\|_2 = \|f\|_2$ (for all mart.)

Step 2. $P(S(f) > \beta\lambda,\ f^* \leq \delta\lambda) \leq \dfrac{40\delta^2}{\beta^2-1} P(S(f)>\lambda)$

holds $\forall \beta>1,\ \lambda>0,\ \delta>0$. Also

$$P(f^* > \beta\lambda,\ S(f) \leq \delta\lambda) \leq \frac{6\delta^2}{(\beta-1)^2} P(f^* > \lambda)$$

(Last 2 inequalities $\Rightarrow$ $\{f^* < \infty\} \underset{\text{a.e.}}{=} \{S(f) < \infty\}$

4/1 MARTINGALES

Theorem: $\{f \text{ converges}\} \underset{a.e.}{=} \{f^* < \infty\} \underset{a.e.}{=} \{S(f) < \infty\}$

$\uparrow$ from last time

Proof. $\subset$ clear

$(\supset)$ Fix n. Let $g = {}^n f$ = martingale with diff seq $(\underbrace{0,0,\dots,0}_{n}, d_{n+1}, \dots)$

Then

$$S^2(g) = S^2({}^n f) = \sum_{k=n+1}^{\infty} d_k^2 \to 0 \text{ if } S(f) < \infty$$

Claim: $P\Big(\limsup_{j,k \to \infty} |f_k - f_j| > 2\beta\lambda,\ S(f) < \infty\Big) = 0$

$$P\Big(\underset{\substack{\uparrow \\ \limsup_{n \le k,j \to \infty} |g_k - g_j| > 2\beta\lambda}}{\qquad\qquad}\Big) \le P(g^* > \beta\lambda,\ S(f) < \infty)$$

$$\le \limsup_{n \to \infty} P(g^* > \beta\lambda,\ S(g) < \infty)$$

$$\le \limsup_{n \to \infty} P(g^* > \beta\lambda,\ S(g) \le \delta\lambda)$$

$$\le \frac{6\delta^2}{(\beta-1)^2}.$$ Now let $\delta \to 0$

Let $\lambda \to 0$ to see $P(\limsup |f_k - f_j| > 0, S(f) < \infty) = 0$
and thus f converges a.e.

▨

To prove: $\|S(f)\|_p \approx \|f^*\|_p \quad 0<p<\infty$

Lemma: $P(\underset{\uparrow}{g} > \beta\lambda, \underset{\nearrow}{f} \le \delta\lambda) \le \varepsilon P(g > \lambda) \quad \forall \lambda > 0$

non-neg. meas $\quad (\beta > 1, \delta > 0, \varepsilon > 0)$

$$\Longrightarrow \|g\|_p^p \le \frac{(\beta/\delta)^p}{1-\beta^p\varepsilon} \|f\|_p^p$$

(provided $\beta^p \varepsilon < 1$)

Proof.

$$\|g\|_p^p = \int_0^\infty p\lambda^{p-1} P(g>\lambda)\, d\lambda$$

$$= \beta^p \int_0^\infty p\lambda^{p-1} P(g>\beta\lambda)\, d\lambda$$

$$= \beta^p \int_0^\infty p\lambda^{p-1} \Big(P(g>\beta\lambda, f \le \delta\lambda) + P(g>\beta\lambda, f>\delta\lambda) \Big)$$

$$\le \beta^p \int_0^\infty p\lambda^{p-1} \big(\varepsilon P(g>\lambda) + P(f>\delta\lambda) \big)$$

$$\le \beta^p \Big[\varepsilon \int_0^\infty p\lambda^{p-1} P(g>\lambda)\, d\lambda + \int_0^\infty p\lambda^{p-1} P(f>\delta\lambda)\, d\lambda \Big]$$

$$= \beta^p \left[\varepsilon \|g\|_p^p + \left\| \frac{f}{\delta} \right\|_p^p \right]$$

$$= \beta^p \varepsilon \|g\|_p^p + \left(\frac{\beta}{\delta}\right)^p \|f\|_p^p$$

Done if $\|g\|_p < \infty$. If $\|g\|_p = \infty$, replace g by $g \wedge n$

▨

Let $\Phi : [0,\infty] \longrightarrow [0,\infty]$ be non-decreasing and continuous and satisfying

