Measure Theory

Joe Miles
University of Illinois

Spring 1978

Notes by Larry Riddle

1/23 MEASURE THEORY

PREVIEW

Riesz Representation Theorem (Weak Version): Let $C_c(\mathbb{R})$ be the set of continuous real-valued functions on $\mathbb{R}$ which vanish off some bounded interval. Suppose $\Lambda: C_c(\mathbb{R}) \to \mathbb{R}$ is a positive linear functional. Associated with Λ there exists a σ-algebra $\mathcal{M}$ containing the Borel sets and a unique positive μ on $\mathcal{M}$ s.t.

$$\Lambda f = \int_{\mathbb{R}} f \, d\mu \qquad \forall f \in C_c(\mathbb{R})$$

Remarks ① Right hand side is a positive linear functional on $C_c(\mathbb{R})$ for any such μ.

② Consider the case of $\Lambda f :=$ Riemann integral of f. We know the conclusion to be true in this case, since in 441 we constructed Lebesgue measure (which is the measure in question) for this Λ.

Radon-Nikodym Theorem: Suppose λ and μ are finite positive measures on a σ-algebra $\mathcal{M}$ of subsets of $\mathbb{R}$ s.t. λ is absolutely continuous with respect to μ, i.e. $\mu(E) = 0 \Rightarrow \lambda(E) = 0$. Then $\exists\, f \in L^1(\mu)$ s.t.

$$\lambda(E) = \int_E f \, d\mu \qquad \forall E \in \mathcal{M}$$

Remarks: ① Note RHS is a measure absolutely continuous w.r.t. μ
- use Dominated Convergence Theorem -

DEFINITION: X set, $\mathcal{M} \subset \mathcal{P}(X)$. We say $\mathcal{M}$ is a σ-algebra if

i) $X \in \mathcal{M}$

ii) $A \in \mathcal{M} \Rightarrow X \setminus A \in \mathcal{M}$

iii) $(A_n) \subset \mathcal{M} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$

Remarks: a) $\phi \in \mathcal{M}$

b) $(A_n) \in \mathcal{M} \Rightarrow \bigcup_{n=1}^{N} A_n \in \mathcal{M} \quad \forall N \in \mathbb{N}$

c) finite and countable intersections are in $\mathcal{M}$

d) $A \in \mathcal{M}, B \in \mathcal{M} \Rightarrow A \setminus B = A \cap X \setminus B \in \mathcal{M}$

DEFINITION: X set, $\mathcal{M}$ a σ-algebra of subsets of X. We say $(X, \mathcal{M})$ is a measurable space (or X if $\mathcal{M}$ is understood)

DEFINITION: Suppose $f : X \longrightarrow Y$, where X is a measurable space, Y a topological space. We say f is measurable if $f^{-1}(V) \in \mathcal{M}$ for all open V in Y.

PROPOSITION: Suppose $X \xrightarrow[\text{meas.}]{f} Y \xrightarrow[\text{cont.}]{g} Z$. Then $g \circ f$ is measurable

Proof. If V is open in Z,

$$(g \circ f)^{-1}(V) = f^{-1}(\underbrace{g^{-1}(V)}_{\text{open in } Y}) \in \mathcal{M}$$

▨

PROPOSITION: Suppose X is a measurable space and $u: X \to \mathbb{R}$, $v: X \to \mathbb{R}$ are measurable. Suppose $\Phi: \mathbb{R}^2 \to Y$ (top. space) is continuous. Let

$$f(x) := \Phi(u(x), v(x))$$

Then $f: X \to Y$ is measurable.

Proof. Let $h: X \to \mathbb{R}^2$ be $h(x) = (u(x), v(x))$. So $f = \Phi \circ h$. Last proposition $\Rightarrow$ sufficient to show h measurable. Let I_a be open interval in $\mathbb{R}$, I_b open interval in $\mathbb{R}$.
$S = I_a \times I_b$

$$h^{-1}(S) = u^{-1}(I_a) \cap v^{-1}(I_b) \in \mathcal{M}$$

If $V \subset \mathbb{R}^2$ is open, then $V = \bigcup S_i$ (countable union) where S_i is a rectangle. Then

$$h^{-1}(V) = \bigcup h^{-1}(S_i) \in \mathcal{M}$$

so h is measurable. ▨

PROPOSITION: X measurable space.

a) Suppose $f: X \to \mathbb{C}$, $f(x) = u(x) + i\,v(x)$ where u and v are measurable real-valued functions. Then f is measurable.

Proof. $\Phi(s,t) := s + it$ $\Phi: \mathbb{R}^2 \to \mathbb{C}$ continuous. Then $f = \Phi \circ (u, v)$, so is measurable

b) Suppose $f(x) = u(x) + i\,v(x)$ is measurable. Then $u(x)$, $v(x)$, and $|f(x)|$ are measurable.

Proof. $u(x) = \mathrm{Re}\,(f(x))$ is composition of a measurable function followed by a continuous function. Similarly for others.

c) If $f: X \to \mathbb{C}$ and $g: X \to \mathbb{C}$ are measurable, then $f+g$ and fg are measurable.

Proof: Case I. f, g are real-valued. Let $\Phi(s,t) = s+t$ or st. Previous proposition $\Rightarrow \Phi(f,g)$ is measurable.

Case II: $f = u_1 + iv_1$, $g = u_2 + iv_2$. Case I $\Rightarrow$ $u_1 + u_2$ measurable and $v_1 + v_2$ measurable. a) $\Rightarrow$ $f+g$ measurable. Also u_1u_2, v_1v_2, u_1v_2, u_2v_1 measurable $\Rightarrow$ $fg = (u_1u_2 - v_1v_2) + i(u_1v_2 + u_2v_1)$ measurable

d) If $E \in \mathfrak{M}$, then χ_E is measurable

Proof: $\chi_E^{-1}(V)$ is either $\emptyset$, X, E, or $X \setminus E$ (all measurable)

e) $f : X \to \mathbb{C}$ measurable. Then $\exists\, \alpha : X \to \mathbb{C}$ α measurable and $|\alpha(x)| = 1$ $\forall x \in X$ s.t.

$$f(x) = \alpha(x)\,|f(x)| \quad \forall x \in X$$

Proof. Let $E = \{x \in X : f(x) = 0\} = f^{-1}(\underbrace{\mathbb{C} \setminus \{0\}}_{\text{open}}) \in \mathfrak{M}$

Let $Y = \mathbb{C} \setminus \{0\}$. Define $\varphi : Y \to$ unit circle by

$$\varphi(z) = \frac{z}{|z|}$$

φ is continuous on Y. $\forall x \in X$

$$f(x) + \chi_E(x) \in Y$$

Now $f + \chi_E : X \to Y$ is measurable. Hence

$$\alpha := \varphi \circ (f + \chi_E) : X \to \text{unit circle}$$

is measurable. Suppose $f(x) = 0$. Nothing to check. If $f(x) \neq 0$, $\chi_E(x) = 0$, so

$$\alpha(x) = \varphi(f(x)) = \frac{f(x)}{|f(x)|} \qquad \blacksquare$$

1/25 MEASURE THEORY

PROPOSITION: X set, $\mathcal{F} \subset \mathcal{P}(X)$, then there is a smallest σ-algebra of subsets containing $\mathcal{F}$

Proof. Let

$$\mathfrak{M}^* := \bigcap_{\mathfrak{M} \in \Omega} \mathfrak{M}$$

where Ω is the collection of all σ-algebras of subsets of X containing $\mathcal{F}$. Clearly $\mathcal{F} \subset \mathfrak{M}^*$. It is easy to check that $\mathfrak{M}^*$ is a σ-algebra.

DEFINITION: If X is a topological space, let $\mathcal{F}$ be the collection of open subsets of X. The smallest σ-algebra containing $\mathcal{F}$ is the collection $\mathcal{B}$ of Borel sets.

DEFINITION: $f : X^{\text{Top}} \longrightarrow Y^{\text{Top}}$ is Borel measurable (Borel function) if f is measurable w.r.t. $\mathcal{B}$

PROPOSITION: X set, $\mathfrak{M}$ σ-algebra in X, $f: X \to Y$ (top.)

a) $\{E \subset Y : f^{-1}(E) \in \mathfrak{M}\}$ is a σ-algebra in Y.

b) If f is measurable (w.r.t. $\mathfrak{M}$) then $f^{-1}(B) \in \mathfrak{M}$ for every Borel set B in Y.

Proof of b). f measurable $\Rightarrow f^{-1}(V) \in \mathfrak{M}$ $\forall$ open V in Y. Thus $\{E : f^{-1}(E) \in \mathfrak{M}\}$ is a σ-algebra (by (a)) containing the open sets of Y, and hence contains the Borel sets of Y. $\boxtimes$

c) If $Y = \mathbb{R}_e$, then if $\forall \alpha \in \mathbb{R}$, $f^{-1}(\alpha, +\infty] \in \mathfrak{M}$, we have that f is measurable.

Proof of c). $\forall \alpha \in \mathbb{R}$, $f^{-1}[\alpha, +\infty] \in \mathfrak{M} \Rightarrow f^{-1}([-\infty, \alpha)) \in \mathfrak{M}$
$\Rightarrow f^{-1}(a,b) \in \mathfrak{M}$ $\forall a<b$ $\Rightarrow f^{-1}(V) \in \mathfrak{M}$ $\forall$ open V $\boxtimes$

d) $\underset{\text{measu.}}{X} \xrightarrow{f} \underset{\text{top.}}{Y} \xrightarrow{g} \underset{\text{top.}}{Z}$. If g is Borel measurable and f is measurable then $g \circ f : X \to Z$ is measurable.

Proof of d) V open in Z.

$$(g \circ f)^{-1}(V) = f^{-1}(\underbrace{g^{-1}(V)}_{\text{Borel set}}) \in \mathfrak{M}$$

$\boxtimes$

NOTE: d) is not true if we assume g is Lebesgue measurable. Recall example from 441.

Suppose $f_n : X \to \mathbb{R}_e$, X measurable space. If each f_n is measurable, then $\sup f_n$ is measurable:

$$\{x \in X : \sup f_n(x) > \alpha\} = \bigcup_n \{x \in X : f_n(x) > \alpha\}$$

Therefore $\overline{\lim} f_n$, $\underline{\lim} f_n$, and $\lim f_n$ (if it exists) are all measurable if each f_n is.

<u>DEFINITION</u>: $f : X \to \mathbb{R}_e$

$$f^+ := \max(f, 0)$$
$$f^{-1} := \max(-f, 0)$$

Note f^+ and f^- are measurable if f is. Certainly

$$f = f^+ - f^-$$
$$|f| = f^+ + f^-$$

Remark: If $f = g - h$, where $g \geq 0$ and $h \geq 0$, then $g \geq f^+$ and $h \geq f^-$, since

$$\left.\begin{array}{c} g \geq 0 \\ g = f + h \geq f \end{array}\right\} \Rightarrow g \geq f^+$$

and $h = g - f \geq f^+ - f = f^-$.

<u>DEFINITION</u>: X measurable. $s : X \to [0, \infty)$ is simple if the range of s is a finite set.

A simple function s has a canonical representation

$$s = \sum_{\alpha \in F} \alpha \chi_{A_\alpha}$$

where F is a finite set and $A_\alpha \cap A_\beta = \emptyset$ $\forall \alpha \neq \beta$.

PROPOSITION: X measurable space. $f: X \longrightarrow [0,\infty]$ measurable. Then $\exists$ simple functions s_n s.t.

(i) $0 \leq s_n \leq s_{n+1} \leq f$

(ii) $s_n(x) \longrightarrow f(x)$ $\forall x \in X$

(iii) s_n measurable

Proof. For $n \in \mathbb{N}^*$, let

$$E_{n,i} := \left\{ x \in X : \frac{i-1}{2^n} \leqslant f(x) < \frac{i}{2^n} \right\} \quad 1 \leq i \leq n2^n$$

$$F_n := \left\{ x \in X : f(x) \geqslant n \right\}$$

Now set

$$s_n := \sum_{i=2}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}} + n \chi_{F_n} \qquad \blacksquare$$

DEFINITION: Suppose $(X, \mathfrak{M})$ is a measurable space. A positive measure μ on $(X, \mathfrak{M})$ is a function $\mu: \mathfrak{M} \longrightarrow [0,\infty]$ s.t. $\mu(A) < \infty$ for some $A \in \mathfrak{M}$ and μ is countably additive. $(X, \mathfrak{M}, \mu)$ is called a measure space.

Elementary consequences:

1. $\mu(\emptyset) = 0$.

Take A s.t. $\mu(A) < \infty$. Then $\mu(A) = \mu(A \cup \bigcup_{n=1}^{\infty}\emptyset) = \mu(A) + \sum_{n=1}^{\infty}\mu(\emptyset)$ and so $\mu(\emptyset) = 0$.

2. μ is finitely additive

3. If $A \subset B$, $A, B \in \mathfrak{M}$, then $\mu(A) \leq \mu(B)$

4. If $(A_n) \subset \mathfrak{M}$, $A_n \subset A_{n+1}$, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty}\mu(A_n)$

Set $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. Then the B_n's are disjoint elements of $\mathfrak{M}$ and

$$\bigcup_{n=1}^{N} B_n = A_N$$

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

Hence $\mu(\bigcup_1^{\infty} A_n) = \mu(\bigcup_1^{\infty} B_n) = \sum_{n=1}^{\infty}\mu(B_n) = \lim_{N\to\infty}\sum_{n=1}^{N}\mu(B_n)$

$$= \lim_{N\to\infty}\mu(\bigcup_{n=1}^{N} B_n) = \lim_{N\to\infty}\mu(A_N)$$

5. If $(A_n) \subset \mathfrak{M}$, $A_{n+1} \subset A_n$, and $\mu(A_1) < \infty$, then $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty}\mu(A_n)$

1/27 MEASURE THEORY

$(X, \mathfrak{M}, \mu)$ measure space

DEFINITION: Suppose $s : X \to [0,\infty)$ is a measurable simple function with canonical representation

$$s = \sum_{\alpha \in F}' \alpha \chi_{E_\alpha}$$

Suppose $E \in \mathfrak{M}$, Then

$$\int_E s \, d\mu := \sum_{\alpha \in F}' \alpha \, \mu(E \cap E_\alpha)$$

(If $s = 0$, $\int_E s \, d\mu := 0$).

Proposition: Suppose $s_1 \leq s_2$ are measurable simple functions on X. Then for $E \in \mathfrak{M}$

$$\int_E s_1 \, d\mu \leq \int_E s_2 \, d\mu$$

Proof. WLOG neither $s_j \equiv 0$. Suppose

$$s_1 = \sum_{i=1}^{N} \alpha_i \chi_{A_i}$$

$$s_2 = \sum_{j=1}^{M} \beta_j \chi_{B_j}$$

For $1 \leq i \leq N$, let $T_i = \{ j : 1 \leq j \leq M \text{ and } B_j \cap A_i \neq \emptyset \}$

Claim: $s_1 \le s_2 \Rightarrow$ (a) $A_i \subset \bigcup_{j \in T_i} B_j \quad 1 \le i \le N$

(b) $\alpha_i \le \beta_j \quad \forall j \in T_i$

Therefore $A_i \cap E = \bigcup_{j \in T_i} (A_i \cap B_j \cap E)$, so

$$\mu(A_i \cap E) = \sum_{j \in T_i} \mu(A_i \cap B_j \cap E)$$

$$\Rightarrow \alpha_i \mu(A_i \cap E) \le \sum_{j \in T_i} \beta_j \mu(A_i \cap B_j \cap E)$$

$$\Rightarrow \sum_{i=1}^{N} \alpha_i \mu(A_i \cap E) \le \sum_{i=1}^{N} \sum_{j \in T_i} \beta_j \mu(A_i \cap B_j \cap E)$$

$$\le \sum_{j=1}^{M} \beta_j \sum_{i=1}^{N} \mu(A_i \cap B_j \cap E)$$

$$\le \sum_{j=1}^{M} \beta_j \mu(B_j \cap E) = \int_E s_2 \, d\mu$$

and so $\int_E s_1 \, d\mu \le \int_E s_2 \, d\mu$. ▨

DEFINITION: Suppose $f : X \to [0, \infty]$ is measurable. Then if $E \in \mathcal{M}$

$$\int_E f \, d\mu := \sup \left\{ \int_E s \, d\mu : s \le f, \ s \text{ measurable simple} \right\}$$

Remarks: ① well-defined by the last proposition

② agrees with 441 definition of $\int f$

"ASIDE" ③ Take case where $\mathcal{M}$ = Lebesgue measure, $X = \mathbb{R}$. Recall if $E \subset \mathbb{R}$, $m_*(E) := \sup\{m(\tilde{E}) : \tilde{E} \subset E, \tilde{E} \in \mathcal{M}\}$. Then $m_*(E) = m^*(E)$ iff $E \in \mathcal{M}$. Take $E \subset [0,1]$. If $A \subset [0,1]$ and $B = [0,1] \setminus A$, then $m_*(A) = 1 - m^*(B)$. Suppose we deleted requirement "f measurable" from last definition. What would happen? Consider $A \subset [0,1]$, $A \notin \mathcal{M}$. Let $f = \chi_A$, $g = \chi_B$. Under our "new" definition, $\int_{[0,1]} f = m_*(A)$ and $\int_{[0,1]} g = m_*(B)$. But

$f + g = 1$ yet $\int f + \int g = m_*(A) + m_*(B) < m_*(A) + m^*(B) = 1 = \int (f+g)$

so $\int f + \int g < \int (f+g)$ THIS IS UNACCEPTABLE!

PROPERTIES : $f, g \geq 0$ on X, measurable

(1) $f \leq g \Rightarrow \int_E f \, d\mu \leq \int_E g \, d\mu \quad \forall E \in \mathcal{M}$

(2) $E \in \mathcal{M}$, $\int_E f \, d\mu = \int_X f \chi_E \, d\mu$.

Proof. Suppose $s \leq f \chi_E$, $s = \sum \alpha \chi_{E_\alpha}$. Note $E_\alpha \subset E$. Then

$$\int_X s \, d\mu = \sum \alpha \mu(E_\alpha) = \sum \alpha \mu(E_\alpha \cap E) = \int_E s \, d\mu$$

and so $\int_E f \, d\mu \geq \int_X f \chi_E \, d\mu$.

Suppose $t \leq f$, $t = \sum \beta \chi_{E_\beta}$. Let

$$t' = \sum \beta \chi_{E_\beta \cap E}$$

Then $t' \leq f\chi_E$ and

$$\int_E t \, d\mu = \sum' \beta \mu(E_\beta \cap E) = \int_X t' \, d\mu$$

and so $\int_E f \, d\mu \leq \int_X f \chi_E \, d\mu$. ▨

(13) $A \subset B \Longrightarrow \int_A f \, d\mu \leq \int_B f \, d\mu$
$A, B \in \mathfrak{M}$

(14) If $f = 0$ on $E \in \mathfrak{M}$, then $\int_E f \, d\mu = 0$

(15) $\mu(E) = 0 \Longrightarrow \int_E f \, d\mu = 0$

(16) $\int_E cf \, d\mu = c \int_E f \, d\mu$, $c \geq 0$.

PROPOSITION: Suppose s, t are simple, measurable on X. For $E \in \mathfrak{M}$, let

$$\varphi(E) := \int_E s \, d\mu$$

Then φ is a measure on $\mathfrak{M}$. Also

$$\int_X (s+t) \, d\mu = \int_X s \, d\mu + \int_X t \, d\mu$$

Proof: Suppose $E_n \in \mathcal{M}$, E_n disjoint. Let

$$E = \bigcup_{n=1}^{\infty} E_n$$

Suppose $s = \sum_{j=1}^{M} \beta_j \chi_{B_j}$. Then

$$\varphi(E) = \sum_{j=1}^{M} \beta_j \mu(B_j \cap E) = \sum_{j=1}^{M} \beta_j \sum_{n=1}^{\infty} \mu(E_n \cap B_j)$$

$$= \sum_{n=1}^{\infty} \sum_{j=1}^{M} \beta_j \mu(E_n \cap B_j) = \sum_{n=1}^{\infty} \int_{E_n} s \, d\mu$$

$$= \sum_{n=1}^{\infty} \varphi(E_n)$$

Notice that $\varphi(\emptyset) = 0$, so $\varphi \not\equiv \infty$ identically.

Let $\beta_0 := 0$ and $B_0 := s^{-1}(0)$, so $X = \bigcup_{0}^{M} B_j$. Suppose

$$t = \sum_{i=1}^{N} \alpha_i \chi_{A_i}$$

and $\alpha_0 := 0$, $A_0 := t^{-1}(0)$, so $X = \bigcup_{0}^{N} A_i$.

For $0 \le i \le N$, $0 \le j \le M$, let

$$E_{ij} = A_i \cap B_j$$

E_{ij} disjoint with union X. Then

$$\int_{E_{ij}} (s+t) \, d\mu = \int_{E_{ij}} s \, d\mu + \int_{E_{ij}} t \, d\mu$$

(all functions constant on E_{ij}). Now add over all i, j.

$$\sum_{j=0}^{M} \sum_{l=0}^{N} \int_{E_{ij}} s \, d\mu = \int_X s \, d\mu \quad \text{by } 1^{st} \text{ part}$$

Similarly for other parts. ▨

<u>MONOTONE CONVERGENCE THEOREM:</u> Given $(X, \mathfrak{M}, \mu)$. Suppose $f_n : X \to [0, \infty]$ are measurable and $f_n \leq f_{n+1}$. If $f := \lim f_n$, then f is measurable and

$$\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu$$

Proof. f is measurable since each f_n is measurable. By previous proposition

$$\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu$$

so $\lim \int_X f_n \, d\mu$ exists. If $\alpha = \lim \int_X f_n \, d\mu$, then $\alpha \leq \int_X f \, d\mu$

since $\int f_n \, d\mu \leq \int f \, d\mu \ \forall n \in \mathbb{N}$. Suppose $s \leq f$, s simple and measurable. Take $0 < c < 1$, and let

$$E_n = \{ x \in X : f_n(x) > c\, s(x) \}$$

Then $E_n \subset E_{n+1}$ and each E_n is measurable. Also

$$\bigcup_{n=1}^{\infty} E_n = X$$

1/30 MEASURE THEORY

(PROOF OF MCT, continued)

Let s be a simple measurable function, $s \le f$. For $n \in \mathbb{N}$ let

$$E_n := \{ x \in X : f_n(x) \ge c\, s(x) \}$$

where $0 < c < 1$. Then each E_n is measurable and $E_n \subset E_{n+1}$.
Claim: $X = \bigcup E_n$. Suppose $f(x) = 0$. Then $x \in E_1$ since $s(x) = 0$.
Suppose $f(x) > 0$. Then $f(x) \ge s(x) > c\, s(x)$. Hence $\exists\, n$ with $f_n(x) > c\, s(x)$, so $x \in E_n$.

For any n,

$$(*) \qquad \alpha \ge \int_X f_n \, d\mu \ge \int_{E_n} f_n \, d\mu \ge c \int_{E_n} s \, d\mu = c\varphi(E_n)$$

[RECALL: $\varphi(E) := \int_E s \, d\mu$, $E \in \mathfrak{M}$, is a measure]. Now

$$\varphi(X) = \lim_{n \to \infty} \varphi(E_n)$$

and so from $(*)$ we have

$$\alpha \ge c\varphi(X) = c \int_X s \, d\mu$$

Let $c \uparrow 1$. Then $\alpha \ge \int_X s \, d\mu$. Hence $\alpha \ge \int_X f \, d\mu$ by definition $\boxtimes$

COROLLARY: (FATOU'S LEMMA) Given $(X, \mathcal{M}, \mu)$ and $f_n : X \to [0,\infty]$ measurable. Then

$$\int_X \underline{\lim}\, f_n \, d\mu \le \underline{\lim} \int_X f_n \, d\mu$$

Proof. Set $g_k := \inf_{n \ge k} f_n$. Then

a) g_k measurable
b) $g_k \le g_{k+1}$
c) $g_k \uparrow \underline{\lim}\, f_n$

Then M.C.T. applied to (g_k) says

$$\int_X \underline{\lim}_{n\to\infty} f_n \, d\mu = \lim_{k\to\infty} \int g_k \, d\mu \le \underline{\lim}_{n\to\infty} \int_X f_n \, d\mu$$

$\blacksquare$

(Aside: Fatou $\Rightarrow$ MCT.
Given $0 \le f_n \le f_{n+1} \cdots$, $f_n \uparrow f$. Then

(PRELIM)

$$\overline{\lim} \int f_n \, d\mu \underset{\substack{\uparrow \\ f_n \uparrow f}}{\le} \int f \, d\mu \underset{\substack{\uparrow \\ \text{FATOU}}}{\le} \underline{\lim} \int f_n \, d\mu$$

Hence $\lim \int f_n = \int f$.)

PROPOSITION: Given $(X, \mathfrak{M}, \mu)$, $f_n : X \to [0,\infty]$, measurable. Set $f = \sum_{n=1}^{\infty} f_n$. Then

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$$

Proof. Suppose $h_1, h_2 \geq 0$ measurable. We know $\exists\, s_n \uparrow h_1$, $t_n \uparrow h_2$, where s_n, t_n measurable simple functions. Then $(s_n + t_n) \uparrow (h_1 + h_2)$. We know

$$\forall n \in \mathbb{N} \quad \int_X (s_n + t_n)\, d\mu = \int_X s_n \, d\mu + \int_X t_n \, d\mu$$

By MCT

$$\begin{array}{ccccc} \int_X (s_n + t_n)\, d\mu & = & \int_X s_n \, d\mu & + & \int_X t_n \, d\mu \\ \downarrow & & \downarrow & & \downarrow \\ \int_X (h_1+h_2)\, d\mu & & \int_X h_1 \, d\mu & & \int_X h_2 \, d\mu \end{array}$$

Hence

$$\int_X (h_1 + h_2)\, d\mu = \int_X h_1 \, d\mu + \int_X h_2 \, d\mu$$

Set

$$g_N := \sum_{n=1}^{N} f_n$$

Then

$$\int g_N \, d\mu = \sum_{n=1}^{N} \int f_n \, d\mu$$

Since $g_N \uparrow f$, MCT $\Rightarrow$

$$\int f\,d\mu = \lim_N \int g_N\,d\mu = \lim_N \sum_{n=1}^{N} \int f_n\,d\mu$$

$$= \sum_{n=1}^{\infty} \int f_n\,d\mu$$

▨

PROPOSITION: Given $(X, \mathcal{M}, \mu)$. Let $f: X \to [0,\infty]$ be measurable. For $E \in \mathcal{M}$, define

$$\varphi(E) = \int_E f\,d\mu$$

Then φ is a measure on $\mathcal{M}$ and furthermore, if $g: X \to [0,\infty]$ is measurable

$$(*) \qquad \int_X g\,d\varphi = \int_X gf\,d\mu$$

Proof. Given $(E_n) \subset \mathcal{M}$ disjoint, let $E = \bigcup_{n=1}^{\infty} E_n$. Must show $\varphi(E) = \sum \varphi(E_n)$. Clearly,

$$f\chi_E = \sum_{n=1}^{\infty} f\chi_{E_n}$$

so by the last proposition

$$\int_E f\,d\mu = \int f\chi_E\,d\mu = \sum_{n=1}^{\infty} \int f\chi_{E_n}\,d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\,d\mu$$

$$\Rightarrow \varphi(E) = \sum_{n=1}^{\infty} \varphi(E_n)$$

First suppose $g = \chi_E$ for some $E \in \mathfrak{M}$.

$$\int_X g\,d\varphi = \int_X \chi_E\,d\varphi = \varphi(E) = \int_E f\,d\mu$$

$$= \int_X f\chi_E\,d\mu = \int_X fg\,d\mu$$

Therefore $(*)$ holds for any simple function. Given a general g, $\exists\, s_n \uparrow g$, s_n simple, measurable.

$$\int_X g\,d\varphi = \lim_n \int_X s_n\,d\varphi = \lim_n \int_X f s_n\,d\mu = \int_X fg\,d\mu$$

$\blacksquare$

<u>DEFINITION</u>: Given $(X, \mathfrak{M}, \mu)$, $f : X \longrightarrow \mathbb{C}$ measurable. Then $f \in L^1(\mu)$ if

$$\int_X |f|\,d\mu < \infty$$

DEFINITION: For $f \in L^1(\mu)$, with $f = u + iv$, define

$$\int_X f\,d\mu := \int_X u^+\,d\mu - \int_X u^-\,d\mu + i\left[\int_X v^+\,d\mu - \int_X v^-\,d\mu\right]$$

REMARK: $u^+ \le |u| \le |f|$ and u^+ is measurable. Similarly for others. $\int f\,d\mu \in \mathbb{C}$

<u>PROPOSITION</u>: If $f, g \in L^1(\mu)$, then $\alpha f + \beta g \in L^1(\mu)$ $\forall \alpha, \beta \in \mathbb{C}$ and

$$\int_X (\alpha f + \beta g)\,d\mu = \alpha \int_X f\,d\mu + \beta \int_X g\,d\mu$$

Proof. $\alpha f + \beta g$ is measurable.

$$|\alpha f + \beta g| \le |\alpha||f| + |\beta||g|$$

$$\Rightarrow \int |\alpha f + \beta g|\,d\mu < \infty$$

Show $\int (f+g) = \int f + \int g$. Sufficient to show for f, g real.

Let $h = f + g$. $f = f^+ - f^-$, $g = g^+ - g^-$, $h = h^+ - h^-$

Then

$$f^+ + g^+ - f^- - g^- = h^+ - h^-$$
$$f^+ + g^+ + h^- = h^+ + f^- + g^-$$

$$\int f^+ d\mu + \int g^+ d\mu + \int h^- d\mu = \int h^+ d\mu + \int f^- d\mu + \int g^- d\mu$$

$$\int f^+ d\mu - \int f^- d\mu + \int g^+ d\mu - \int g^- d\mu = \int h^+ d\mu - \int h^- d\mu$$

2/1 MEASURE THEORY

HOMEWORK: Chap 1 #1, 9, 12 Due Monday, Feb. 13. Look at 7, 8

PROPOSITION: $(X, \mathcal{M}, \mu)$. Suppose $f \in L^1(\mu)$. Then

$$\left| \int_X f\, d\mu \right| \leq \int_X |f|\, d\mu$$

Proof. Set $z = \int_X f\, d\mu$. $\exists\, \alpha \in \mathbb{C}$ with $|\alpha| = 1$ s.t. $\alpha z = |z|$. Then if $u = \operatorname{Re} \alpha f$, $|u| \leq |f|$, and so

$$\left| \int_X f\, d\mu \right| = \alpha \int_X f\, d\mu = \int_X \alpha f\, d\mu = \int_X u\, d\mu$$

$$\leq \int_X |f|\, d\mu$$

because $\int \alpha f\, d\mu$ is real!

$\blacksquare$

DOMINATED CONVERGENCE THEOREM: Given $(X, \mathcal{M}, \mu)$.

f_n measurable

Suppose $f_n : X \to \mathbb{C}$ are such that $|f_n(x)| \leq g(x)$ $\forall x \in X$ $\forall n$ for some $g \in L^1(\mu)$. Suppose $f_n(x) \to f(x)$ $\forall x \in X$. Then $f \in L^1(\mu)$ and

$$\int_X f_n\, d\mu \to \int_X f\, d\mu$$

Moreover, $\int |f_n - f|\, d\mu \to 0$

Proof. Note $|f_n| \le g$, f_n measurable, $g \in L^1(\mu) \Rightarrow f_n \in L^1(\mu)$. Also note that $|f(x)| \le g(x)$. Thus $f \in L^1(\mu)$ and

$$2g - |f_n - f| \ge 0$$

and is measurable, so Fatou's lemma $\Rightarrow$

$$\underset{\text{finite}}{\int 2g\,d\mu} \le \underline{\lim} \int (2g - |f_n - f|)\,d\mu$$
$$= \int 2g - \overline{\lim} \int |f_n - f|\,d\mu$$

$$\Rightarrow \quad \overline{\lim} \int |f_n - f|\,d\mu = 0$$

$$\Rightarrow \quad \lim_{n\to\infty} \int |f_n - f|\,d\mu = 0$$

But

$$\left| \int_X f_n\,d\mu - \int_X f\,d\mu \right| = \left| \int (f_n - f)\,d\mu \right| \le \int |f_n - f|\,d\mu \to 0$$

$\blacksquare$

Sets of Measure Zero

__Theorem__: Given $(X, \mathfrak{M}, \mu)$. Let $\mathfrak{M}^*$ be the collection of all sets $E \subset X$ s.t. $\exists\, A \subset E \subset B$ with $A, B \in \mathfrak{M}$ and $\mu(B \setminus A) = 0$. For $E \in \mathfrak{M}^*$, let $\mu(E) := \mu(A)$. Then $\mathfrak{M}^*$ is a σ-algebra containing $\mathcal{M}$ and μ (extended to $\mathcal{M}^*$) is a measure on $\mathcal{M}^*$.

DEFINITION: $\mathcal{M}^*$ is called the μ-completion of $\mathcal{M}$ and μ (on $\mathcal{M}^*$) is said to be complete (i.e. if $E \in \mathcal{M}^*$, $Y \subset E$, and $\mu(E) = 0$, then $Y \in \mathcal{M}^*$)

[[If $E \in \mathcal{M}$, $\mu(E) = 0$, $Y \subset E$, consider $\emptyset \subset Y \subset E$. Shows $Y \in \mathcal{M}^*$. If $E \notin \mathcal{M}$, $\exists\, B \in \mathcal{M}$, $E \subset B$, $\mu(B) = 0$ Consider $\emptyset \subset Y \subset B$. Shows $Y \in \mathcal{M}^*$]]

Proof of Theorem: $\mathcal{M}^*$ is a σ-algebra. $X \in \mathcal{M}^*$ since $\mathcal{M} \subset \mathcal{M}^*$. Suppose $E \in \mathcal{M}^*$. $\exists\, A \subset E \subset B$ with $A, B \in \mathcal{M}$, $\mu(B \setminus A) = 0$. Then $X \setminus B \subset X \setminus E \subset X \setminus A$ and $X \setminus B, X \setminus A \in \mathcal{M}$, $\mu((X \setminus B) \setminus (X \setminus A)) = \mu(B \setminus A) = 0$. Suppose $(E_n) \in \mathcal{M}^*$. $\exists\, A_n \subset E_n \subset B_n$, $A_n, B_n \in \mathcal{M}$, $\mu(B_n - A_n) = 0$. Then

$$\underbrace{\bigcup^{\infty} A_n}_{\in \mathcal{M}} \subset \bigcup E_n \subset \underbrace{\bigcup^{\infty} B_n}_{\in \mathcal{M}}$$

and

$$\mu\left(\bigcup B_n - \bigcup A_n\right) \leq \mu\left(\bigcup (B_n - A_n)\right) \leq \sum \mu(B_n - A_n) = 0$$

Hence $\bigcup E_n \in \mathcal{M}^*$.

NOTE: $\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$.

Next show μ is well-defined on $\mathfrak{M}^*$. Suppose $E \in \mathfrak{M}^*$ and

$$A_1 \subset E \subset B_1 \qquad \mu(B_1 - A_1) = 0$$
$$A_2 \subset E \subset B_2 \qquad \mu(B_2 - A_2) = 0$$

Must show $\mu(A_1) = \mu(A_2)$.

$$A_1 - A_2 \subset E_1 - A_2 \subset B_2 - A_2$$

$$\Longrightarrow \mu(A_1 - A_2) = 0$$

Then $\mu(A_1) = \mu(A_1 - A_2) + \mu(A_1 \cap A_2) = \mu(A_1 \cap A_2) \underset{\uparrow}{=} \mu(A_2)$

symmetry

Left to show μ is countably additive on $\mathfrak{M}^*$. Suppose $E_n \in \mathfrak{M}^*$, E_n disjoint

$$A_n \subset E_n \subset B_n \qquad A_n, B_n \in \mathfrak{M};\ \mu(B_n - A_n) = 0$$

$$\mu(\cup E_n) = \mu(\cup A_n) = \sum \mu(A_n) = \sum \mu(E_n)$$

$\blacksquare$

<u>Observations</u>: If $f: X \to \mathbb{C}$; $g: X \to \mathbb{C}$ and $f, g \in L^1(\mu)$ and $f = g$ a.e. $[\mu]$, then

$$\int_X f\, d\mu = \int_X g\, d\mu$$

Proof. Show $\int_X (f-g)\,d\mu = 0$.

Let $\mathrm{Re}(f-g) = u$. Then $u^+ = 0$ a.e., so $\int_X u^+\,d\mu = 0$. Similarly for others. ▨

Suppose μ is complete, $S \subset X$ with $\mu(X-S) = 0$. Then $f : X \to Y$ (top. space) is measurable iff $\forall$ open $V \subset Y$, $f^{-1}(V) \cap S \in \mathfrak{M}$, since

$$f^{-1}(V) = (f^{-1}(V) \cap S) \cup (\underbrace{f^{-1}(V) \cap (X-S)}_{\in \mathfrak{M}})$$

$\in \mathfrak{M}$ since subset of set of measure 0

$(X, \mathfrak{M}, \mu)$ complete

PROPOSITION: Suppose f_n are complex-valued measurable functions defined a.e. on X. Suppose

$$\sum_{n=1}^{\infty} \int_X |f_n|\,d\mu < \infty$$

Then $\sum f_n$ converges a.e. on X to some $f \in L^1(\mu)$ and

$$\int_X f\,d\mu = \sum_{n=1}^{\infty} \int f_n\,d\mu$$

Proof: Let $S_n \subset X$ be domain of f_n. So $\mu(X - S_n) = 0$. Let $S = \bigcap S_n$. Then $\mu(X-S) = 0$. Let

$$\varphi(x) := \sum_{n=1}^{\infty} |f_n(x)| \quad \forall x \in S$$

Corollary of MCT $\Longrightarrow$

so $\varphi \in L^1$ $\rightarrow$ $$\int_S \varphi \, d\mu = \sum \int_S |f_n| \, d\mu = \sum \int_X |f_n| \, d\mu < \infty$$

Definition of $\int \varphi \, d\mu \Longrightarrow \varphi < +\infty$ a.e. on S. Hence $\sum f_n(x)$ converges a.e. on S to $f(x)$. Certainly

$$|f| \leq \varphi(x) \in L^1(\mu)$$

and so $f \in L^1(\mu)$. Let $g_N = \sum_{n=1}^{N} f_n$. On S, $|g_n| \leq \varphi$

DCT $\Longrightarrow$

$$\int_S f \, d\mu = \int_S \sum^{\infty} f_n = \int_S \lim g_N \, d\mu = \lim \int_S g_N \, d\mu$$

$$= \lim \sum^{N} \int f_n$$

$$= \sum_{1}^{\infty} \int_S f_n \, d\mu$$

OR

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$$

2/3 MEASURE THEORY

Suppose $(X, \mathcal{M}, \mu)$ is a measure space with completion $(X, \mathcal{M}^*, \mu^*)$. Suppose $f : X \to [0, \infty]$ is $\mathcal{M}$-measurable. Then f is also $\mathcal{M}^*$-measurable.

QUESTION: Is

$$\int_X f \, d\mu = \int_X f \, d\mu^* \; ?$$

ANSWER - Yes

Suppose $s \le f$, s simple and $\mathcal{M}$-measurable. Then s is also $\mathcal{M}^*$-measurable. Also

$$\int_X s \, d\mu = \int_X s \, d\mu^*$$

Hence $\int_X f \, d\mu \le \int_X f \, d\mu^*$

Now suppose $\tilde{s}$ is simple and $\mathcal{M}^*$-measurable. Say

$$\tilde{s} = \sum_{i=1}^{N} \alpha_i \chi_{E_i}$$

where $\alpha_i > 0$ and $E_i \in \mathcal{M}^*$. But $\exists A_i \in \mathcal{M}$, $A_i \subset E_i$ and $\mu(A_i) = \mu^*(E_i)$. Let

$$s_1 = \sum \alpha_i \chi_{A_i}$$

Note that $s_1 \le \tilde{s} \le f$ and s_1 is $\mathfrak{M}$-measurable. Moreover,

$$\int_X s_1 \, d\mu = \int_X \tilde{s} \, d\mu^*$$

Hence $\int_X f \, d\mu^* \le \int_X f \, d\mu$.

PROPOSITION: (1) $f : X \to [0,\infty]$, f measurable. If $E \in \mathfrak{M}$ such that

$$(*) \qquad \int_E f \, d\mu = 0$$

Then $f = 0$ μ-a.e. on E.

Proof. Let $\Delta_n = \{x \in E : f(x) > 1/n\}$ $\forall n \in \mathbb{N}$. Then $\mu(\Delta_n) = 0$ by $(*)$, so $\mu\{x \in E : f(x) \neq 0\} = \mu(\bigcup \Delta_n) = 0$

(2) Suppose $f \in L^1(\mu)$, $f : X \to \mathbb{C}$. Suppose

$$\int_E f \, d\mu = 0 \quad \forall E \in \mathfrak{M}$$

Then $f = 0$ a.e. on X.

Proof. Write $f = u + iv$. Let

$$E = \{x \in X : u(x) > 0\} \in \mathfrak{M}$$

Then $\int_E f\,d\mu = 0 \Rightarrow \int_E u\,d\mu = 0 \Rightarrow \int_E u^+\,d\mu = 0$

$\Rightarrow u^+ = 0$ a.e. on E $\Rightarrow u^+ = 0$ a.e. on X, etc.

(3) Suppose $f \in L^1(\mu)$ and

$$\left| \int_X f\,d\mu \right| = \int_X |f|\,d\mu$$

Then $\exists\, \alpha \in \mathbb{C}$, $|\alpha| = 1$ s.t. $\alpha f = |f|$ a.e.