① $\Phi(0) = 0$
② $\Phi(2\lambda) \leq c\, \Phi(\lambda) \qquad \lambda > 0$

Example: $\Phi(\lambda) = \lambda^p \qquad 0 < p < \infty$
$\Phi(\lambda) = \log(\lambda + 1)$
$\Phi(\lambda) = \lambda \log(\lambda + 1)$

Proposition: $P(g > \beta\lambda, \; f \leq \delta\lambda) \leq \varepsilon P(g > \lambda) \;\; \forall \lambda$

$$\Rightarrow E\,\Phi(g) \leq C\, E\,\Phi(f)$$

Proof. $E\,\Phi(\beta^{-1} g) = \int_0^\infty P(g > \beta\lambda)\, d\Phi(\lambda)$

$$\leq \varepsilon \int_0^\infty P(g > \lambda)\, d\Phi(\lambda) + \int_0^\infty P(f > \delta\lambda)\, d\Phi(\lambda)$$

$$= \varepsilon E \Phi(g) + E \Phi(\delta^{-1}\xi)$$

Now

$$E \Phi(g) = E \Phi(\beta \beta^{-1} g) \leq \underset{\uparrow}{\gamma} E \Phi(\beta^{-1} g)$$

depends on β $\left(\Phi(\beta\lambda) \leq \gamma \Phi(\lambda) \right)$

$$\leq \gamma\varepsilon E \Phi(g) + \gamma \underset{\uparrow}{\eta} E \Phi(\xi)$$

depends on δ $\left(\Phi(\delta^{-1}\lambda) \leq \eta \Phi(\lambda) \right)$

$$\therefore \quad \Phi(g) \leq \frac{\gamma\eta}{1-\gamma\varepsilon} E \Phi(\xi)$$

(Assuming $\gamma\varepsilon < 1$)

1. Let $(\Omega, \mathcal{A}, P)$ be a probability space and G a finite group of transformations from Ω to Ω ($\varphi \in G$ $\Longrightarrow$ φ is 1-1 onto and $\varphi^{-1} \in G$; the composition of two functions in G is in G). In addition, suppose that each φ in G is measure-preserving (if $A \in \mathcal{A}$, then $\varphi^{-1}(A) \in \mathcal{A}$ and $P(\varphi^{-1}(A)) = P(A)$). Let $\mathcal{B}$ be the class of invariant sets in $\mathcal{A}$:

$$\mathcal{B} = \{A \in \mathcal{A} : \varphi^{-1}(A) = A, \ \varphi \in G\}$$

Show that

(i) $\mathcal{B}$ is a σ-field,

(ii) if f is integrable or nonnegative $\mathcal{A}$-measurable then

$$E(f \mid \mathcal{B}) = \frac{\sum_{\varphi \in G} f(\varphi)}{|G|} \quad \text{a.e.}$$

Here $|G|$ denotes the number of elements in G.

2. Let $f = (f_1, f_2, \ldots)$ be an L^1-bounded nonnegative submartingale. Let $F_n = \sup_{k \geq n} E(f_k \mid \mathcal{A}_n)$. Show that $F = (F_1, F_2, \ldots)$ is a martingale satisfying

(i) $f_n \leq F_n$ a.e., $n \geq 1$,

(ii) $\|f\|_1 = \|F\|_1$,

(iii) F is the smallest martingale with property (i), in the sense that a martingale G satisfying $f \leq G$ a.e. must also satisfy $F \leq G$ a.e..

HINT. $E(f_k \mid \mathcal{A}_n) \leq E(f_{k+1} \mid \mathcal{A}_n)$ a.e. so that $F_n = \lim_{k \to \infty} E(f_k \mid \mathcal{A}_n)$ a.e.

3. Let $f = (f_1, f_2, \ldots)$ be an L^1- bounded martingale: f is a martingale and

$$\|f\|_1 = \sup_n \|f_n\|_1 < \infty.$$

Show that $f = g + h$ where g and h are nonnegative martingales. HINT. Use 2.