Proof. Set $z = \int_X f\,d\mu$. $\exists\, \alpha$, $|\alpha| = 1$, with $\alpha z = |z|$.

$$\left| \int_X f\,d\mu \right| = \alpha \int_X f\,d\mu = \int_X \alpha f\,d\mu = \int_X u\,d\mu \quad [u = \operatorname{Re} \alpha f]$$

$u \leq |f|$

$$\leq \int_X |f|\,d\mu$$

equality holds here by assumption

Hence $\int_X (|f| - u)\,d\mu = 0$ and $|f| - u \geq 0$ on X. By (1)

$|f| = u = \operatorname{Re} \alpha f$ a.e. on X.

$$|\alpha f| = \operatorname{Re} \alpha f \Rightarrow |f| = \alpha f \qquad \blacksquare$$

Review of Topology

X: topological space

<u>**Definition**</u>: $f : X \to [-\infty, \infty]$ is $\left\{\begin{matrix} \text{upper semi-continuous} \\ \text{lower semi-continuous} \end{matrix}\right\}$

if for every $\alpha \in \mathbb{R}$ $\left\{ x \in X : \begin{matrix} f(x) < \alpha \\ f(x) > \alpha \end{matrix} \right\}$ is open

Observations: (1) f is continuous iff f is both usc and lsc.

(2) The $\left\{\begin{matrix} \inf \\ \sup \end{matrix}\right\}$ of any family of $\left\{\begin{matrix} \text{usc} \\ \text{lsc} \end{matrix}\right\}$ functions is $\left\{\begin{matrix} \text{usc} \\ \text{lsc} \end{matrix}\right\}$

(3) χ_A is $\left\{\begin{matrix} \text{usc} \\ \text{lsc} \end{matrix}\right\}$ if A is $\left\{\begin{matrix} \text{closed} \\ \text{open} \end{matrix}\right\}$

<u>**Definition**</u>: $f : X \to \mathbb{C}$. The support of f is the closure of $\{x \in X : f(x) \neq 0\}$

<u>**Notation**</u>: If X is top. space, $C_c(X)$ denotes the collection of all complex-valued continuous functions on X with compact support

$C_c(X)$ is a vector space.

We write $K \prec f$ to mean

(1) K compact set in $\mathbb{C}$
(2) $f : X \longrightarrow [0,1]$, $f \in C_c(X)$
(3) $f(x) = 1 \quad \forall x \in K$

We write $f \prec V$ to mean
(1) V open
(2) $f \in C_c(X)$
(3) $\operatorname{supp} f \subset V$

<u>LEMMA</u>: X locally compact T_2-space. Suppose $K \subset U$ where K is compact and U is open. Then $\exists$ open V with compact closure s.t.

$$K \subset V \subset \overline{V} \subset U$$

Proof. Since K is compact and X is locally compact, $\exists$ G open, $G \supset K$ s.t. $\overline{G}$ is compact. Done if $X = U$. If $X \neq U$, consider closed set $C = X - U$. Consider $p \in C$. Since K is compact and X is T_2, $\exists$ open set $W_p \supset K$ s.t. $p \notin \overline{W_p}$. Consider the collection of closed sets

$$C \cap \overline{G} \cap \overline{W_p}$$

for $p \in C$. This collection has an empty intersection. Since $\overline{G}$ is compact, $\exists$ finite number of these sets with empty intersection. Suppose

$$(*) \qquad C \cap \overline{G} \cap \overline{W_{p_1}} \cap \dots \cap \overline{W_{p_M}} = \emptyset$$

Let $V := G \cap W_{p_1} \cap \dots \cap W_{p_M}$. Then V is open, $K \subset V$, $\overline{V} \subset U$ by $(*)$ since $C = X - U$, and $\overline{V}$ is compact since $\overline{V} \subset \overline{G}$

compact.

2/6 MEASURE THEORY

Recall that if X is a locally compact T_2 space and

$$\underset{\text{compact}}{K} \subset \underset{\text{open}}{U}$$

then $\exists$ open V with $\bar{V}$ compact and

$$K \subset V \subset \bar{V} \subset U$$

URYSOHN'S LEMMA: X loc. compact T_2 space. If compact $K \subset$ open U, then $\exists\, f : X \longrightarrow [0,1]$ s.t.

$$K \prec f \prec U$$

Proof. Let $r_0 = 0$, $r_1 = 1$, $(r_n)_{n=2}^{\infty}$ an enumeration of the rationals in $(0,1)$. $\exists$ open V_0 with $\overline{V_0}$ compact s.t.

$$K \subset V_0 \subset \overline{V_0} \subset U$$

Also, $\exists$ open V_1 with $\overline{V_1}$ compact s.t.

$$K \subset V_1 \subset \overline{V_1} \subset V_0 \subset \overline{V_0} \subset U$$

Suppose we have already defined V_{r_i} open with $\overline{V_{r_i}}$ compact for $0 \leq i \leq n$ and furthermore

$$r_i < r_j \Rightarrow \overline{V_{r_j}} \subset V_{r_i}$$

We specify $V_{r_{n+1}}$ as follows: Let r_i be the largest member of $\{r_0, \dots, r_n\}$ s.t. $r_i < r_{n+1}$. Let r_j be the smallest member of $\{r_0, \dots, r_n\}$ s.t. $r_j > r_{n+1}$. So

$$r_i < r_{n+1} < r_j$$

Hence $\overline{V_{r_j}} \subset V_{r_i}$. Let $V_{r_{n+1}}$ be open with $\overline{V_{r_{n+1}}}$ compact such that $\overline{V_{r_j}} \subset V_{r_{n+1}} \subset \overline{V_{r_{n+1}}} \subset V_{r_i}$

By induction we obtain a sequence of open sets V_r, $r \in \mathbb{Q} \cap [0,1]$ s.t. $\overline{V_r}$ is compact and

$$(*) \qquad r < s \Rightarrow \overline{V_s} \subset V_r$$

Define

$$f_r(x) := \begin{cases} r & \text{if } r \in V_r \\ 0 & \text{if } x \in X - V_r \end{cases}$$

Note f_r is lower semi-continuous. Let $f := \sup_r f_r$, so f is lsc.

Define

$$g_s(x) = \begin{cases} 1 & x \in \overline{V_s} \\ s & x \in X - \overline{V_s} \end{cases}$$

g_s is upper semi-continuous. Let

$$g := \inf_s g_s$$

Then g is usc.

$$K \subset V_r \ \forall r \Rightarrow f(x) = 1 \ \forall x \in K$$
$$\overline{V_r} \subset \overline{V_0} \subset U \ \forall r \Rightarrow f(x) = 0 \ \forall x \in X - \overline{V_0}$$

Hence $\operatorname{sup} f \subset \overline{V_0}$, a compact subset of U.

We must show f is continuous. It is sufficient to show $f = g$.

Suppose $f_r(x) > g_s(x)$ for some $x \in X$. Then $r > s$ and $x \in V_r$, $x \notin \overline{V_s}$. This contradicts construction of V_r (see (*)) Hence $\forall x \in X$ $f_r(x) \le g_s(x) \Rightarrow f(x) \le g(x)$. Suppose $f(x) < g(x)$. $\exists r, s \in \mathbb{Q} \cap [0,1]$ s.t.

$$f(x) < r < s < g(x)$$

$f_r(x) < r \Rightarrow x \notin V_r$ and $g_s(x) > s \Rightarrow x \in \overline{V_s}$. Again this contradicts (*). Hence $f = g$.

$\blacksquare$

COROLLARY: X loc. compact T_2 space, K compact $K \subset \bigcup_{i=1}^{N} V_i$, V_i open. Then $\exists h_i \prec V_i$ s.t. $\sum_{i=1}^{n} h_i = 1$ on K

Proof. For each $x \in K$, $\exists$ open W_x, $\overline{W_x}$ compact, $x \in W_x$ and $\overline{W_x} \subset$ some V_i. This gives open covering of K, so

$$K \subset \bigcup_{j=1}^{M} W_{x_j}$$

For $1 \leq i \leq N$, let $H_i = \bigcup \{ \overline{W_{x_j}} : \overline{W_{x_j}} \subset V_i \}$ (finite union)

H_i is compact and $K \subset \bigcup_{i=1}^{N} H_i$. Clearly $H_i \subset V_i$

By Urysohn's lemma, $\exists\, g_i$ s.t. $H_i \prec g_i \prec V_i$

Let
$$h_1 = g_1$$
$$h_2 = (1-g_1) g_2$$
$$\vdots$$
$$h_n = (1-g_1)(1-g_2)\cdots(1-g_{n-1})\, g_n$$

Trivially $\operatorname{supp} h_i \subset \operatorname{supp} g_i \subset V_i$.

CLAIM: $\sum_1^n h_i = 1 - \prod_1^n (1-g_i)$ $\left(\Rightarrow \sum h_i = 1 \text{ on } K \right)$

Proof. (By induction). Suppose $h_1 + \cdots + h_k = 1 - (1-g_1)\cdots(1-g_k)$
Add h_{k+1}:

$$\sum^{k+1} h_i = 1 - \prod^{k} (1-g_i) + g_{k+1} \prod_{i=1}^{k} (1-g_i)$$

$$= 1 - \left(\prod_{i=1}^{k} (1-g_i) \right) (1-g_{k+1})$$

$\blacksquare$

RIESZ REPRESENTATION THEOREM (weak version)

X loc. compact T_2 space. Suppose $\Lambda : C_c(X) \to \mathbb{C}$ is a positive linear functional. Then there is a σ-algebra $\mathfrak{M}$ of subsets of X and a unique positive measure μ on $\mathfrak{M}$ s.t.

(a) $\int_X f \, d\mu = \Lambda(f) \quad \forall f \in C_c(X)$

(b) $\mu(K) < \infty \quad K$ compact

(c) $\mu(E) = \inf \{\mu(V) : V \text{ open}, V \supset E\} \quad \forall E \in \mathfrak{M}$

(d) $\mu(E) = \sup \{\mu(K) : K \text{ compact}, K \subset E\} \quad \forall$ open E and $\forall E \in \mathfrak{M}$ with $\mu(E) < \infty$

(e) μ is complete

Proof of uniqueness: Suppose μ_1 and μ_2 are positive measures on $\mathfrak{M}$ which satisfies (a)-(e). By (c) and (d) it is sufficient to show $\mu_1(K) = \mu_2(K)$ $\forall$ compact $K \subset X$.

Given $\varepsilon > 0$. By (b) $\mu_2(K) < \infty$ and by (c) $\exists$ open $V \supset K$ s.t. $\mu_2(V) < \mu_2(K) + \varepsilon$. By Urysohn's lemma $\exists f$ s.t. $K \prec f \prec V$.

$$\mu_1(K) = \int_X \chi_K \, d\mu_1 \leq \int_X \chi_K f \, d\mu_1$$

$$\leq \int_X f \, d\mu_1 = \Lambda(f) = \int_X f \, d\mu_2$$

$$\leq \int_X \chi_V \, d\mu_2 = \mu_2(V) \leq \mu_2(K) + \varepsilon$$

Hence $\mu_1(K) \leq \mu_2(K)$. By symmetry $\mu_2(K) \leq \mu_1(K)$, so $\mu_1(K) = \mu_2(K)$.

2/8 MEASURE THEORY

Proof of Riesz Representation theorem (continued)

(K will always be compact, V always open)

Definition of $\mathfrak{M}$ and μ.

For V open, let $\mu(V) := \sup\{\Lambda f : f \prec V\}$. Note that μ is monotone, i.e. $V_1 \subset V_2 \Rightarrow \mu(V_1) \leq \mu(V_2)$. So for any $E \subset X$ let

$$\mu(E) := \inf\{\mu(V) : E \subset V\}$$

This is well-defined by monotonicity.

Let $\mathfrak{M}_F$ be the collection of all $E \subset X$ such that

1) $\mu(E) < \infty$
2) $\mu(E) = \sup\{\mu(K) : K \subset E\}$

Let $\mathfrak{M}$ be the collection of all $E \subset X$ s.t. $E \cap K \in \mathfrak{M}_F$ $\forall K$ (compact)

Observations: μ is monotone $(A \subset B \Rightarrow \mu(A) \leq \mu(B))$

This implies that μ is complete on $\mathfrak{M}$. Suppose $\mu(A) = 0$. Then clearly $A \in \mathfrak{M}_F$, and so $A \in \mathfrak{M}$

If $f \leq g$, f, g real-valued in $C_c(X)$, then $\Lambda f \leq \Lambda g$.

STEP I : $E_i \subset X$, $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$

First show $\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$. Consider $f \prec V_1 \cup V_2$. Let $\operatorname{supp} f = K \subset V_1 \cup V_2$. By corollary to Urysohn's lemma, $\exists\, g_i \prec V$ and $g_1 + g_2 = 1$ on K.
$i = 1, 2$

CLAIM: $f = fg_1 + fg_2$. Trivial for $x \in K$. But off K $f = 0$ so all terms are 0.

$\operatorname{supp} fg_1 \subset \operatorname{supp} g_1 \subset V_1 \Rightarrow fg_1 \prec V_1$. Similarly $fg_2 \prec V_2$

Now $\Lambda f = \Lambda fg_1 + \Lambda fg_2 \leq \mu(V_1) + \mu(V_2)$, and so $\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$

In the general case there is nothing to prove if some E_i has $\mu(E_i) = \infty$.

If all the $\mu(E_i)$'s are finite, given $\varepsilon > 0$ $\forall n \in \mathbb{N}$ $\exists\, V_n$ s.t.

$$\mu(V_n) < \mu(E_n) + \frac{\varepsilon}{2^n}$$

Let $V := \bigcup_{n=1}^{\infty} V_n \supset E := \bigcup_{n=1}^{\infty} E_n$. Suppose $f \prec V$. Then $\operatorname{supp} f \subset \bigcup_{i=1}^{N} V_i$ for some $N \Rightarrow f \prec \bigcup_{i=1}^{N} V_i$

$$\Lambda f \leq \mu\left(\bigcup_{i=1}^{N} V_i\right) \leq \sum_{i=1}^{N} \mu(V_i)$$

$$\leq \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon$$

Hence $\mu(V) \leq \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon$. But $E \subset V$, so $\mu(E) \leq \mu(V)$
Hence

$$\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

STEP II: If K is compact, $K \in \mathfrak{M}_F$ and

$$(*) \qquad \mu(K) = \inf \{ \Lambda f : K \prec f \}$$

First note that $(*)$ implies that $K \in \mathfrak{M}_F$

Suppose $K \prec f$. Select $\alpha \in (0,1)$ and let

$$V_\alpha = \{ x \in X : f(x) > \alpha \}$$

V_α is open and $K \subset V_\alpha$. Furthermore, if $g \prec V_\alpha$, then $\alpha g \leq f$. For if $x \in V_\alpha$, then $f(x) > \alpha \geq \alpha g(x)$. If $x \notin V_\alpha$, then $g(x) = 0 \leq f(x)$. Now $K \subset V_\alpha$, so $\mu(K) \leq \mu(V_\alpha) = \sup \{ \Lambda g : g \prec V_\alpha \} \leq \frac{1}{\alpha} \Lambda f$

Let $\alpha \to 1$; then

$$\mu(K) \leq \Lambda f$$

Hence $\mu(K) \leq \inf \{ \Lambda f : K \prec f \}$. In particular this shows that $\mu(K) < \infty$. Then if $\varepsilon > 0$, $\exists V \supset K$ s.t.

$$\mu(V) < \mu(K) + \varepsilon$$

By Urysohn's lemma $\exists f$ s.t. $K \prec f \prec V$. Then

$$\Lambda f \leq \mu(V) < \mu(K) + \varepsilon$$

and so $\inf\{\Lambda f : K \prec f\} \leq \mu(K)$

STEP III: V open, $\mu(V) < \infty \Rightarrow V \in \mathfrak{M}_F$

Must show $\mu(V) = \sup\{\mu(K) : K \subset V\}$. Suppose $\beta < \mu(V)$. Then $\exists\, f \prec V$ s.t. $\Lambda f > \beta$. Let $K = \operatorname{supp} f$. Consider open $W \supset K$. Certainly $f \prec W$, so

$$\mu(W) \geq \Lambda f > \beta$$

Hence $\mu(K) \geq \beta$. Since $K \subset V$, and $\beta < \mu(V)$ is arbitrary

$$\sup\{\mu(K) : K \subset V\} \geq \mu(V)$$

But trivially $\mu(K) \leq \mu(V)$ if $K \subset V$, so $\sup\{\mu(K) : K \subset V\} \leq \mu(V)$.

2/13 MEASURE THEORY

(PROOF OF RIESZ REPRESENTATION - CONTINUED)

STEP IV. Suppose $(E_i) \subset \mathfrak{M}_F$, disjoint. Let $E = \bigcup E_i$. Then

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$$

and if $\mu(E) < \infty$, then $E \in \mathfrak{M}_F$

Proof. First suppose $K_1 \cap K_2 = \phi$. Show $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(E_2)$
Urysohn's lemma with $K = K_1$ and $V = X - K_2$ says that $\exists f$
s.t. $K_1 \prec f \prec X - K_2$. $K_1 \cup K_2$ is compact, and so if $\varepsilon > 0$,
step II $\Rightarrow$ $\exists g$ with $K_1 \cup K_2 \prec g$ and

$$\Lambda g < \mu(K_1 \cup K_2) + \varepsilon$$

Certainly $g = \underbrace{(1-f)g}_{\succ K_2} + \underbrace{fg}_{\succ K_1}$. Hence

$$\mu(K_1 \cup K_2) > \Lambda g - \varepsilon = \Lambda(1-f)g + \Lambda fg - \varepsilon$$

$$\geq \mu(K_2) + \mu(K_1) - \varepsilon$$

$$\Longrightarrow \mu(K_1) + \mu(K_2) \leq \mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2)$$

Therefore $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$

General case: By step I

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

$E_i \in \mathfrak{M}_F \Rightarrow \exists$ compact $H_i \subset E_i$ s.t. $\mu(H_i) > \mu(E_i) - \varepsilon/2^i$

$$\forall N \in \mathbb{N},\ \mu(E) \geq \mu\left(\bigcup_{i=1}^{N} H_i\right) = \sum_{i=1}^{N} \mu(H_i) \geq \sum_{i=1}^{N} \mu(E_i) - \varepsilon$$

$$\Rightarrow \mu(E) \geq \sum_{i=1}^{\infty} \mu(E_i)$$

Suppose $\mu(E) < \infty$. Take $\varepsilon > 0$. $\exists N$ s.t.

$$\sum_{i=1}^{N} \mu(E_i) > \mu(E) - \varepsilon$$

Let $K = \bigcup_{i=1}^{N} H_i$. K is compact, $K \subset E$, and

(as above)

$$\mu(K) = \sum_{i=1}^{N} \mu(H_i) > \mu(E) - 2\varepsilon$$

Hence $E \in \mathfrak{M}_F$

STEP V: Suppose $E \in \mathfrak{M}_F$, $\varepsilon > 0$. $\exists K \subset E \subset V$ s.t. $\mu(V - K) < \varepsilon$.

Proof. $\mu(E) < \infty \Rightarrow \exists V \supset E$ s.t. $\mu(V) < \mu(E) + \varepsilon/2$
$E \in \mathfrak{M}_F \Rightarrow \exists K \subset E$ s.t. $\mu(E) < \mu(K) + \varepsilon/2$. Then

$$\mu(V) - \varepsilon/2 < \mu(E) < \mu(K) + \varepsilon/2$$

Since $V-K \subset V$, $\mu(V) < \infty$ and $V-K$ is open, by III $V-K \in \mathfrak{M}_F$, so

$$V = \underset{\in \mathfrak{M}_F}{K} \cup \underset{\in \mathfrak{M}_F}{(V-K)} \quad \text{(disjoint union)}$$

$$\Rightarrow \mu(V) = \mu(K) + \mu(V-K)$$

$$\Rightarrow \mu(V-K) = \mu(V) - \mu(K) < \varepsilon$$

STEP VI : $A, B \in \mathfrak{M}_F \Rightarrow A \cap B \in \mathfrak{M}_F,\ A \cup B \in \mathfrak{M}_F,\ A-B \in \mathfrak{M}_F$

Proof. $\text{V} \Rightarrow \exists\, K_1 \subset A_1 \subset V_1,\ \mu(V_1 - K_1) < \varepsilon$
$\exists\, K_2 \subset B_2 \subset V_2,\ \mu(V_2 - K_2) < \varepsilon$

$$A - B \subset V_1 - K_2 \subset (V_1 - K_1) \cup (K_1 - V_2) \cup (V_2 - K_2)$$

$$\Rightarrow \mu(A-B) \le \mu(V_1 - K_1) + \mu(K_1 - V_2) + \mu(V_2 - K_2)$$

$$< \mu(K_1 - V_2) + 2\varepsilon$$

Note $K_1 - V_2$ is compact subset of $A-B$, and so $A - B \in \mathfrak{M}_F$

$$A \cup B = A \cup (B-A) \quad \text{disjoint}$$

Since $\mu(A\cup B) < \infty$ by I, we see by IV that $A\cup B \in \mathcal{M}_F$

$A\cap B = A - (A-B)$ difference of sets in $\mathcal{M}_F$, so by the 1st part of proof $A\cap B \in \mathcal{M}_F$

STEP VII: $\mathcal{M}$ is a σ-algebra containing the Borel sets

Proof. 1) Suppose $E \in \mathcal{M}$. Show $X - E \in \mathcal{M}$.

$$(X-E)\cap K = \underbrace{K}_{\in \mathcal{M}_F} - \underbrace{(K\cap E)}_{\in \mathcal{M}_F \text{ since } E\in\mathcal{M}} \in \mathcal{M}_F \quad (\text{by VI})$$

2) Suppose $(E_i) \in \mathcal{M}$. Show $(\bigcup E_i)\cap K \in \mathcal{M}_F$. Let

$$B_1 = E_1 \cap K \in \mathcal{M}_F$$

(induction) $$B_n = (E_n \cap K) - \underbrace{\bigcup_{i=1}^{n-1} B_i}_{\in \mathcal{M}_F \text{ by induction assumption}} \in \mathcal{M}_F$$

B_n's disjoint, in $\mathcal{M}_F$, and

$$\bigcup_{n=1}^{\infty} B_n = \left(\bigcup_{i=1}^{\infty} E_i\right) \cap K$$

$$\bigcup_{n=1}^{\infty} B_n \subset K \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} B_n\right) < \infty \underset{(\text{IV})}{\Longrightarrow} \bigcup B_n \in \mathcal{M}_F$$

3) Suppose C closed. Show $C \in \mathcal{M}$ (Then $X \in \mathcal{M}$ and $\mathcal{M}$ contains all Borel sets)

But $C \cap K \in \mathcal{M}_F$ since $C \cap K$ is compact and all compact sets are in $\mathcal{M}_F$. Hence $C \in \mathcal{M}$

STEP VIII: $\mathcal{M}_F$ is precisely the collection of members of $\mathcal{M}$ of finite measure.

Proof. Suppose $E \in \mathcal{M}_F$. Then certainly $\mu(E) < \infty$. Then if K is compact, $E \cap K \in \mathcal{M}_F$ by steps II and VI. Hence $E \in \mathcal{M}$.

Suppose $E \in \mathcal{M}$, $\mu(E) < \infty$. Show $E \in \mathcal{M}_F$.
$\exists$ open $V \supset E$ s.t. $\mu(V) < \infty$. By steps III and V, $\exists$ compact K with $K \subset V$ and $\mu(V-K) < \varepsilon$. $E \cap K \in \mathcal{M}_F \Rightarrow$
$\exists$ H compact and $H \subset E \cap K$

$$\mu(H) > \mu(E \cap K) - \varepsilon$$

Now

$$E \subset (E \cap K) \cup (V-K)$$

$$\Rightarrow \mu(E) \leq \mu(E \cap K) + \mu(V-K) \leq \mu(E \cap K) + \varepsilon$$

$$\leq \mu(H) + 2\varepsilon$$

Hence compact $K \subset E$ and $\mu(K) \geq \mu(E) - 2\varepsilon$. Therefore $E \in \mathfrak{M}_F$

2/15 MEASURE THEORY

Chap. 2 : 4, 11, 14, 17, 20 (2/27)

(RIESZ Rep. THEOREM CONT.)

STEP IX : μ is a measure on $\mathfrak{M}$

Suppose $E_i \in \mathfrak{M}$, E_i disjoint. Step I $\Rightarrow$ it is only necessary to show

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} \mu(E_i)$$

Trivial if $\exists E_i$ s.t. $\mu(E_i) = +\infty$. So suppose $\mu(E_i) < \infty$ $\forall i$. By step VIII, $E_i \in \mathfrak{M}_F$ $\forall i$. But step IV is countable additivity on $\mathfrak{M}_F$, so we're done.

STEP X : $\Lambda f = \int f \, d\mu \quad \forall f \in C_c(X)$

Sufficient to do this for f real-valued, since both sides are linear functionals. Sufficient to show for all $f \in C_c(X)$ (real-valued) that

$$\Lambda f \leq \int f \, d\mu$$

for then

$$-\Lambda f = \Lambda(-f) \leq \int -f \, d\mu = -\int f \, d\mu$$

Given f real-valued, $f \in C_c(X)$, let $K = \operatorname{supp} f$. We have $f(X) \subset [a,b]$ for some $a<b$. Let $\varepsilon > 0$. Consider a partition $y_0 < a < y_1 < y_2 < \dots < y_n = b$ where

$$y_i - y_{i-1} < \varepsilon$$

Let

$$E_i = \{x \in X : y_{i-1} < f(x) \le y_i\} \cap K$$

E_i is Borel, hence $E_i \in \mathfrak{M}$. Also $\bigcup_{i=1}^n E_i = K$ and the E_i's are disjoint.

$\exists$ open $W_i \supset E_i$ s.t. $\mu(W_i) < \mu(E_i) + \varepsilon/n$. Let

$$R_i = \{x \in X : f(x) < y_i + \varepsilon\}$$

R_i is open and $R_i \supset E_i$. Let $V_i = R_i \cap W_i$. V_i open and $V_i \supset E_i$. Certainly

$$\mu(V_i) < \mu(E_i) + \varepsilon/n$$

Now

$$\bigcup_{i=1}^n V_i \supset \bigcup_{i=1}^n E_i = K$$

Corollary of Urysohn's lemma $\Rightarrow \exists\, h_i \prec V_i$ s.t. $\sum_{i=1}^n h_i = 1$ on K. Then

$$f = \sum_{i=1}^n f h_i$$

Since $K \prec \sum_{i=1}^{n} h_i$, by step II

$$\mu(K) \leq \Lambda\left(\sum_{i=1}^{n} h_i\right)$$

We also have $f h_i \leq (y_i + \varepsilon) h_i$ on X since on V_i $f < y_i + \varepsilon$ and off V_i $h_i = 0$. Also note that $f(x) > y_i - \varepsilon$ for $x \in E_i$

$$\Lambda(f) = \sum_{i=1}^{n} \Lambda(f h_i) \underset{(\Lambda \text{ positive})}{\leq} \sum_{i=1}^{n} (y_i + \varepsilon) \Lambda h_i$$

$$= \sum_{i=1}^{n} \underbrace{\left(|a| + y_i + \varepsilon\right)}_{\text{positive}} \Lambda h_i - |a| \sum_{i=1}^{n} \Lambda h_i$$

$$\leq \sum_{i=1}^{n} (|a| + y_i + \varepsilon) \Lambda h_i - |a| \mu(K)$$

$$\leq \sum_{i=1}^{n} (|a| + y_i + \varepsilon)\left[\mu(E_i) + \varepsilon/n\right] \quad \left[\Lambda h_i < \mu(V_i) < \mu(E_i) + \varepsilon/n\right]$$

$$- |a| \mu(K)$$

$$= \sum_{i=1}^{n} \left[(y_i + \varepsilon)(\mu(E_i)) + \frac{\varepsilon}{n}(|a| + y_i + \varepsilon)\right]$$

$$\left[\mu(K) = \sum \mu(E_i)\right]$$

$$= \sum_{i=1}^{n} (y_i - \varepsilon)\mu(E_i) + \frac{\varepsilon}{n} \sum_{i=1}^{n} (|a| + y_i + \varepsilon) + 2\varepsilon \mu(K)$$

$$\leq \int_X f\,d\mu + \varepsilon(|a|+|b|+\varepsilon) + 2\varepsilon\mu(K) \qquad \left[\!\left[f \geq \sum_{i=1}^{n}(y_i-\varepsilon)\chi_{E_i} \right]\!\right]$$

Let $\varepsilon \to 0$ to obtain $\Lambda f \leq \int_X f\,d\mu$

$\boxed{}\ \boxed{}\ \boxed{}\ \boxed{}\ \boxed{}$!!

DEFINITIONS: Borel measure is a measure on the Borel sets. A Borel measure is outer regular if $\forall$ Borel E,

$$\mu(E) = \inf\{\mu(V) : V \supset E,\ V \text{ open}\}$$

and is inner regular if $\forall$ Borel E

$$\mu(E) = \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}$$

A Borel measure is regular if it is both inner and outer regular.

DEFINITION: If X is a topological space, we say X is σ-compact if $X = \bigcup_{n=1}^{\infty} K_n$ for K_n compact.

DEFINITION: If X is a measure space $(X, \mathcal{M}, \mu)$ we say X is σ-finite if $X = \bigcup_{i=1}^{\infty} E_i$ for $\mu(E_i) < \infty$

<u>THEOREM</u>: Same hypothesis as Riesz Rep. Th^m, but add X is σ-compact. Then the μ of the conclusion satisfies

a) $\forall \varepsilon > 0$ $\forall E \in \mathfrak{M}$, $\exists$ closed F, open V s.t. $F \subset E \subset V$ and $\mu(V-F) < \varepsilon$

b) μ is regular

c) $E \in \mathfrak{M} \Rightarrow \exists F_\sigma \subset E \subset G_\delta$ s.t. $\mu(G_\delta - F_\sigma) = 0$

Proof. Given $X = \bigcup_{n=1}^{\infty} K_n$, K_n compact, let $E \in \mathfrak{M}$. Then $\mu(E \cap K_n) < \infty$ so $\exists$ open $V_n \supset E \cap K_n$ s.t.

$$\mu(V_n - (E \cap K_n)) < \varepsilon/2^n$$

Let $V = \bigcup V_n$. Then $V - E \subset \bigcup_{n=1}^{\infty} (V_n - (E_n \cap K))$, so

$$\mu(V-E) \leq \sum_{n=1}^{\infty} \mu(V_n - (E_n \cap K)) < \varepsilon$$

Apply to $X-E$ as well to get open $W \supset X-E$ with

$$\mu(W-(X-E)) < \varepsilon$$

Set $F = X - W$. Then F is closed and $F \subset E \subset V$. Also $\mu(E-F) = \mu(W-(X-E)) < \varepsilon$, so

$$\mu(V-F) < 2\varepsilon$$

2/17 MEASURE THEORY

THEOREM: Same hypothesis as RRT, but add that X is σ-compact. Then $\mathfrak{M}$ and μ of conclusion of RRT

a) $E \in \mathfrak{M} \Rightarrow \forall \varepsilon > 0 \ \exists$ closed F, open V s.t. $F \subset E \subset V$ and $\mu(V-F) < \varepsilon$

b) μ is regular

c) $E \in \mathfrak{M} \Rightarrow \exists$ an F_σ-set A, G_δ-set B s.t. $A \subset E \subset B$ and $\mu(B-A) = 0$

Proof. a) done

b) We have $X = \bigcup_{n=1}^{\infty} K_n$, K_n compact. Suppose F closed. Certainly

$$F = \bigcup_{n=1}^{\infty} (K_n \cap F)$$

and $\bigcup_{n=1}^{N} (K_n \cap F)$ is compact. Then

$$(*) \qquad \mu(F) = \lim_{N \to \infty} \mu\left(\bigcup_{n=1}^{N} (K_n \cap F)\right)$$

Given $E \in \mathfrak{M}$, (a) $\Rightarrow \exists$ closed $F \subset E$ s.t. $\mu(E-F) < \varepsilon$. Combined with $(*)$, we see

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$$

Thus μ is inner regular. But μ is outer regular from RRT.

(c) $A_n \subset E \subset B_n$, A_n closed, B_n open and $\mu(B_n - A_n) < 1/n$

Set

$$A = \bigcup_{n=1}^{\infty} A_n$$

$$B = \bigcap_{n=1}^{\infty} B_n$$

Then $A \subset E \subset B$ and $\mu(B-A) \leq \mu(B_n - A_n) < 1/n \ \forall n \in \mathbb{N}^*$, and so $\mu(B-A) = 0$

$\blacksquare$

<u>THEOREM</u>: Suppose X is a locally compact T_2-space. Suppose λ is a positive Borel measure on X such that $\lambda(K) < \infty$ for every compact K. <u>Suppose every open subset of X is σ-compact</u>. Then λ is regular.

Proof. Define

$$\Lambda f := \int_X f \, d\lambda \quad \forall f \in C_c(X)$$

Since $\lambda(K) < \infty$ for $K = \operatorname{supp} f$, $|\Lambda f| \leq M \lambda(K) < \infty$, $M = \max|f(x)|$

Hence Λ is a positive linear functional. By the RRT, there is a positive measure μ s.t. $\forall f \in C_c(X)$

$$\int_X f \, d\lambda = \int_X f \, d\mu$$

We know μ is regular since X is σ-compact.

Suppose V is open. We want to show $\lambda(V)=\mu(V)$. By our hypothesis, $V=\bigcup_{n=1}^{\infty} H_n$, H_n compact. By Urysohn's lemma, $\exists f_1$ s.t. $H_1 \prec f_1 \prec V$. Set $K_1 = \operatorname{supp} f_1$. Certainly $H_1 \subset K_1$. Suppose $f_1, \dots, f_n$ have been defined where $\operatorname{supp} f_j =: K_j$, and K_j is compact and $K_j \subset V$. Choose f_{n+1} s.t.

$$\underbrace{(H_1 \cup \dots \cup H_n \cup K_1 \cup \dots \cup K_n)}_{\text{compact}} \prec f_{n+1} \prec V$$

Claim $f_n \uparrow \chi_V$. Note $f_{n+1}=1$ on $K_n = \operatorname{supp} f_n$ and so $f_{n+1} \geq f_n$ everywhere since $0 \leq f_n \leq 1$ everywhere. Since $\bigcup H_n = V$, we see $f_{n+1} \uparrow \chi_V$.

We apply Monotone convergence theorem twice

$$\lambda(V) = \int_X \chi_V \, d\lambda = \lim \int_X f_n \, d\lambda = \lim \int_X f_n \, d\mu$$

$$= \int_X \chi_V \, d\mu = \mu(V)$$

Suppose E is a Borel set. Suppose $V \supset E$, V open. Then $\lambda(E) \leq \lambda(V) = \mu(V)$. μ is regular, so taking inf over all $V \supset E$, V open

$$\lambda(E) \leq \mu(E)$$

Given $\varepsilon>0$, $\exists$ closed F, open V s.t. $F\subset E\subset V$ and $\mu(V-F)<\varepsilon$. Then

$$\lambda(E) \geq \lambda(F) = \lambda(V) - \lambda(V-F) = \mu(V) - \mu(V-F)$$

(not $\neq \infty$) since λ, μ agree on open sets

$$\geq \mu(E) - \varepsilon$$

and so $\lambda(E) \geq \mu(E)$. Therefore $\lambda(E)=\mu(E)$ for every Borel set E, so λ is regular since μ is regular.

LEBESGUE MEASURE ON R^1

THEOREM: There exists a positive complete regular measure m on a σ-algebra $\mathfrak{M}$ containing the Borel sets in $\mathbb{R}$ s.t.

a) $m(I) =$ length of I (I interval)

b) $E\in\mathfrak{M}$ if and only if $\exists$ F_σ-set A, G_δ-set B s.t. $A\subset E\subset B$ and $m(B-A)=0$

c) $m(x+E) = m(E)$ $\forall x\in\mathbb{R}$

d) If μ is a positive Borel measure on $\mathbb{R}$ which is translation invariant and $\mu(K)<\infty$ $\forall K$ compact, then $\exists$ $c>0$ s.t. $\mu(E) = c\, m(E)$ $\forall$ Borel set E.

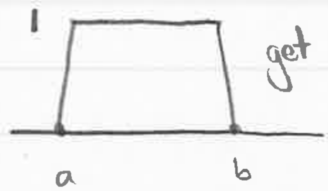
Proof. Define

$$\Lambda f := R\int_{\mathbb{R}} f(x)\,dx \quad \forall f\in C_c(\mathbb{R})$$

(Riemann integral). Λ is a positive linear functional. By RRT there is an m which is regular, complete measure on $\mathfrak{M} \supset$ Borel sets.

a) $m(I) = \sup\{\Lambda f : f \prec (a,b)\} = b-a$

$I = (a,b)$



get as close as you want

$m(\{x_0\}) = 0 \implies m(I) = \ell(I)$ for any interval I

b) shown in previous theorem

c) I open interval $\implies m(x+I) = m(I)$. If V is open,

$$V = \bigcup_{n=1}^{\infty} I_n$$

I_n disjoint, open, so $m(x+V) = m\left(\bigcup_{n=1}^{\infty}(x+I_n)\right) = m(V)$.
$m(E) = \inf\{m(V) : V \supset E, V \text{ open}\} \implies m(x+E) = m(E)$.

d) Let $\mu(0,1) = c > 0$ (WLOG since $c = 0 \implies \mu(\mathbb{R}) = 0 \implies \mu(E) = 0 = c\,m(E)\ \forall E$)

Suppose I = some interval, length $1/n$. Translation invariance $\implies \mu(I) = c/n$. μ is regular by the previous theorem. Also $\mu\{x_0\} = 0$ by translation invariance since otherwise we could show $\mu(0,1) = \infty$. Therefore $\mu(I) = c$ for any interval of length 1.

Given V open, $V = \bigcup I_n$, I_n disjoint, $\ell(I_n)$

is the reciprocal of some integer. Then

$$\mu(V) = c\, m(V) \quad \text{for every open } V$$

Since μ is regular, we get $\mu(E) = c\, m(E) \quad \forall$ Borel E

2/20 MEASURE THEORY

Remark: Consider the counting measure μ

$$\mu(E) = \# E$$

Certainly μ is not a scaler multiple of Lebesgue measure. Note $\mu(K) = \infty$ for lots of compact K.

Remark: Note $\int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} f\,dm \quad \forall f \in C_c(\mathbb{R})$

Riemann integral

In fact, we know everything necessary about Lebesgue measure now to show that every Riemann integrable function on $[a,b]$ is Lebesgue integrable (with the same value)

Recall

THEOREM: If E is a Lebesgue measurable set in R, $\varepsilon > 0$, $m(E) < \infty$, then $\exists$ open disjoint intervals $I_1, \dots, I_n$ s.t.

$$m\left(E \,\Delta\, \bigcup_{i=1}^{N} I_i\right) < \varepsilon$$

Sketch of proof. $\exists$ open $V \supset E$ s.t. $m(V) < m(E) + \varepsilon/2$ $\Rightarrow m(V - E) < \varepsilon/2$. V open $\Rightarrow V = \bigcup_{n=1}^{\infty} I_n$, I_n

disjoint intervals. $\exists N$ s.t. $\sum_{N+1}^{\infty} m(I_n) < \varepsilon/2$

$$\bigcup_1^N I_n - E \subset V - E \quad \text{measure} < \varepsilon/2$$

$$E - \bigcup_1^N I_n \subset \bigcup_{N+1}^{\infty} I_n \quad \text{measure} < \varepsilon/2$$

<u>LUSIN'S THEOREM</u>: X locally compact Hausdorff space. $(X, \mathfrak{M}, \mu)$ of sort produced by R.R.T. Suppose $f: X \to \mathbb{C}$ and f is $\mathfrak{M}$-measurable. Suppose $\exists A \subset X$, $\mu(A) < \infty$ s.t. $f(x) = 0$ $\forall x \in X - A$. Then $\exists g \in C_c(X)$ s.t.

$$\mu\{x \in X : f(x) \neq g(x)\} < \varepsilon$$

Moreover, if $\sup_X |f(x)| < \infty$, then g can be chosen so that $\sup |g(x)| \leq \sup |f(x)|$

Proof. First suppose $0 \leq f \leq 1$ and A is compact. $\exists$ simple functions $s_n \uparrow f$. Recall

$$t_n := s_n - s_{n-1} = 2^{-n} \chi_{T_n} \quad n > 1$$

$$t_1 := s_1 = 2^{-1} \chi_{T_1}$$

Then $f = \sum_{n=1}^{\infty} t_n$. Note $T_n \subset A$. A compact, X locally compact $\Rightarrow A \subset V \subset \overline{V}$ (compact). Since $\mu(T_n) < \infty$, $\exists$ compact K_n, open V_n s.t.

$$K_n \subset T_n \subset V_n \subset V \quad \text{and} \quad \mu(V_n - K_n) < \varepsilon 2^{-n}$$

Urysohn's lemma $\Rightarrow \exists\, h_n$ s.t. $K_n \prec h_n \prec V_n$. Set

$$g := \sum_{n=1}^{\infty} 2^{-n} h_n$$

Certainly g is continuous (uniform limit of continuous functions). Each $h_n = 0$ outside $V_n \subset V \Rightarrow \operatorname{supp} g \subset \overline{V}$ and hence compact. On K_n, $2^{-n} h_n = t_n$. Off V_n, $2^{-n} h_n = 0 = t_n$. Therefore $2^{-n} h_n = t_n$ off $V_n - K_n$

$$\Rightarrow g = f \text{ off } \bigcup_{n=1}^{\infty} (V_n - K_n)$$

But $\mu\left(\bigcup_{n=1}^{\infty} (V_n - K_n)\right) < \varepsilon$.

Remove simplifying assumptions. First suppose f is bounded. Work with the real and imaginary parts separately. For appropriate M and a,

$$\frac{\operatorname{Re} f}{M} + a \quad \text{takes values in } [0,1)$$

Then $M(g - a)$ should work for $\operatorname{Re} f$.

Now remove condition that A is compact. There exists a compact $K \subset A$ s.t. $\mu(A - K) < \varepsilon$. Then $f \chi_A$ agrees with off the set $A - K$ of measure $< \varepsilon$. We can obtain a suitable g for $f \chi_A$

For the general case, suppose f is unbounded and let

$$B_n := \{x \in X : |f(x)| > n\}$$

Then $\bigcap_{n=1}^{\infty} B_n = \emptyset$ (since f is complex-valued) and $B_n \subset A$, $\mu(A) < \infty$, so that $\lim_{n\to\infty} \mu(B_n) = 0$. $\exists\, n$ s.t. $\mu(B_n) < \varepsilon$.