4. Suppose that $f = (f_1, f_2, \ldots)$ is a sequence of integrable functions on a probability space such that

$$E(f_{n+1} \mid f_n) = f_n \text{ a.e.}, \quad n \geq 1,$$

where $E(f_{n+1} \mid f_n)$ denotes the conditional expectation of f_{n+1} given the σ-field generated by f_n. (Note that a martingale satisfies all this and more.) Show that, for $n \geq 1$,

$$\|f_{n+1}\|_1 - \|f_n\|_1$$

$$= 2 \int_{\{f_n \leq 0\}} f_{n+1}^{+} + 2 \int_{\{f_n > 0\}} f_{n+1}^{-} .$$

5. Apply 4 to simple random walk stopped at ± b, where b is a positive integer, and show that the expected number of 0's in the sequence

$$\left(f_0(\omega) = 0,\ f_1(\omega),\ f_2(\omega), \ldots\right)$$

is b.

$$\int_{(f_n \le 0)} f_{n+1}^{+} = 1 \cdot P(f_n = 0, f_{n+1} = 1) = \frac{1}{2} P(f_n = 0)$$

$$\int_{f_n > 0} f_{n+1}^{-} = 0$$

$$\#\text{zeros} = \sum_{k=0}^{\infty} \chi_{(f_k = 0)}$$

6. Let $f = (f_1, f_2, \ldots)$ be a nonnegative L^1-bounded submartingale with limit f_∞. Then

$$\|f_\infty\|_1 \leq \|f\|_1$$

as we have seen. Show that if

$$\|f_\infty\|_1 = \|f\|_1 ,$$

then f is uniformly integrable.

HINT. Show first that $\|f_n - f_\infty\|_1 \to 0$. Note that

$$|f_n - f_\infty| = 2(f_\infty - f_n)^+ - (f_\infty - f_n)$$

and $(f_\infty - f_n)^+$, $n \geq 1$, is dominated by f_∞.

7. Suppose that f and g are nonnegative measurable functions on $(\Omega, \mathcal{a}, P)$ and β is a positive number such that, for all $\lambda > 0$,

$$(*) \quad \lambda P(g > \beta\lambda) \leq \int_{\{f > \lambda\}} f .$$

Show that

$$\|g\|_p^p \leq \beta^p q \|f\|_p^p$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$.

Note that (*) gives a slightly different result* then the inequality

$$\lambda P(g > \lambda) \leq \int_{\{g > \lambda\}} f .$$

* even for $\beta = 1$

8. Suppose that $(\Omega, \mathcal{A}, P)$ is a probability space, Π_n is a finite measurable partition of Ω, and the partitions become finer with increasing n (if $B \in \Pi_n$, then B is the finite union of sets in Π_{n+1}) so that $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}$ where $\mathcal{A}_n$ is the σ-field generated by Π_n. Let $\varphi : \mathcal{A} \to [0, \infty)$ be finitely additive and

$$f_n = \sum_{\substack{A \in \Pi_n \\ P(A) > 0}} \frac{\varphi(A)}{P(A)} I_A .$$

Show that $f = (f_1, f_2, \ldots)$ is a nonnegative supermartingale. Under what conditions on φ is f a martingale?

9. Let $f = (f_1, f_2, \ldots)$ be an almost everywhere convergent sequence of $\mathcal{A}$-measurable functions with integrable maximal function f^* and let $(\mathcal{A}_n)$ be a monotone sequence of sub-σ-fields of $\mathcal{A}$. Show that

$$E(f_n \mid \mathcal{A}_n) \longrightarrow E(f_\infty \mid \mathcal{A}_\infty) \quad \text{a.e.}$$

as $n \to \infty$, where f_∞ denotes the almost everywhere limit of f and

$$\mathcal{A}_\infty = \begin{cases} \bigvee_{k=1}^{\infty} \mathcal{A}_k & \text{if } \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \\ \bigcap_{k=1}^{\infty} \mathcal{A}_k & \text{if } \mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots. \end{cases}$$

10. Show that if f is an L^1-bounded martingale and g is the transform of f by $(0, f_1, f_2, \ldots)$, then g converges almost everywhere.

11. Use the above to prove Austin's result: If f is an L^1-bounded martingale with difference sequence $d = (d_1, d_2, \ldots)$, then $\sum_{k=1}^{\infty} d_k^2 < \infty$ almost everywhere.

12. tentative project proposal.