Then $f(1-\chi_{B_n})$ agrees with f off the set B_n, and off B_n

$$|f(1-\chi_{B_n})| < n$$

On B_n, $f(1-\chi_{B_n}) = 0$. Now find g for $f(1-\chi_{B_n})$. This agrees with f off a set of measure $< 2\varepsilon$

Suppose $\sup |f(x)| = R < \infty$. Let

$$\varphi(z) = \begin{cases} z & |z| \le R \\ \dfrac{z}{|z|} R & |z| > R \end{cases}$$

This is continuous. We have $g \in C_c(X)$ s.t. $\mu\{x \in X : f(x) \ne g(x)\} < \varepsilon$.

Let $g_1 = \varphi \circ g$. The set $\{x \in X : f(x) \ne g_1(x)\} \subset \{x \in X : f(x) \ne g(x)\}$ and so $\mu\{x \in X : f(x) \ne g_1(x)\} < \varepsilon$. Clearly $\sup |g_1(x)| \le R$

$\blacksquare$

2/22 MEASURE THEORY

QUESTION: Suppose $f: X \to \mathbb{R}$ measurable (X locally compact Hausdorff) Does there exist a continuous g on X s.t. g approximates f well (in some sense) and, for instance, $g \le f$?

ANSWER <u>NO</u> Consider $X = \mathbb{R}$ with Lebesgue measure

$$f = \chi_{C_\alpha} \qquad C_\alpha \subset [0,1] \text{ Cantor set of positive measure}$$

Claim: Suppose $g: [0,1] \to R$ is lower semicontinuous and $g \le f$ on $[0,1]$. Then $g \le 0$

If $g(x_0) > 0$, then $g > 0$ on an open interval I containing x_0

But $I \not\subset C_\alpha$ ↯ ($\exists$ point where $f(x) = 0$ but $g(x) > 0$)

Thus the best approximation to f from below is $\tilde{g} = 0$

But note $f - \tilde{g} = 1$ on a set of measure $1 - \alpha > 0$

$$\int f - \int \tilde{g} = 1 - \alpha > 0$$

THEOREM: (VITALI - CARATHÉODORY) Suppose X is a locally compact Hausdorff space and $(X, \mathfrak{M}, \mu)$ of sort produced by RRT. Suppose $f: X \to \mathbb{R}$ is in $L^1(\mu)$. Then if $\varepsilon > 0$ there is an upper-semicontinuous u which is bounded above, a lower-semicontinuous v which is bounded below, such that

$$u \leq f \leq v$$

and
$$\int_X (u-v)\, d\mu < \varepsilon$$

Proof. First suppose $f \geq 0$. $\exists$ simple $s_n \uparrow f$. Let $s_0 = 0$ and

$$t_n := s_n - s_{n-1} \quad \forall n \in \mathbb{N}$$

Then $f = \sum_{n=1}^{\infty} t_n$ (converges everywhere on X). In fact

$$f = \sum_{i=1}^{\infty} c_i \chi_{E_i}$$

$c_i > 0$, E_i measurable (not in general disjoint). $f \in L^1(\mu)$ implies

$$\infty > \int_X f\, d\mu \underset{\text{MCT}}{=} \sum_{i=1}^{\infty} c_i \mu(E_i)$$

Hence $\mu(E_i) < \infty$, so because μ is "regular" on sets of finite measure, $\exists K_i \subset E_i \subset V_i$ s.t. K_i is compact, V_i open, and

$$c_i \mu(V_i - K_i) < \frac{\varepsilon}{2^{i+1}}$$

Also, $\exists N \in \mathbb{N}$ s.t. $\sum_{i=N+1}^{\infty} c_i \mu(E_i) < \varepsilon/2$.

Let

$$v = \sum_{i=1}^{\infty} c_i \chi_{V_i}$$

$$u = \sum_{i=1}^{N} c_i \chi_{K_i}$$

Certainly $u \leq f \leq v$.

$$v - u = \sum_{i=1}^{N} c_i (\chi_{V_i} - \chi_{K_i}) + \sum_{i=N+1}^{\infty} c_i \chi_{V_i}$$

$$\leq \sum_{i=1}^{\infty} c_i (\chi_{V_i} - \chi_{K_i}) + \sum_{i=N+1}^{\infty} c_i \chi_{E_i}$$

$$\Longrightarrow \int_X (v-u)\, d\mu \leq \sum_{i=1}^{\infty} c_i \mu(V_i - K_i) + \sum_{i=N+1}^{\infty} c_i \mu(E_i)$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Claim: v is lower semi-continuous. Suppose $v(x_0) > \alpha$
$\exists$ M s.t.

$$\sum_{i=1}^{M} c_i \chi_{V_i}(x_0) > \alpha$$

V_i open $\Rightarrow \sum_{i=1}^{M} c_i \chi_{V_i}(x) > \alpha$ on a nbhd of x_0

Claim: u is upper semi-continuous. Suppose $u(x_0) < \alpha$

$$\sum_{i=1}^{N} c_i \chi_{K_i}(x_0) < \alpha$$

Let $I = \{ i : 1 \le i \le n,\ x_0 \in K_i \}$. $\bigcup_{\substack{i=1 \\ i \notin I}}^{n} K_i$ closed not containing x_0, so on $X - \bigcup_{\substack{i=1 \\ i \notin I}}^{n} K_i$, we have $u(x) \le u(x_0) < \alpha$

General Case: $f = f^+ - f^-$. Obtain

$$u_1 \le f^+ \le v_1$$
$$u_2 \le f^- \le v_2$$

Then $\underbrace{u_1 - v_2}_{u} \le \underbrace{f^+ - f^-}_{f} \le \underbrace{v_1 - u_2}_{v}$. Certainly

$$\int_X [(v_1 - u_2) - (u_1 - v_2)]\, d\mu < 2\varepsilon$$

$u \le u_1$ bdd above, $v \ge -u_2$ bdd below. u is usc if we show the sum of two usc is usc

[Suppose h_1, h_2 are u.s.c. Show $h_1 + h_2$ is u.s.c.
Given $\alpha \in \mathbb{R}$. For real r, let

$$E_r = \{x \in X : h_1(x) < r\} \cap \{x \in X : h_2(x) < \alpha - r\}$$
(open)

Claim : $A := \{x \in X : h_1(x) + h_2(x) < \alpha\} = \bigcup_{r \in \mathbb{R}} E_r$

Clearly $\bigcup_{\mathbb{R}} E_r \subset A$. Given $x \in A$, $\exists r \in \mathbb{R}$ s.t.

$$0 < r - h_1(x) < \alpha - h_1(x) - h_2(x)$$

$$\Longrightarrow x \in E_r$$

]

L^p - SPACES

DEFINITION: $f : (a,b) \to \mathbb{R}$ is convex if

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y)$$

$\forall \lambda \in [0,1]$, $\forall x, y \in (a,b)$

Reminder: (1) f is continuous, differentiable off a countable set

(2) if $a < s < t < u < b$, then

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(t)}{u - t}$$

(3) Jensen's inequality for $\mathbb{R}$

<u>THEOREM (JENSEN'S INEQUALITY)</u> Suppose $(X, \mathcal{M}, \mu)$ is a measure space, $\mu(X) = 1$. Suppose $f : X \to (a,b)$ is in $L^1(\mu)$ and $\varphi : (a,b) \to \mathbb{R}$ is convex. Then

$$\varphi\left(\int_X f\,d\mu\right) \leq \int_X (\varphi \circ f)\,d\mu$$

Remarks: (1) First notice $a < \int_X f\,d\mu < b$ since $\mu(X) = 1$ and $f(x) - a > 0 \ \forall x \in X$

(2) Also, φ convex $\Rightarrow$ φ continuous $\Rightarrow$ $\varphi \circ f$ measurable.

(3) $\int \varphi \circ f\,d\mu$ could be $+\infty$

2/24 MEASURE THEORY

Proof of Jensen's inequality: Set $t = \int_X f \, d\mu \in (a,b)$. If $a < s < t < u < b$, then

$$(*) \qquad \frac{\varphi(t) - \varphi(s)}{t-s} \leq \frac{\varphi(u) - \varphi(t)}{u-t}$$

Let

$$\beta := \sup_{s<t} \frac{\varphi(t) - \varphi(s)}{t-s}$$

(left derivative of φ at t). For all $y \in (a,b)$

$$\varphi(y) \geq \varphi(t) + \beta(y-t)$$

from definition of β and $(*)$. Therefore $x \in X$ implies

$$\varphi(f(x)) \geq \underbrace{\varphi(t) + \beta(f(x)-t)}_{\in L^1(\mu)}$$

Hence $(\varphi \circ f)^- \in L^1(\mu)$. Now we consider

$$\int_X (\varphi \circ f) \, d\mu := \int_X (\varphi \circ f)^+ \, d\mu - \int_X (\varphi \circ f)^- \, d\mu$$

$$\geq \int_X \left(\varphi(t) + \beta(f(x)-t)\right) d\mu$$

$$= \varphi(t) + \beta\left(\int f\,d\mu - t\right) \quad \left(\text{used here } \mu(X)=1\right)$$

$$= \varphi(t) = \varphi\left(\int_X f\,d\mu\right)$$

$\blacksquare$

EXAMPLE: $X = (0,1)$ μ Lebesgue measure

$$f(x) := \frac{1}{\sqrt{x}} \in L^1(\mu) \qquad \varphi(t) = e^t$$

Then
$$\int (\varphi \circ f)\,d\mu = \int_0^1 e^{1/\sqrt{x}}\,dx \geq \int_0^1 \frac{1}{x^2}\,dx = \infty$$

L^p - SPACES

THEOREM: Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Suppose $f: X \to [0,\infty]$ and $g: [0,\infty)$ are measurable. Suppose $1 < p < \infty$ and $1/p + 1/q = 1$. Then

(i) (HÖLDER)
$$\int_X fg\,d\mu \leq \left(\int f^p\,d\mu\right)^{1/p} \left(\int g^q\,d\mu\right)^{1/q}$$

(ii) (MINKOWSKI)
$$\left(\int_X (f+g)^p\,d\mu\right)^{1/p} \leq \left(\int f^p\,d\mu\right)^{1/p} + \left(\int g^p\,d\mu\right)^{1/p}$$

Proof. (i) Let $A = \left(\int f^p \, d\mu \right)^{1/p}$

$B = \left(\int g^q \, d\mu \right)^{1/q}$

If $A = 0$, result trivial since this implies $f = 0$ a.e. If $B = \infty$, RHS is ∞, so inequality okay. So only case needing serious discussion is $0 < A, B < \infty$.

Set $F := f/A$

$G := g/B$

Note

$$\int F^p \, d\mu = \frac{1}{A^p} \int_X f^p \, d\mu = 1$$

$$\int G^q \, d\mu = \frac{1}{B^q} \int_X g^q \, d\mu = 1$$

Suppose $x \in X$ s.t. $F(x)G(x) > 0$. $\exists \; s, t \in \mathbb{R}$ s.t.

$$F(x) = e^{s/p}, \quad G(x) = e^{t/q}$$

Then $F(x)\,G(x) = e^{s/p + t/q} \leq \frac{1}{p} e^s + \frac{1}{q} e^t = \frac{1}{p} F(x)^p + \frac{1}{q} G(x)^q$

e^x convex

$1/p + 1/q = 1$

In fact this holds for all x, and so

$$\int_X FG\,d\mu \leq \frac{1}{p}\int F^p d\mu + \frac{1}{q}\int G^q d\mu = \frac{1}{p}+\frac{1}{q} = 1$$

$$\Rightarrow \int_X fg\,d\mu \leq AB$$

(ii) Apply Holder to f and $(f+g)^{p-1}$

$$\int_X f(f+g)^{p-1} d\mu \leq \left(\int_X f^p d\mu\right)^{1/p} \left(\int (f+g)^{(p-1)q} d\mu\right)^{1/q}$$

Also

$$\int_X g(f+g)^{p-1} d\mu \leq \left(\int g^p d\mu\right)^{1/p} \left(\int (f+g)^{(p-1)q} d\mu\right)^{1/q}$$

Note $(p-1)q = p$. Adding

$$(*) \quad \int_X (f+g)^p d\mu \leq \left(\int (f+g)^p d\mu\right)^{1/q} \left[\left(\int f^p d\mu\right)^{1/p} + \left(\int g^p d\mu\right)^{1/p}\right]$$

If $\int (f+g)^p d\mu = 0$, result is trivial. If RHS of (ii) is $+\infty$, result is trivial. Now t^p is a convex function for $0<t<\infty$ and so

$$\left(\frac{f+g}{2}\right)^p \leq \frac{1}{2}\left(f^p + g^p\right)$$

Hence we may assume $\int (f+g)^p d\mu < \infty$.

Now divide (*) by $\left(\int (f+g)^p d\mu\right)^{1/q}$ and use $1-1/q=1/p$ to obtain result

$\blacksquare$

DEFINITION: $(X, \mathcal{M}, \mu)$ measure space. $L^p(\mu)$ $(0<p<\infty)$ is the set of all complex-valued measurable f s.t.

$$\|f\|_p := \int |f|^p d\mu < \infty$$

(If $X = \mathbb{N}$, μ counting measure, $\mathcal{M} = \mathcal{P}(\mathbb{N})$, then we denote $L^p(\mu)$ by ℓ_p)

DEFINITION: $(X, \mathcal{M}, \mu)$ measure space, $f: X \longrightarrow [0,\infty]$
Let

$$S := \{\alpha \in \mathbb{R} : \mu\left(f^{-1}(\alpha, +\infty]\right) = 0\}$$

If $S = \phi$, set $\beta = +\infty$. If $S \neq \phi$, set $\beta := \inf S$. β is called the essential supremum of f. Note $\beta \in S$ since

$$\mu\left(f^{-1}(\beta, \infty]\right) = \bigcup_{n=1}^{\infty} \mu\left(f^{-1}(\beta + 1/n, \infty]\right) = 0$$

Now let $L^\infty(\mu)$ be the set of all complex-valued measurable f such that

$$\|f\|_\infty := \operatorname{ess\,sup} |f| < \infty$$

Remark: Suppose $f \in L^\infty(\mu)$. For $0 \le \lambda < \infty$, then

$$|f(x)| \le \lambda \text{ a.e.} \iff \lambda \geq \|f\|_\infty$$

Proof. Suppose $|f(x)| \le \lambda$ a.e. Then $\lambda \in S$ (for $|f|$) and so $\operatorname{ess\,sup} |f| \le \lambda \implies \|f\|_\infty \le \lambda$.

Suppose $\|f\|_\infty \le \lambda$. Then $\lambda \in S \implies |f(x)| \le \lambda$ a.e.

THEOREM: Suppose $1 \le p$, $1/p + 1/q = 1$. Suppose $f : X \to \mathbb{C}$ is in L^p and $g : X \to \mathbb{C}$ is in L^q. Then $fg \in L^1$ and

$$\|fg\|_1 \le \|f\|_p \|g\|_q$$

Proof. For $1 < p < \infty$ this is Hölder's inequality. Suppose $p = 1$, so $g \in L^\infty(\mu)$. Then

$$|f(x) g(x)| \le |f(x)| \|g\|_\infty \text{ a.e.}$$

$$\implies \int |fg| \le \left(\int |f| \right) \|g\|_\infty < \infty$$

$$\implies \|fg\| \le \|f\|_1 \|g\|_\infty$$

$\boxtimes$

THEOREM: $f, g \in L^p(\mu)$, $1 \le p \le \infty$. Then

$$\|f+g\|_p \le \|f\|_p + \|g\|_q$$

Proof. If $1 < p < \infty$, Minkowski.

If $p = 1$: $|f+g| \le |f| + |g|$ - integrate

If $p = \infty$: $|f(x)| \le \|f\|_\infty$ a.e.

$|g(x)| \le \|g\|_\infty$ a.e.

$$\Longrightarrow \quad |f| + |g| \le \|f\|_\infty + \|g\|_\infty \quad \text{a.e.}$$

$$\Longrightarrow \quad \|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$$

2/27 MEASURE THEORY Ch. 3 8, 14, 15, 20 3/13

If $f, g \in C_c(\mathbb{R})$, let

$$d(f,g) := \int_{-\infty}^{\infty} |f(x) - g(x)|\, dx$$

What is the completion of $C_c(\mathbb{R})$ with this metric?

"DEFINITION" Given $(X, \mathcal{M}, \mu)$. We define $f \sim g$ if $f = g$ a.e. for f, g measurable. The "new" $L^p(\mu)$ is the space of equivalence classes under the above equivalence relation, of the "old" $L^p(\mu)$.

$$\|\tilde{f}\|_p = \|f\|_p \quad \text{for any } f \in \tilde{f}$$

The "new" $L^p(\mu)$ is a normed vector space. This gives a metric defined by

$$d(f,g) := \|f - g\|_p$$

THEOREM: For $1 \leq p \leq \infty$, $L^p(\mu)$ is complete.

Proof. $1 \leq p < \infty$. Let (f_n) be Cauchy in L^p. We want to find $f \in L^p$ such that $\|f_n - f\|_p \to 0$.

(f_n) Cauchy $\implies \exists N_i > 0$ such that $N_{i+1} > N_i$ and

$$n, m > N_i \implies \|f_n - f_m\|_p < 1/2^i$$

Suppose $n_i > N_i$. Then

$$\| f_{n_{i+1}} - f_{n_i} \| < 1/2^i$$

Consider

$$g_k := |f_{n_1}| + \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|$$

$$g = |f_{n_1}| + \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|$$

Then $g_k \longrightarrow g$. Moreover

$$\|g_k\|_p \leqslant \|f_{n_1}\|_p + \sum_{i=1}^{k} 2^{-i} < 1 + \|f_{n_1}\|_p$$

By Fatou's lemma

$$\int_X g^p \, d\mu \leqslant \varliminf_{k\to\infty} \int_X g_k^p \, d\mu < (1 + \|f_{n_1}\|_p)^p < \infty$$

Then g is measurable and $\int_X g^p d\mu < \infty \Rightarrow g(x) < \infty$ a.e. $[\mu]$
Define $f : X \to \mathbb{C}$ as follows

$$f(x) := \begin{cases} f_{n_1}(x) + \sum_{i=1}^{\infty} f_{n_{i+1}}(x) - f_{n_i}(x) & \text{if } g(x) < \infty \\ 0 & \text{if } g(x) = \infty \end{cases}$$

Then f is measurable, and $f(x) = \lim_{i\to\infty} f_{n_i}(x)$ a.e.

Claim: $f \in L^p(\mu)$ and $\|f_n - f\|_p \to 0$.

Let $\varepsilon > 0$. $\exists N > 0$ s.t. $n,m > N \Rightarrow \|f_n - f_m\|_p < \varepsilon$. Let $m > N$. Then by Fatou

$$(*) \int_X |f - f_m|^p \, d\mu \leq \varliminf_{i \to \infty} \int |f_{n_i}(x) - f_m(x)|^p \, d\mu \leq \varepsilon^p$$

Hence $f - f_m \in L^p \Rightarrow f \in L^p$. Moreover $(*)$ shows that $\|f - f_m\|_p \to 0$.

$p = \infty$: Let (f_n) be Cauchy in L^∞. For $n,m \in \mathbb{N}$, let

$$B_{nm} := \{x \in X : |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}$$

Definition of $\|\cdot\|_\infty \Rightarrow \mu(B_{nm}) = 0$. Let

$$B := \bigcup_{n=1}^\infty \bigcup_{m=1}^\infty B_{nm}$$

Then $\mu(B) = 0$. Off B, $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$. Hence $(f_n(x))$ is uniformly Cauchy on $X - B$, so $\exists f$ on $X - B$ such that $f_n \to f$ uniformly on $X - B$. Let $f := 0$ on B. Then f is measurable. For large enough n, $\|f_n - f\|_\infty \leq 1$, so $\|f\|_\infty \leq 1 + \|f_n\|_\infty < \infty$. Hence $f \in L_\infty(\mu)$. But also $\|f_n - f\|_\infty \to 0$ since $f_n \to f$ uniformly on $X - B$.

$\square$

THEOREM: $(X, \mathcal{M}, \mu)$ measure space, $1 \le p < \infty$. Let

$$S := \left\{ s : s \text{ simple, complex-valued} \atop s(A) = 0 \text{ for some } \mu(X-A) < \infty \right\}$$

Then S is dense in $L^p(\mu)$.

Proof. First suppose $f \in L^p(\mu)$ and $f \ge 0$. Then there exist simple s_n, $0 \le s_n \le f_n$ with $s_n \uparrow f$ on X. But $\int f^p \, d\mu < \infty$ implies s_n vanishes off a set of finite measure. Hence $s_n \in S$.

Now $0 \le (f - s_n)^p \le f^p$ on X and $f - s_n \to 0$ a.e. on X. Also $\int f^p \, d\mu < \infty \Rightarrow f^p \in L^1(\mu)$. Hence the dominated convergence theorem says that

$$\lim_{n \to \infty} \int_X (f - s_n)^p \, d\mu = 0$$

$$\Longrightarrow \| f - s_n \|_p \longrightarrow 0$$

For a general f, $f = (\operatorname{Re} f)^+ - (\operatorname{Re} f)^- + i\left[(\operatorname{Im} f)^+ - (\operatorname{Im} f)^- \right]$ approximate each term on right separately. ▨

REMARK: This is false if $p = \infty$.

Take $f = 1$ on $\mathbb{R}$, μ = Lebesgue measure. If $s \in S$ then $\| f - s \|_\infty \ge 1$.

PROPOSITION: Suppose $(X, \mathcal{M}, \mu)$ is a measure space where μ has the properties of the conclusion of the RRT. Then $C_c(X)$ is dense in $L^p(\mu)$ for $1 \leq p < \infty$

Proof. Take $s \in S$. Lusin's theorem implies if $\varepsilon > 0$ $\exists\, g \in C_c(X)$ s.t.

$$(1) \quad \mu(\underbrace{\{x \in X : g(x) \neq s(x)\}}^{E}) < \varepsilon$$

$$(2) \quad \|g\|_\infty \leq \sup_{s \in X} |s(x)| \leq \|s\|_\infty$$

Then

$$\int_X |g-s|^p \, d\mu = \int_E |g-s|^p \, d\mu \leq (2\|s\|_\infty)^p \varepsilon$$

and so $\|g-s\|_p \leq 2\|s\|_\infty \varepsilon^{1/p}$

Given $f \in L^p(\mu)$, $\exists\, s \in S$ such that $\|f-s\|_p < \eta/2$ by the previous theorem. The above calculation shows that $\exists\, g \in C_c(X)$ s.t. $\|s-g\|_p < \eta/2$, and so $\|f-g\| \leq \eta$. ▨

This shows that $C_c(\mathbb{R})$ is a dense subset of $L^1(\mu)$ and so $L^1(\mu)$ is the completion of $C_c(\mathbb{R})$.

3/1 MEASURE THEORY

Remarks: ① Consider the proof that $L^p(\mu)$ is complete, $1 \le p \le \infty$. In the proof we also showed that if $f_n \to f$ in $L^p(\mu)$, then there is a subsequence f_{n_k} s.t. $f_{n_k}(x) \to f(x)$ a.e. $[\mu]$

② We showed that $C_c(X)$ is a dense subset of $L^p(\mu)$ where $(X, \mathcal{M}, \mu)$ is a measure space satisfying conclusions of RRT. This statement could not possibly be true if there was no relationship between the topology on X and the measurable sets $\mathcal{M}$. For example, consider $\mathbb{R}$ with the usual topology, and let μ be the counting measure on the subsets of $\mathbb{R}$. Then $C_c(\mathbb{R}) \not\subset L^1(\mu)$.

Recall: If X is a metric space, define, for Cauchy sequences (x_n) and (y_n) in X,

$$(x_n) \sim (y_n) \quad \text{if} \quad \lim_{n\to\infty} d(x_n, y_n) = 0$$

Let S = set of equivalence classes. If $s \in S$, $t \in S$, let

$$\tilde{d}(s,t) = \lim_{n\to\infty} d(x_n, y_n)$$

where $(x_n) \in s$, $(y_n) \in t$. Check

① $\lim d(x_n, y_n)$ exists

② $\tilde{d}(s,t)$ well-defined

③ $\tilde{d}$ is a metric on S

(4) $(S, \tilde{d})$ is complete

Regard $X \subset S$ in following sense. Suppose $a \in X$. The constant Cauchy sequence $a, a, a, \ldots$ belongs to an equivalence class $\tilde{a} \in S$. Identify a with $\tilde{a}$. Check

(5) X is dense in S

(6) Any complete metric space Z of which X is a dense subset is isometric to $(S, \tilde{d})$

Recall: For $1 \leq p < \infty$, $f, g \in C_c(\mathbb{R}^k)$. Define

$$d_p(f,g) = \|f-g\|_p$$

We know $(C_c(\mathbb{R}^k), d_p)$ is a metric space which is a dense subset of L^p (Lebesgue measure on $\mathbb{R}^k$) which is itself a complete metric space. Thus $L^p(\mathbb{R}^k)$ is the completion of $(C_c(\mathbb{R}^k), d_p)$

<u>QUESTION</u>: What is the completion of $(C_c(\mathbb{R}^k), d_\infty)$ where

$$d_\infty(f,g) := \sup_{x \in \mathbb{R}^k} |f(x) - g(x)|$$

Take $f = 1$ on $\mathbb{R}^k$. Then $\|f-g\|_\infty \geq 1 \quad \forall g \in C_c(\mathbb{R}^k)$ and so $C_c(\mathbb{R}^k)$ is not dense in $L^\infty(\mathbb{R}^k)$.

Suppose $g_n \in C_c(\mathbb{R}^k)$, $g_n \to g$ in $L^\infty(\mathbb{R}^k)$

$$\|g_n - g\|_\infty \to 0 \implies g_n \text{ is uniformly Cauchy on } \mathbb{R}^k$$

Hence $g_n \to h$ uniformly on $\mathbb{R}^k$, h continuous, and so $h = g$ a.e, i.e. g is equal a.e. to a function continuous everywhere.

( is not an acceptable g)

<u>DEFINITION</u>: X locally compact T_2-space. $f : X \to \mathbb{C}$ vanishes at ∞ if $\forall \varepsilon > 0$ $\exists$ compact K s.t. $|f(x)| < \varepsilon$ if $x \in X - K$.

$C_0(X)$ denotes the space of continuous functions vanishing at ∞

<u>THEOREM</u>: X loc. compact T_2-space. For $f, g \in C_c(X)$ let

$$d(f,g) := \sup_{x \in X} |f(x) - g(x)|$$

The completion of $(C_c(X), d)$ is $(C_0(X), d)$.

Proof. Clearly $C_c(X) \subset C_0(X)$. Must show $C_c(X)$ is dense in $C_0(X)$ and $C_0(X)$ is complete.

Choose $f \in C_0(X)$ and let $\varepsilon > 0$. $\exists$ K compact in X s.t. $|f(x)| < \varepsilon$ if $x \in X - K$. By Urysohn's lemma there is a $g \in C_c(X)$ s.t. $K \prec g \prec X$. Set $h := fg$. Certainly h is continuous, and $\operatorname{supp} h \subset \operatorname{supp} g$, and so is compact.

$$h - f = f(1-g) = 0 \quad \text{on } K$$
$$|h - f| < \varepsilon \qquad \text{on } X - K$$

Hence $d(f,h) < \varepsilon$, so $C_c(X)$ is dense in $C_0(X)$.

Suppose (f_n) Cauchy in $C_0(X)$. Let $\varepsilon > 0$. Since (f_n) is uniformly Cauchy on X, there is a continuous $f : X \to \mathbb{C}$ s.t. $f_n \to f$ uniformly. Then $\exists N$ s.t.

$$\sup_{x \in X} |f_N(x) - f(x)| < \varepsilon/2$$

There is a compact K s.t. $|f_N(x)| < \varepsilon/2$ for $x \in X - K$. Hence $|f(x)| < \varepsilon$ for $x \in X - K$, so $f \in C_0(X)$. Therefore $C_0(X)$ is complete.

Things to look out for in Hilbert space chapter

Riesz-Fisher thm
Parseval's thm
Bessel Inequality
Fejer thm
Characterization of the continuous linear functionals on Hilbert space

3/3 Measure Theory

DEFINITION: Suppose H is a vector space over $\mathbb{C}$. If there is a function $(\cdot|\cdot) : H\times H \to \mathbb{C}$ satisfying the following conditions, we say H is an inner product space

(i) $(x|y) = \overline{(y|x)}$
(ii) $(x_1+x_2|y) = (x_1|y) + (x_2|y)$
(iii) $(\alpha x|y) = \alpha(x|y)$
(iv) $(x|x) \geq 0$
(v) $(x|x) = 0 \iff x = 0$

Properties: (a) $(0|y) = (y|0) = 0$ [(i) and (iii)]
(b) For a fixed y, the map $(\cdot|y)$ is a linear functional on H.
(c) $(x|\alpha y) = \bar{\alpha}(x|y)$
$(x|y_1+y_2) = (x|y_1) + (x|y_2)$

DEFINITION: For $x\in H$ let $\|x\| := (x|x)^{1/2}$

SCHWARZ INEQUALITY: $|(x|y)| \leq \|x\|\,\|y\|$

Proof: Let $A = \|x\|$, $B = |(x|y)|$, $C = \|y\|$. There is an $\alpha \in \mathbb{C}$ s.t. $\alpha(y|x) = B$. For every $r \in \mathbb{R}$,

$$0 \leq (x-\alpha r y \,|\, x - \alpha r y) = (x|x) - \alpha r(y|x) - \bar{\alpha} r(x|y) + r^2(y|y)$$

$$\Rightarrow 0 \leq A^2 - rB - r\bar{B} + r^2 C^2 \quad \forall r \in \mathbb{R}$$

If $C=0$, then $B=0$, so result holds. If $C \neq 0$, then

$$(2B)^2 - 4A^2C^2 \leq 0$$

(otherwise quadratic is <0 for some r)

$$\Rightarrow \quad B \leq AC$$

<u>Triangle Inequality</u> — $\|x+y\| \leq \|x\| + \|y\|$.

$$\|x+y\|^2 = (x+y\,|\,x+y) = (x|x) + (y|x) + (x|y) + (y|y)$$

$$= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\,(x|y).$$

$$\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \qquad [\![\text{Schwartz}]\!]$$

$$= (\|x\| + \|y\|)^2$$

DEFINITION: If $x, y \in H$, let

$$d(x,y) := \|x-y\|$$

This defines a metric on H. H is called a Hilbert space if H is complete in this metric.

EXAMPLE: (a) Consider a measure space $(X, \mathcal{M}, \mu)$. For $f \in L^2(\mu)$, $g \in L^2(\mu)$, define

$$(f|g) := \int_X f \bar{g} \, d\mu$$

(Note that Holder $\Rightarrow f\bar{g} \in L^1(\mu)$) The Hilbert space norm obtained from this inner product is just the L^2 norm, and so $L^2(\mu)$ is a Hilbert space.

(b) Let $\mathbb{C}^n := \{(x_1, \dots, x_n)^{*x} : x_k \in \mathbb{C}\}$. Define

$$(x|y) := \sum_{k=1}^{n} x_k \overline{y_k}$$

This is $L^2(\mu)$ where $X = \mathbb{N}_n^*$ and μ is the counting measure. This is also a Hilbert space

(c) $H = C[0,1]$. $f, g \in H$

$$(f|g) := \int_0^1 f(t) \overline{g(t)} \, dt$$

This gives an inner product space. Let $h = \chi_{(1/2, 1]} \in L^2[0,1]$. $\exists$ continuous g_n s.t. $\|g_n - h\|_2 \to 0$. Then $(g_n) \subset H$ and is Cauchy in (H, d). If $g_n \to g$ in (H, d), then $g = h$ a.e. But no continuous function on $[0,1]$ can equal h a.e.

Remark: The map $(\cdot|y)$ is a continuous functional on H

$$|(x_1|y)-(x_2|y)| = |(x_1-x_2|y)| \leq \|x_1-x_2\|\,\|y\|$$

The maps $(x|\cdot)$ and $\|\cdot\|$ are also continuous.

<u>DEFINITION</u>: $M \subset H$ is a closed subspace if it is a vector space which is closed in the topology of H.

<u>THEOREM</u>: Suppose $E \subset H$ is a closed convex set. Then E contains a unique element of smallest norm.

Proof. Note

$$(x+y\,|\,x+y) = (x|x)+(y|y)+(y|x)+(x|y)$$
$$(x-y\,|\,x-y) = (x|x)+(y|y)-(y|x)-(x|y)$$

and so $\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+\|y\|^2$.
Suppose $x,y\in E$. Let

$$\delta := \inf_{z\in E}\|z\|$$

Since E is convex, $\frac{1}{2}(x+y)\in E$. Therefore

$$\frac{1}{4}\|x-y\| = \frac{1}{2}\|x\|^2+\frac{1}{2}\|y\|^2-\frac{1}{4}\|x+y\|^2$$

$$\leq \frac{1}{2}\|x\|^2+\frac{1}{2}\|y\|^2-\delta^2$$

$$\|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 - 4\delta^2$$

If $\|x\| = \|y\| = \delta$, then $\|x-y\| = 0$ from the above, so $x = y$.
This shows uniqueness.

$$\exists\, (y_n) \subset E \text{ s.t. } \|y_n\| \to \delta$$

$$\|y_n - y_m\|^2 \leq 2\|y_n\|^2 + 2\|y_m\|^2 - 4\delta^2 \xrightarrow[n,m \to \infty]{} 0$$

Hence (y_n) is Cauchy, so $\exists\, x_0 \in E$ s.t. $y_n \to x_0$

E closed

$\|\cdot\|$ continuous $\Rightarrow$ $\|x_0\| = \lim \|y_n\| = \delta$ ▨

<u>**DEFINITION**</u>: $x \perp y$ means $(x|y) = 0$ (x is "orthogonal" to y). If $x \in H$

$$x^\perp := \{ y \in H : (x|y) = 0 \}$$

[$x^\perp$ is a closed subspace (= inverse image of $\{0\}$ under $(x|\cdot)$)]
If $M \subset H$ is a subspace,

$$M^\perp := \bigcap_{x \in M} x^\perp$$

[$M^\perp$ is also a closed subspace]

3/6 MEASURE THEORY

THEOREM: Suppose H is a Hilbert space, M closed subspace of H. Then $\exists\, P : H \to M$ and $Q : H \to M^\perp$ such that $\forall x \in H$, $x = Px + Qx$. P and Q are unique. Moreover

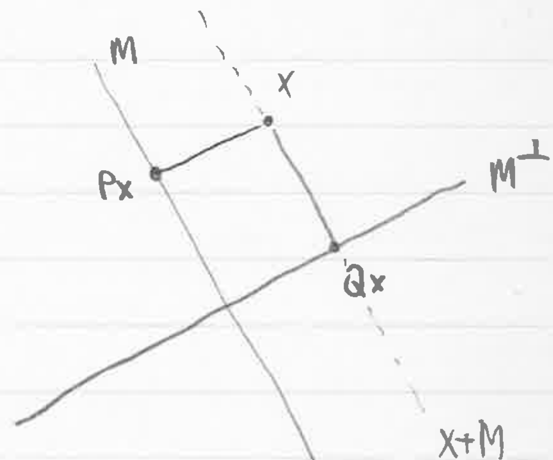
i) $x \in M \Rightarrow Px = x$, $Qx = 0$

ii) $x \in M^\perp \Rightarrow Qx = x$, $Px = 0$

iii) $\| Px - x \| = \inf \{ \| y - x \| : y \in M \}$

iv) $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$

v) P, Q are linear



Proof. Note $x + M$ is closed and convex. Let Qx be the unique element of $x+M$ with smallest norm. Let $Px := x - Qx$. Since $Qx \in x + M$, $Px \in M$. Want to show $Qx \in M^\perp$.

Let $z := Qx$. Select $y \in M$, $\|y\| = 1$, and show $(z|y) = 0$. For any scalar α, $z - \alpha y \in x + M$. Hence for all α

$$(z|z) \leq (z - \alpha y \mid z - \alpha y) = (z|z) - \alpha (y|z) - \bar{\alpha} (z,y) + |\alpha|^2$$

Set $\alpha := (z|y)$. Then

$$0 \leq -|\alpha|^2 - |\alpha|^2 + |\alpha|^2 = -|\alpha|^2$$

and so $\alpha = 0$. Therefore $Qx \in M^\perp$.

Uniqueness: Suppose $x = x_1 + x_2$, where $x_1 \in M$, $x_2 \in M^\perp$

Then $Px - x_1 = x_2 - Qx$. But $Px - x_1 \in M$ and $x_2 - Qx \in M^\perp$ and $M \cap M^\perp = \{0\}$. Therefore $x_1 = Px$ and $x_2 = Qx$.

(i), (ii) follows immediately from uniqueness since $x = x + 0$

(iii) follows from definition of Qx

(iv) $(x|x) = (Px + Qx \mid Px + Qx) = (Px|Px) + (Qx|Qx)$

(v)
$$\alpha x = P(\alpha x) + Q(\alpha x)$$
$$\beta y = P(\beta y) + Q(\beta y)$$
$$\alpha x + \beta y = P(\alpha x + \beta y) + Q(\alpha x + \beta y)$$

Subtract
$$0 = \underbrace{P(\alpha x + \beta y) - P(\alpha x) - P(\beta y)}_{\in M} + \underbrace{Q(\alpha x + \beta y) - Q(\alpha x) - Q(\beta y)}_{\in M^\perp}$$

Hence P and Q are linear $\square$

Example: $H = L^2[-\pi, \pi]$ $M = C[-\pi, \pi]$.
M is a dense subspace of H (not closed). Hence $M^\perp = (0)$ so we can't write $x = Px + Qx$ for $x \notin M$ with $Px \in M$ and $Qx \in M^\perp$.

COROLLARY: If M is a closed subspace of H, $M \neq H$, then $M^\perp \neq \{0\}$.

Proof. Let $x \notin M$. Then $Px \neq x$, so $Qx \neq 0$.

(*) THEOREM: Suppose $L : H \to \mathbb{C}$ is linear and continuous. Then there is a unique $y \in H$ such that

$$Lx = (x|y) \quad \forall x \in H$$

Proof. If $L = 0$ then $y = 0$ works. Note $y = 0$ is the only choice since $Ly = \|y\|^2 \neq 0$ if $y \neq 0$.

Suppose $L \neq 0$. Set

$$M := \{ x \in H : Lx = 0 \}$$

Then M is a closed proper subspace of H. Set $z \in M^{\perp}$, $\|z\| = 1$. Define for $x \in H$,

$$u_x := (Lx)z - (Lz)x$$

Note that $L(u_x) = 0$, so $u_x \in M$. Therefore

$$0 = (u_x|z) = Lx\,(z|z) - Lz\,(x|z)$$

$$= Lx - (x|(\overline{Lz})z)$$

Set $y := (\overline{Lz})z$. Then the above shows that $\forall x \in H$, $Lx = (x|y)$.

$\blacksquare$

DEFINITION: H Hilbert space. $\{u_\alpha : \alpha \in A\} \subset A$ is an orthonormal family if

$$(u_\alpha \mid u_\beta) = \begin{cases} 0 & \alpha \neq \beta \\ 1 & \alpha = \beta \end{cases}$$

If $(u_\alpha : \alpha \in A)$ is an orthonormal family, then for any $x \in H$, $(x \mid u_\alpha)$ is called the α^{th} Fourier coefficient of x (relative to $(u_\alpha : \alpha \in A)$)

Classical case: $H = L^2\left([-\pi, \pi], \frac{d\theta}{2\pi}\right)$ ← Lebesgue measure divided by 2π

Let $u_n(t) := e^{int}$ for $n \in \mathbb{Z}$.
Gives an orthonormal family
If $f \in H$, its n^{th} Fourier coefficient is

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} f(t)\, dt$$

The Fourier series of f is $\sum_n \hat{f}(n) e^{int}$.

PROPOSITION: Suppose $\{u_i : i \in \mathbb{N}_n\}$ is an orthonormal set in H. Let

$$x = \sum_{i=0}^{n} c_i u_i$$

Then $c_i = (x \mid u_i)$ and $\|x\|^2 = \sum_{i=0}^{n} |c_i|^2$. In particular $(u_i : i \in \mathbb{N}_n)$ is linearly independent

3/8 MEASURE THEORY

Recall: If $(u : u \in F)$ is a finite orthonormal family in H and

$$x = \sum_F c_u u$$

then $c_u = (x|u)$ and $\|x\|^2 = \sum_F |c_u|^2$.

Rephrase as follows: Given F a finite orthonormal family, let $M = \text{span}\{u : u \in F\}$. The map from M into ℓ^2 (counting measure on $\mathbb{N}_k$) (where $k = |F|$), given by

$$\forall x \in M \qquad x \longmapsto \left((x|u_1), (x|u_2), \ldots (x|u_k) \right)$$

is norm preserving.

<u>THEOREM</u>: Suppose F is a finite orthonormal family in H
For every $x \in H$,

$$(*) \qquad \left\| x - \sum_F (x|u)\, u \right\| \leq \left\| x - \sum_F \lambda_u u \right\|$$

for any family $(\lambda_u : u \in F)$ of scalars. Equality holds if and only if $\lambda_u = (x|u) \;\; \forall u \in F$

The projection of x into the (necessarily closed) subspace M of H spanned by F is $\sum_F (x|u) u$. If $\delta = d(x, M)$, then

$$(**) \qquad \sum_F |(x|u)|^2 = \|x\|^2 - \delta^2$$

Proof. (*) is equivalent to

$$(x|x) - \sum_F \overline{(x|u)}(x|u) - \sum_F (x|u)(u|x) + \sum_F (x|u)\overline{(x|u)}$$
$$\leq (x|x) - \sum_F \lambda_u (u|x) - \sum_F \overline{\lambda_u}(x|u) + \sum_F \lambda_u \overline{\lambda_u}$$

which is equivalent to

$$2\,\mathrm{Re} \sum_F \lambda_u (u|x) \leq \sum_F |(x|u)|^2 + \sum_F |\lambda_u|^2$$

Now

$$\mathrm{Re} \sum_F \lambda_u (u|x) \underset{(\dagger)}{\leq} \sum_F |\lambda_u|\,|(u|x)| \leq \left(\sum_F |\lambda_u|^2\right)^{1/2} \left(\sum |(u|x)|^2\right)^{1/2}$$

Schwartz inequality in $\ell^2(\mathbb{N}_{|F|})$

$$\leq \frac{1}{2}\left(\sum_F |\lambda_u|^2 + \sum_F |(u|x)|^2 \right)$$

geo. mean $\leq$ arith. mean

Equality holds iff $\lambda_u (u|x) \geq 0$ $(\dagger)$ and $|\lambda_u| = c\,|(u|x)|$ (Schwartz)

and $c = 1$ (geo = arith). Hence $\lambda_u (u|x) \geq 0$ and $|\lambda_u| = |(u|x)|$. Therefore

$$\lambda_u = \overline{(u|x)} = (x|u) \quad \forall u \in F$$

If $x \notin M$, $(*) \Rightarrow \mathrm{dist}(x, M) \geq \|x - \sum (x|u)u\| > 0$

Hence M is closed.

Recall that if $P : H \to M$ is the projection onto M (so $x = Px + Qx$, $Qx \in M^\perp$) , then $\|x - Px\| \le \|x - y\|$ $\forall y \in M$ so by (*) , $Px = \sum_i (x|u)u$. Also

$$\delta^2 = \|x - \sum_F (x|u)u\|^2 = \|x\|^2 - \sum_F |(x|u)|^2 \qquad \boxtimes$$

<u>COROLLARY</u>: If $(u_\alpha : \alpha \in A)$ is an orthonormal family in H, then

$$\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 \le \|x\|^2 \qquad \begin{pmatrix}\text{Bessel's}\\ \text{Inequality}\end{pmatrix}$$

(where $\hat{x}(\alpha)$ is the α^{th} Fourier coefficient of x)

[[$\sum_{\alpha \in A}$ is sup of all sums over finite subsets of A]]

Proof: Follows from (**) $\boxtimes$

<u>COROLLARY</u>: Only countably many $\hat{x}(\alpha) \neq 0$ for any particular $x \in H$.

Let $\ell^2(A) = L^2(A$, counting measure$)$ Notice that Bessel's inequality tells us that the mapping from H into $\ell^2(A)$ given by $x \longrightarrow \hat{x}$ is a linear norm-decreasing mapping

Riesz - Fischer Theorem: Let H be a Hilbert space and $(u_\alpha : \alpha\in A)$ an orthonormal family. Given $\varphi \in \ell^2(A)$, then $\exists\, x\in H$ such that $\hat{x} = \varphi$ (In other words, $x \to \hat{x}$ maps H onto $\ell^2(A)$)

Proof. For $n\in\mathbb{N}$, let

$$A_n := \{\alpha\in A : |\varphi(\alpha)| > 1/n\}$$

Since $\varphi\in \ell^2(A)$, A_n is finite. Define

$$x_n := \sum_{\alpha\in A_n} \varphi(\alpha)\, u_\alpha \quad \text{(finite sum)}$$

(1) Claim: $\hat{x}_n = \varphi \chi_{A_n}$

If $\beta\in A_n$, $\hat{x}_n(\beta) = (x_n | u_\beta) = \varphi(\beta)$ ✓
If $\beta\notin A_n$, $\hat{x}_n(\beta) = (x_n | u_\beta) = 0$

(2) Claim: $\hat{x}_n \to \varphi$ pointwise on A

If $\varphi(\beta) = 0$, then $\hat{x}_n(\beta) = 0$ since $\beta\notin A_n$ for any n. If $\varphi(\beta)\neq 0$, then $\hat{x}_n(\beta) = \varphi(\beta)$ eventually.

Claim (1) also shows that $|\hat{x}_n - \varphi|^2 \le |\varphi|^2$ on A. Since $\hat{x}_n - \varphi \to 0$ pointwise on A and is dominated by an integrable function ($|\varphi|^2$), the DCT says that $\|\hat{x}_n - \varphi\|_2 \to 0$ in $\ell^2(A)$. Hence $\hat{x}_n$ is Cauchy in $\ell^2(A)$. But

$$\| \hat{x}_n - \hat{x}_m \|_2 = \| x_n - x_m \|_H \quad [\![x_k \text{ finite sum }]\!]$$

Hence (x_n) is Cauchy in H, and so converges to some $x \in H$.
For any $\alpha \in A$

$$\hat{x}(\alpha) = (x \mid u_\alpha) = \lim_{n\to\infty} (x_n \mid u_\alpha) = \lim_{n\to\infty} \hat{x}_n(\alpha) = \varphi(\alpha)$$

Hence $\hat{x} = \varphi$. $\boxtimes$

3/10 MEASURE THEORY

THEOREM: Suppose H is a Hilbert space and $\{u_\alpha : \alpha \in A\}$ is an orthonormal family in H. TFAE

i) $\{u_\alpha : \alpha \in A\}$ is a maximal orthonormal family

ii) The set S of finite linear combinations of members of this family is dense in H

iii) $x \in H \Rightarrow \|x\|^2 = \sum_{\alpha \in A} |\hat{x}(\alpha)|^2$ (Parseval's Theorem)

iv) $\forall x, y \in H$, $(x|y) = \sum_{\alpha \in A} \hat{x}(\alpha)\overline{\hat{y}(\alpha)}$

Proof. i) $\Rightarrow$ (ii) Suppose ii) does not hold, i.e. $M := cl(S) \neq H$. Note that M is a subspace and closed. Since $M \neq H$, $M^\perp \neq \{0\}$. Let $u \in M^\perp$, $u \neq 0$. Then $u/\|u\| \in M^\perp$ and

$$\left(\frac{u}{\|u\|} \,\middle|\, u_\alpha\right) = 0 \quad \forall \alpha$$

Adjoin $u/\|u\|$ to $\{u_\alpha : \alpha \in A\}$ to obtain a larger orthonormal family.

(ii) $\Rightarrow$ (iii) Given $\varepsilon > 0$, $x \in H$, (ii) says $\exists\, F \subset A$ finite and $(c_\alpha : \alpha \in F) \subset \mathbb{C}$ s.t.

$$\left\|x - \sum_{\alpha \in F} c_\alpha u_\alpha\right\| < \varepsilon$$

Recall that the best approximation is with $(x|u_\alpha)$, so that

$$\left\|x - \sum_{\alpha \in F} (x|u_\alpha) u_\alpha\right\| \leq \left\|x - \sum_{\alpha \in F} c_\alpha u_\alpha\right\| < \varepsilon$$

Then

$$\|x\| \leq \Big\| \sum_F (x|u_\alpha) u_\alpha \Big\| + \varepsilon = \Big(\sum_{\alpha\in F} |(x|u_\alpha)|^2 \Big)^{1/2} + \varepsilon$$

and so

$$(\|x\| - \varepsilon)^2 \leq \sum_{\alpha \in F} |(x|u_\alpha)|^2 \underset{\substack{\uparrow \\ \text{Bessel}}}{\leq} \|x\|^2$$

Therefore $\|x\|^2 = \sum_{\alpha\in A} |(x|u_\alpha)|^2$.

(iii) $\Rightarrow$ (iv) What (iii) says is that $\|x\| = \|\hat{x}\|_2 \quad \forall x \in H$
Let $\lambda \in \mathbb{C}$. Then

$$(x+\lambda y \,|\, x+\lambda y) = (\widehat{x+\lambda y} \,|\, \widehat{x+\lambda y}) = (\hat{x}+\lambda\hat{y} \,|\, \hat{x}+\lambda\hat{y})$$

$$\Rightarrow \qquad \lambda(y|x) + \bar{\lambda}(x|y) = \lambda(\hat{y}|\hat{x}) + \bar{\lambda}(\hat{x}|\hat{y})$$

$$\Rightarrow \qquad \operatorname{Re} \lambda(y|x) = \operatorname{Re} \lambda(\hat{y}|\hat{x}) \quad \forall \lambda \in \mathbb{C}$$

Setting $\lambda = 1$ and then $\lambda = i$, we see that $(y|x) = (\hat{y}|\hat{x})$, i.e.

$$(y|x) = (\hat{y}|\hat{x}) = \sum_{\alpha\in A} \hat{y}(\alpha)\overline{\hat{x}(\alpha)}$$

(iv) $\Rightarrow$ (i) Suppose $(u_\alpha : \alpha \in A)$ is not maximal. Then
$\exists\, u \notin (u_\alpha : \alpha\in A)$ such that $\hat{u}(\alpha) = (u|u_\alpha) = 0 \quad \forall \alpha \in A$.
$u \neq 0$

Hence

$$(u|u) \neq 0 = \sum_{\alpha\in A} \hat{u}(\alpha)\,\overline{\hat{u}(\alpha)}$$

so (4) does not hold.

$\boxed{/\!/}$

Summary: If $(u_\alpha : \alpha \in A)$ is a maximal orthonormal family, then the mapping from H onto $\ell^2(A)$ given by $x \to \hat{x}$ is a Hilbert space isomorphism.

Remark: Every orthonormal family in H is contained in some maximal orthonormal family. Hence any Hilbert space is isomorphic to $\ell^2(A)$ for some A.

<u>Classical Case</u>

$$H = L^2\left([-\pi,\pi],\ \frac{d\theta}{2\pi}\right)$$

$\uparrow$ normalized Lebesgue measure

$$T = \{z \in \mathbb{C} : |z| = 1\}$$

$C(T)$ = continuous complex-valued functions on T
($\Leftrightarrow$ cont. complex-valued functions on $\mathbb{R}$ with period 2π)

Claim: $\{e^{int} : n \in \mathbb{Z}\}$ is an orthonormal family in H

$$\left(e^{int} \middle| e^{imt}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int} e^{-imt}\, dt = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

Let $S_N(x,f) = S_N(x) := \sum_{k=-N}^{N} \hat{f}(k) e^{ikx}$ for $f \in L^2[-\pi,\pi]$

S_N is the N^{th} partial sum of the Fourier series of f.

Fejér's Theorem: Suppose $f \in C(T)$. Let

$$\sigma_N(x,f) = \sigma_N(x) := \frac{1}{N+1}\sum_{k=0}^{N} S_k(x)$$

Then $\sigma_N \to f$ uniformly on $[-\pi,\pi]$

Proof later

Clearly $\sigma_N(x) = \sum_{j=-N}^{N} c_j e^{ijx}$ for some choice of c_j's

trigonometric polynomial of degree N

We know $C(T)$ are dense in H. Then by Fejér's theorem S = set of finite linear combinations of $\{e^{inx} : n \in \mathbb{Z}\}$ (trig poly) is dense in H, and so $\{e^{inx} : n \in \mathbb{Z}\}$ is maximal.

Suppose $f \in L^2[-\pi,\pi]$

$$\hat{f}(n) = (f | e^{inx}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int}\, dt$$

Then $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$. Also, if $g \in L^2[-\pi,\pi]$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\overline{g(t)}\, dt = \sum_{n=-\infty}^{\infty} \hat{f}(n)\,\overline{\hat{g}(n)}$$

For $f \in L^2[-\pi,\pi]$

$$\widehat{f - S_N}(k) = \begin{cases} \hat{f}(k) & |k| > N \\ 0 & |k| \le N \end{cases}$$

By Parseval's theorem

$$\| f - S_N \|_2^2 = \sum_{|k|>N} |\hat{f}(k)|^2 \to 0 \quad \text{as } N \to \infty$$

Hence $S_N \to f$ in $L^2[-\pi,\pi]$, so $\exists\, S_{n_j}$ s.t. $S_{N_j}(x) \to f(x)$ a.e.

What trig polynomial of degree N best approximates f in L^2 sense?

Answer - S_N

$$\| f - S_N \| \le \| f - \sum_{k=-N}^{N} c_k e^{ikx} \| \quad \forall c_k \in \mathbb{C}$$

3/13 MEASURE THEORY

PROPOSITION: Define the Dirichlet kernel by

$$D_m(x) := \sum_{k=-m}^{m} e^{ikx}$$

for $m \in \mathbb{N}$. Define the Fejér kernel by

$$K_n(x) := \frac{1}{n+1} \sum_{m=0}^{n} D_m(x)$$

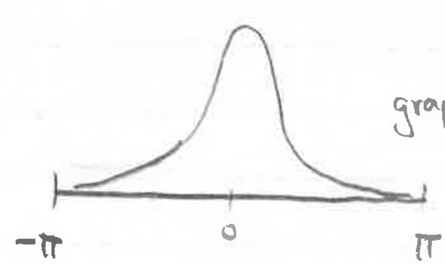
for $n \in \mathbb{N}$. Then

(1) $$D_m(x) = \frac{\sin\left(m+\frac{1}{2}\right)x}{\sin\frac{x}{2}}$$

(2) $$K_n(x) = \frac{1}{n+1} \frac{1-\cos(n+1)x}{1-\cos x}$$

(3) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_m(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1$$

(4) $0 \le K_n(x)\ \forall x$ and $K_n(x) \le \frac{2}{n+1}\frac{1}{1-\cos\delta}$ for $\delta \le |x| \le \pi$



graph of K_n

Proof.

$$1*) \quad (e^{ix}-1)\, D_m(x) = e^{i(m+1)x} - e^{-imx}$$

Multiply this by $e^{-ix/2}$:

$$2i \sin \frac{x}{2}\, D_m(x) = e^{i(m+1/2)x} - e^{-i(m+1/2)x} = 2i \sin(m+1/2)x$$

$$\Rightarrow D_m(x) = \frac{\sin(m+1/2)x}{\sin x/2}$$

(*) also implies that

$$(n+1)\, K_n(x)(e^{ix}-1) = \sum_{m=0}^{n} \left(e^{i(m+1)x} - e^{-imx}\right)$$

$$= \sum_{j=-n}^{n+1} c_j\, e^{ijx} \qquad c_j = \begin{cases} 1 & 1 \leq j \leq n+1 \\ -1 & -n \leq j \leq 0 \end{cases}$$

Hence

$$(n+1)\, K_n(x)(e^{ix}-1)(e^{-ix}-1) = -e^{i(n+1)x} - e^{-i(n+1)x} + 2$$

$$= 2 - 2\cos(n+1)x$$

$$\Rightarrow (n+1)\, K_n(x) = \frac{2-2\cos(n+1)x}{2-2\cos x} = \frac{1-\cos(n+1)x}{1-\cos x}$$

(4) now follows immediately from (2). (3) also easy.

$\blacksquare$

FEJÉR'S THEOREM: Suppose $f \in C(\mathbb{T})$ (i.e. f is continuous, complex-valued, period 2π). Let

$$S_N(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{ikx}$$

and

$$\sigma_N(x) = \frac{1}{N+1} \sum_{k=0}^{N} S_k(x)$$

Then $\sigma_N \to f$ uniformly on $[-\pi, \pi]$

Proof.

$$S_N(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{ikx} = \sum_{k=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt \right) e^{ikx}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{k=-N}^{N} e^{ik(x-t)} dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_N(x-t)\, dt \qquad [\![u = x-t]\!]$$

$$= \frac{1}{2\pi} \int_{x+\pi}^{x-\pi} f(x-u) D_N(u) (-du)$$

($f(x-u)D_N(u)$ has period 2π, so may replace x by 0 in limits)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\,D_N(t)\,dt$$

Then

$$\sigma_n(x) = \frac{1}{n+1}\sum_{N=0}^{n} S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\left[\frac{1}{n+1}\sum_{N=0}^{n} D_N(t)\right]dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\,K_n(t)\,dt$$

Hence

$$\sigma_n(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(x-t)-f(x)\right]K_n(t)\,dt$$

(since $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)\,dt = 1$), and so

$$|\sigma_n(x) - f(x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)|\,K_n(t)\,dt$$

(since $K_n(t) \geq 0$!) Since f is continuous, $\exists M$ s.t. $|f(y)| \leq M$ $\forall y \in [-\pi,\pi]$. Also, given $\varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon/2$$

Since $K_n(t) \leq \frac{2}{n+1} \frac{1}{1-\cos\delta}$ $\forall$ $\delta \leq |t| \leq \pi$, $\exists$ $L \in \mathbb{N}$ such that $\forall n \geq L$,

$$\delta \leq |t| \leq \pi \Rightarrow K_n(t) < \frac{\varepsilon}{4M}$$

Thus, for all $n \geq L$

$$\frac{1}{2\pi}\int_{-\delta}^{\delta} |f(x-t) - f(x)|\, K_n(t)\, dt \leq \frac{\varepsilon}{2}$$

$$\frac{1}{2\pi}\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) \leq 2M \cdot \frac{1}{2\pi} \int_{\delta \leq |t| \leq \pi} K_n(t)\, dt$$

$$\leq 2M \cdot \frac{1}{2\pi} \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{4\pi} < \frac{\varepsilon}{2}$$

Hence $\forall x$

$$|\sigma_n(x) - f(x)| \leq \varepsilon \quad \forall n \geq L$$

$\blacksquare$

Note

$$|S_N(x) - f(x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)|\, |D_N(t)|\, dt$$

and $\int_{-\pi}^{\pi} |D_N(t)|\,dt > c \cdot \log N$

<u>COROLLARY</u>: $\{e^{inx} : n \in \mathbb{Z}\}$ is a maximal orthonormal system in $L^2[-\pi,\pi]$.

Add to our list of observations on $L^2[-\pi,\pi]$:

Given $(c_n : n \in \mathbb{Z}) \in \ell^2(\mathbb{Z})$, i.e. $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$, $\exists\, f \in L^2[-\pi,\pi]$ s.t.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,e^{-int}\,dt$$

(Riesz-Fischer)

Question - Does $S_N(x) \longrightarrow f(x)$ say for $f \in C(T)$?

Not true $\forall f$ and $\forall x$. However, it is true if $f \in BV[-\pi,\pi]$ (in fact uniform convergence)

THEOREM: If $f \in L^2[-\pi,\pi]$, then $S_N(f) \to f(x)$ a.e.

(Proof mid 1960's)

3/15 MEASURE THEORY

BANACH SPACES

Examples of Banach spaces

1) $L^p(\mu)$ $1 \le p \le \infty$
2) Hilbert spaces
3) $\mathbb{C}$
4) $C(T)$ with supremum norm

RECALL: BAIRE CATEGORY THEOREM If X is a complete metric space and $\mathcal{O}_n$ is a seq. of dense open sets. Then $\bigcap \mathcal{O}_n$ is dense (and hence non-empty)

<u>COROLLARY</u>: X complete metric space, G_n seq. of dense G_δ-sets. Then $\bigcap G_n$ is dense

Proof: Each $G_n = \bigcap_{i=1}^{\infty} \mathcal{O}_{n,i}$, each $\mathcal{O}_{n,i}$ open, dense

$$\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \bigcap_{i=1}^{\infty} \mathcal{O}_{n,i} \quad \text{dense in } X$$

$\blacksquare$

<u>UNIFORM BOUNDEDNESS THEOREM</u> (BANACH-STEINHAUS)
Suppose X is a Banach space and Y is a normed linear space. $\{\Lambda_\alpha : \alpha \in A\} \subset \mathcal{B}(X,Y)$. Then one of the following (dramatically different) alternatives must occur

(1) $\exists M>0$ s.t. $\|\Lambda_\alpha\| \leq M \quad \forall \alpha \in A$

(2) $\sup_{\alpha \in A} \|\Lambda_\alpha x\| = \infty$ for a dense G_δ subset of X

(Hence pointwise boundedness $\Rightarrow$ uniform boundedness)

Proof. Define $\varphi : X \longrightarrow [0,\infty]$ by

$$\varphi(x) = \sup_{\alpha \in A} \|\Lambda_\alpha x\|$$

Define for $n \in \mathbb{N}$

$$V_n := \{x \in X : \varphi(x) > n\}$$

Note for a fixed $\alpha \in A$, $\Lambda_\alpha x$ is a continuous function of x. Hence $\|\Lambda_\alpha x\|$ is continuous. Therefore $\sup \|\Lambda_\alpha x\|$ is lower semi-continuous. Hence V_n is open.

Suppose $\exists N \in \mathbb{N}$ s.t. V_N is not dense. Then $\exists x_0 \in X$ and $r>0$ such that

$$\|x\| \leq r \implies x + x_0 \notin V_N$$

Therefore $\|x\| \leq r \implies \varphi(x+x_0) \leq N \Rightarrow \|\Lambda_\alpha(x+x_0)\| \leq N \quad \forall \alpha \in A$

In particular $\|\Lambda_\alpha(x_0)\| \leq N \quad \forall \alpha \in A$. Hence if $\|x\| \leq r$

$$\|\Lambda_\alpha x\| = \|\Lambda_\alpha(x+x_0) - \Lambda_\alpha(x)\| \leq 2N \quad \forall \alpha \in A$$

Therefore if $\|u\| = 1$

$$\| \Lambda_\alpha u \| \leq \frac{2N}{r} \quad \forall \alpha \in \Lambda$$

and so (1) holds $(M = 2N/r)$

If each V_N is dense, then $E = \bigcap_{n=1}^{\infty} V_n$ is dense

If $x \in E$, then $\varphi(x) > n \quad \forall n \in \mathbb{N}$, i.e. $\varphi(x) = \infty$. Hence (2) holds.

$\boxed{}$

Question: Suppose $f \in C(T)$. Does $S_n(x,f) \to f(x)$ $\forall x \in [-\pi, \pi]$?

Answer: No

Recall $S_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_n(t)\, dt$, where

$$D_n(t) = \frac{\sin(n+1/2)t}{\sin t/2}$$

Define $\Lambda_n : C(T) \to \mathbb{C}$ by

$$\Lambda_n(f) := S_n(0,f) \quad \forall f \in C(T)$$

$$|\Lambda_n(f)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(-t) D_n(t)\, dt \right|$$

$$\leq \|f\|_\infty \|D_n\|_1$$

Hence $\|\Lambda_n\| \leq \|D_n\|_1$

We will show that $\|\Lambda_n\| = \|D_n\|_1 \to \infty$ as $n \to \infty$. Hence by the uniform boundedness principal, there is a dense set of f in $C(\mathbb{T})$ s.t.

$$\sup_n |\Lambda_n f| = \sup_n |S_n(0,f)| = \infty$$

and so for this large collection we have $S_n(0,f) \not\to f(0)$.

3/17 MEASURE THEORY

Consider $\Lambda_n : C(T) \longrightarrow \mathbb{C}$ given by

$$\Lambda_n(f) = S_n(f,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) D_n(0-t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) D_n(t)\,dt$$

By Hölder's inequality

$$|\Lambda_n f| \leq \|f\|_\infty \|D_n\|_1$$

and so $\|\Lambda_n\| \leq \|D_n\|_1$

CLAIM: $\|\Lambda_n\| = \|D_n\|_1 \longrightarrow \infty$

$$\|D_n\|_1 = \frac{1}{\pi}\int_0^{\pi} \frac{|\sin(n+1/2)t|}{\sin t/2}\,dt$$

$$\geq \frac{2}{\pi}\int_0^{\pi} \frac{|\sin(n+1/2)t|}{t}\,dt \qquad \left[\!\left[\sin x \leq x \text{ for } x \geq 0 \right]\!\right]$$

$$= \frac{2}{\pi}\int_0^{(n+1/2)\pi} \frac{|\sin u|}{u}\,du$$

$$\geq \frac{2}{\pi}\sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi} \frac{|\sin u|}{u}\,du$$

$$\geq \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin u| \, du$$

$$= \frac{4}{\pi^2} \sum_{k=1}^{n} \frac{1}{k} \longrightarrow \infty \quad \text{as } n \to \infty$$

Define

$$g_n(t) = \begin{cases} 1 & \text{if } D_n(t) \geq 0 \\ -1 & \text{if } D_n(t) < 0 \end{cases}$$

g_n is a step function. It's elementary that $\exists\, f_j \in C(T)$ such that $\|f_j\|_\infty = 1$ and $f_j(t) \to g_n(t)$ as $j \to \infty$ for each $t \in [-\pi, \pi]$. Now

$$\|\Lambda_n\| = \sup_{\|f\|_\infty = 1} |\Lambda_n f|$$

and $\Lambda_n f_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_j(t) D_n(t)\, dt$. By D.C.T, as $j \to \infty$

$$\Lambda_n f_j \longrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(t) D_n(t)\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt$$

$$= \|D_n\|$$

Hence $\|\Lambda_n\| \geq \|D_n\|$. But we saw earlier that $\|D_n\| \geq \|\Lambda_n\|$.

This establishes the claim. Now the Uniform Boundedness principle says $\exists$ dense G_δ-set $E \subset C(T)$ s.t.

$$\sup_{n\in\mathbb{N}} |S_n(f,0)| = +\infty \quad \forall f \in E$$

and so $S_n(f,0)$ does not converge

There is nothing special about 0. $\forall x \in [-\pi,\pi]$ $\exists$ a dense G_δ-set $E_x \subset C(T)$ s.t.

$$\sup |S_n(f,x)| = \infty \quad \forall f \in E_x$$

Let $S^*(f,x) = \sup_n |S_n(f,x)|$. Fix f. Then $S^*(f,x)$ is a supremum of continuous functions and so is lower semicontinuous. It is the case that for each $f \in C(T)$ $\{x : S^*(f,x) = \infty\}$ is a G_δ-set.

Let (x_n) be a dense sequence in $[-\pi,\pi]$. Associate with each x_i a set $E_{x_i} \subset C(T)$ s.t. E_{x_i} is a dense G_δ-set and

$$S^*(f,x_i) = \infty \quad \forall f \in E_{x_i}$$

Let $E = \bigcap_{i=1}^{\infty} E_{x_i}$. E is a dense G_δ set. Suppose $f \in E$

$$S^*(f,x_i) = \infty \quad \forall i \in \mathbb{N}$$

Then for each $f \in E$, $\{x \in [-\pi,\pi] : S^*(f,x) = \infty\}$ is a dense

G_δ - set

Summary: There is a dense G_δ set $E \subset C(T)$ s.t. for every $f \in E$, $S_n(f,x)$ diverges for all $x \in F$, where F is a dense G_δ set in $[-\pi, \pi]$.

Remark: If X is a complete metric space with no isolated points, a dense G_δ is uncountable.

Open Mapping Theorem

Suppose X and Y are both Banach spaces. Suppose $\Lambda : X \to Y$ is a bounded linear transformation onto Y. Let $U = \{x \in X : \|x\| \le 1\}$ and $V = \{y \in Y : \|y\| \le 1\}$. Then $\exists\, \delta > 0$ s.t.

$$\delta V \subset \Lambda(U)$$

Remark: It follows from the linearity of Λ that the image of every open set in X is an open set in Y.

Observation: Suppose X is a complete metric space. If $X = \bigcup E_n$, then $\exists\, n$ s.t. $\operatorname{int}(\overline{E_n}) \neq \emptyset$ (Baire Cat. Thm)

Proof: Since Λ is onto

$$Y = \bigcup_{k=1}^{\infty} \Lambda(kU)$$

Y complete $\Rightarrow$ $\exists\, k$ s.t. $\overline{\Lambda(kU)}$ contains W open, $W \neq \emptyset$.
$\exists\, y_0 \in W, \eta > 0$ s.t. $\|y\| \leq \eta \Rightarrow y_0 + y \in W$. $\exists\, x_i' \in kU$ s.t. $\Lambda x_i' \to y_0$. For $\|y\| \leq \eta$, $\exists\, x_i'' \in kU$ s.t. $\Lambda x_i'' \to y_0 + y$. Let $x_i = x_i'' - x_i'$. Then $\Lambda x_i \to y$
$(x_i) \subset (2k)U$

If $\|y\| = \eta$, $\exists (x_i) \subset (2k)U$ s.t. $\Lambda x_i \to y$
Let $\delta = \eta/2k$. If $\|y\| = \eta$, $\exists (x_i)$ s.t. $\|x_i\| \leq \delta^{-1}\|y\|$ s.t. $\Lambda x_i \to y$. But now Λ linear $\Rightarrow$ $\forall y$, $\exists (x_i)$ s.t. $\|x_i\| \leq \delta^{-1}\|y\|$ and $\Lambda x_i \to y$

(*) For any $\varepsilon > 0$, $y \in Y$, $\exists\, x \in X$ s.t. $\|x\| < \delta^{-1}\|y\|$ s.t. $\|\Lambda x - y\| < \varepsilon$.

Suppose $\|y\| < \delta$. By (*) $\exists\, x_1$, $\|x_1\| < 1$ s.t. $\|\Lambda x_1 - y\| < \frac{1}{2}\delta\varepsilon$. Suppose $x_1, \dots, x_n$ have been chosen s.t.

$$\|y - \Lambda x_1 - \Lambda x_2 - \dots - \Lambda x_n\| < 2^{-n}\delta\varepsilon$$

Choose, by (*), $x_{n+1} \in X$, $\|x_{n+1}\| < 2^{-n}\varepsilon$ s.t.

$$\|(y - \Lambda x_1 - \dots - \Lambda x_n) - \Lambda x_{n+1}\| < 2^{-(n+1)}\delta\varepsilon$$

Let $s_n = x_1 + \dots + x_n$. Then S_n is Cauchy in X since $\|x_{n+1}\| < 2^{-n}\varepsilon$. Therefore $s_n \to x$. $\Lambda S_n \to \Lambda x$. But $\Lambda S_n \to y$ also, so $y = \Lambda x$. Now $\|x\| < 1 + \varepsilon$, so

$$\delta V \subset \Lambda((1+\varepsilon)U)$$

and so

$$(1+\varepsilon)^{-1}\,\delta V \subset \Lambda(U) \quad \forall \varepsilon>0$$

$$\Rightarrow \quad \delta V \subset \Lambda(U)$$

$\blacksquare$

3/27 ANALYSIS

<u>COROLLARY</u> : $\Lambda : X \longrightarrow Y$ 1-1, onto, linear, and bounded. X, Y Banach spaces. Then $\exists\, \delta > 0$ s.t.

$$\| \Lambda x \| \geq \delta \|x\|$$

$\forall x \in X$ (and so Λ^{-1} is bounded, with $\|\Lambda^{-1}\| \leq 1/\delta$).

Proof. Let δ be as in Open Mapping Theorem. If $\|\Lambda x\| < \delta$, then $\|x\| < 1$, and so if $\|x\| \geq 1$, we have $\|\Lambda x\| \geq \delta$. In particular

$$\left\| \Lambda \left(\frac{x}{\|x\|} \right) \right\| \geq \delta \quad \forall x \neq 0$$

$$\Longrightarrow \| \Lambda x \| \geq \delta \|x\| \quad \forall x \in X$$

$\boxtimes$

<u>RIEMANN</u> - <u>LEBESQUE</u> <u>LEMMA</u> : If $f \in L^1[-\pi, \pi]$, then

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt \longrightarrow 0$$

as $|n| \to \infty$.

Proof. There is a continuous g on $[-\pi, \pi]$ such that

$$\|g - f\|_1 < \varepsilon$$

WLOG, assume $g(-\pi) = g(\pi)$ (can modify g on a small set)
Thus $g \in C(\mathbb{T})$, so by Fejér's theorem, there is a trig. polynomial P s.t.

$$\|P - g\|_1 \leq \|P - g\|_\infty < \varepsilon$$

Hence $\|f - P\|_1 < 2\varepsilon$.

Suppose $|n| > \deg P$. Then

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \big(f(t) - P(t)\big) e^{-int}\, dt$$

↑ contributes 0 to integral

and so

$$|\hat{f}(n)| \leq \|f - P\|_1 \, \|e^{-int}\|_\infty = \|f - P\|_1 < 2\varepsilon$$

whenever $|n| > \deg P$.

$\boxtimes$

QUESTION: If $(a_n : n \in \mathbb{Z}) \to 0$ as $|n| \to \infty$, does there exist $f \in L^1[-\pi, \pi]$ s.t. $\hat{f}(n) = a_n$?

ANSWER: NO

Recall, Riesz-Fischer theorem tells us that every (a_n) s.t. $\sum a_n^2 < \infty$ is of the form $a_n = \hat{f}(n)$ for some $f \in L^2[-\pi, \pi]$

THEOREM: Define $\Lambda : L^1[-\pi,\pi] \to c_0(\mathbb{Z})$ given by

$$(\Lambda f)_n := \hat{f}(n)$$

$\Big((\hat{f}(n)) \in c_0$ by Riemann-Lebesgue Lemma $\Big)$ Then Λ is a bounded 1-1 linear transformation, but Λ is not onto.

Proof.

$$\|\Lambda f\| = \sup_n |\hat{f}(n)| \underset{\text{Hölder}}{\leqslant} \|f\|_1 \qquad \forall f \in L^1$$

Hence $\|\Lambda\| \leq 1$.

Suppose $f \in L^1[-\pi,\pi]$ and $\hat{f}(n) = 0 \ \forall n \in \mathbb{Z}$. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \, dt = 0$$

$\forall n$, and so

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, p(t) \, dt = 0$$

for any trig. polynomial p. Suppose $g \in C(\mathbb{T})$. There exist trig. polynomials P_n s.t. $\|P_n - g\|_\infty \to 0$. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) g(t) \, dt = \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(g-P_n)}_{\to 0} + \underbrace{\frac{1}{2\pi} \int f P_n}_{= 0}$$

Therefore

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt = 0 \qquad \forall g \in C(T)$$

In fact

$$(*) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,g(t)\,dt = 0 \qquad \forall g \text{ cont. on } [-\pi,\pi]$$

(can modify g on a small set so $g(\pi) = g(-\pi)$).

By Lusin's theorem, for any measurable $E \subset [-\pi,\pi]$ $\exists$ cont. g_n such that $\|g_n\|_\infty \leq 1$ and $g_n \to \chi_E$ a.e. Therefore

$$\frac{1}{2\pi}\int_E f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\chi_E \underset{\text{D.C.T.}}{=} \lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} fg_n \underset{(*)}{=} 0$$

and so $f = 0$ a.e. Hence Λ is 1-1.

If Λ was onto, then the corollary to the Open Mapping theorem would imply that Λ^{-1} is bounded. But consider

$$D_n(t) = \sum_{k=-n}^{n} e^{ikt}$$

$\|D_n\|_1 \to \infty$, while $\|\hat{D}_n\|_{c_0} = 1$. Thus there is no $\delta > 0$ s.t.

$$\forall n \quad \|\Lambda D_n\|_{c_0} \geq \delta\,\|D_n\|$$

Therefore Λ can not be onto.

∎

3/29 ANALYSIS

DEFINITION: X vector space over $\mathbb{C}$. We say $f : X \to \begin{Bmatrix} \mathbb{R} \\ \mathbb{C} \end{Bmatrix}$ is $\begin{Bmatrix} \text{real} \\ \text{complex} \end{Bmatrix}$ - linear if $f(x+y) = f(x) + f(y)$ and $f(\alpha x) = \alpha f(x)$ for every $\begin{Bmatrix} \text{real} \\ \text{complex} \end{Bmatrix}$ scalar α.

Remark: If f is complex-linear, then $\operatorname{Re} f$ is real-linear

LEMMA: X vector space over $\mathbb{C}$

(1) Suppose $f : X \to \mathbb{C}$ is complex-linear. Let $u = \operatorname{Re} f$. Then $\forall x \in X$, $f(x) = u(x) - i\,u(ix)$

(2) If $u : X \to \mathbb{R}$ is real-linear, then $f(x) := u(x) - i\,u(ix)$ is complex-linear.

(3) X normed linear space over $\mathbb{C}$. If u is a real-linear bounded functional, the complex-linear functional $f(x) = u(x) - i\,u(ix)$ satisfies $\|f\| = \|u\|$

Proof. $u = \operatorname{Re} f \Rightarrow \|u\| \leq \|f\|$. Suppose $x \in X$. $\exists\, \alpha \in \mathbb{C}$ s.t. $|\alpha| = 1$ and $\alpha f(x) = |f(x)|$

since $f(\alpha x)$ is Real

$$|f(x)| = \alpha f(x) = f(\alpha x) = u(\alpha x) \leq \|u\| \, \|\alpha x\| = \|u\| \, \|x\|$$

Hence $\|f\| \leq \|u\|$.

<u>HAHN-BANACH THEOREM</u>: X normed linear space (over $\mathbb{R}$ or $\mathbb{C}$). Let M be a proper subspace. Suppose f is a bounded functional on M. Then f extends to a bounded functional on X, say F, with $\|F\| = \|f\|$.

Specifically, we want to treat these cases:
(1) Field of scalars $= \mathbb{R}$, f real-linear
(2) Field of scalars $= \mathbb{C}$, f real-linear
(3) Field of scalars $= \mathbb{C}$, f complex-linear

Proof. Assume f is real-linear (case (1)). Nothing to prove if $\|f\| = 0$, so WLOG $\|f\| = 1$. Consider $x_0 \in X - M$, and set

$$M_1 := sp(M \cup \{x_0\})_{\mathbb{R}} = \{ x + \lambda x_0 : x \in M, \lambda \in \mathbb{R} \}$$

Note each member of M_1 is uniquely expressible in the form $x + \lambda x_0$ for $x \in M$ and $\lambda \in \mathbb{R}$. Thus it makes sense to define $f: M_1 \to \mathbb{R}$ by

$$f(x + \lambda x_0) = f(x) + \lambda \alpha$$

where α is a fixed real number at our disposal. Then f is a real-linear functional on M_1 which agrees with its old self on M.

Question: do there a choice of α so that $\|f\|=1$, regarding f as defined on M_1? That is, is there a real α s.t.

$$(*) \quad |f(x)+\lambda\alpha| = |f(x+\lambda x_0)| \leq \|x+\lambda x_0\|$$

Note: if $x\in M$, $y\in M$, then

$$f(x)-f(y) = f(x-y) \leq |f(x-y)| \leq \|x-y\| \leq \|x-x_0\|+\|y-x_0\|$$

$$\Longrightarrow \quad f(x)-\|x-x_0\| \leq f(y)+\|y-x_0\| \quad \forall x,y\in M$$

Choose $\alpha\in R$ s.t.

$$\sup_{x\in M}\left(f(x)-\|x-x_0\|\right) \leq \alpha \leq \inf_{y\in M}\left(f(y)+\|y-x_0\|\right)$$

Given $x\in M$ and $\lambda\in\mathbb{R}$, we want to show $(*)$. WLOG $\lambda\neq 0$.
Let $y=-x/\lambda \in M$

$$f(x)+\lambda\alpha = f(-\lambda y)+\lambda\alpha = -\lambda\left(f(y)-\alpha\right)$$

$$\begin{aligned} |f(x)+\lambda\alpha| &= |\lambda|\,|f(y)-\alpha| \\ &\leq |\lambda|\,\|y-x_0\| \\ &= \|-\lambda y+\lambda x_0\| \\ &= \|x+\lambda x_0\| \end{aligned}$$

Thus f has a norm-preserving extension to M_1

Let P be the collection of order pairs (M', f') where M' is a subset closed under addition and multiplication by real scalars, $M' \supset M$, $f': M' \to R$ is real linear and $\|f'\| = 1$. Partially order P as follows

$$(M', f') \leq (M'', f'') \text{ iff } M' \subset M'' \text{ and } f''|_{M'} = f'$$

Hausdorff Maximality Theorem says $\exists$ a maximal totally ordered subset Ω of P

Let

$$\tilde{M} = \bigcup \{M' : (M', f') \in \Omega\}$$

① $\tilde{M}$ is a subspace of X

② Define $F: \tilde{M} \to \mathbb{R}$ by $F(x) := f'(x)$ if $x \in M'$

F is well-defined and linear. If $x \in \tilde{M}$,

$$|F(x)| = \underbrace{|f'(x)|}_{\text{some } f'} \leq \|x\|$$

Hence F is bounded. Finally $F|_M = f$ since each f' has this property, so in fact $\|F\| = 1$.

The fact that Ω is a maximal chain implies that $\tilde{M} = X$, for otherwise we could repeat first part of proof with $\tilde{M}$ to produce a larger chain. Hence F is the desired extension.

3/31 ANALYSIS

Case ②: X vector space over $\mathbb{C}$; $f: M \to \mathbb{R}$ real linear.
Simply regard X and M as a vector space over $\mathbb{R}$. Then follows from case ①

Case ③: X vector space over $\mathbb{C}$; $f: M \to \mathbb{C}$ linear.
Let $u := \operatorname{Re} f$ on M. Then u is a real-linear functional.
$\forall x \in M$

$$f(x) = u(x) - i\,u(ix)$$

and $\|f\| = \|u\|$. By case 2, there is an extension $U: X \to \mathbb{R}$ of u with $\|U\| = \|u\|$. Set

$$F(x) := U(x) - i\,U(ix)$$

$\forall x \in X$. Then F is complex linear and $\|F\| = \|U\| = \|u\| = \|f\|$.
Moreover, if $x \in M$,

$$F(x) = U(x) - i\,U(ix) = u(x) - i\,u(ix) = f(x)$$

$\blacksquare$

COROLLARIES:

① X normed linear space. M subspace. Then

$$x \in \overline{M} \text{ if and only if } \left(f(M) = 0 \Rightarrow f(x) = 0 \quad \forall f \in X^* \right)$$

Proof. Suppose $f \in X^*$, $f(M) = 0$, and $x \in \overline{M}$. Then by continuity, $f(x) = 0$.

Suppose $x_0 \notin \overline{M}$. Then $\exists\, \delta > 0$ s.t. $\|x - x_0\| \geq \delta \ \forall x \in M$. Let $M_1 = \text{sp}(M \cup \{x_0\}) = \{x + \lambda x_0 : x \in M, \lambda \in \mathbb{F}\}$. Define $f \in M_1^*$ by

$$f(x + \lambda x_0) := \lambda$$

$f(M) = 0$

Note $f(x_0) = 1$. Must check f is actually bounded. If $\lambda \neq 0$ then $\|x_0 + x/\lambda\| \geq \delta$. Hence

$$|f(x + \lambda x_0)| = |\lambda| \leq \frac{\|\lambda x_0 + x\|}{\delta} \Longrightarrow \|f\| \leq \frac{1}{\delta}$$

By Hahn-Banach, we can extend f to $F \in X^*$. Then $F(M) = 0$ and $F(x_0) = 1$.

$\blacksquare$

(2) X normed linear space, $x_0 \neq 0$. $\exists\, f \in X^*$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$

Proof: Let $M = \text{sp}\{x_0\}$. This is a subspace of X. Define $f : M \to \mathbb{F}$ by $f(\lambda x_0) = \lambda \|x_0\|$. Then $f \in M^*$ and $\|f\| = 1$. Extend to all of X.

$\blacksquare$

Complex Measures

Definition: Suppose $\mathcal{M}$ is a σ-algebra of subsets of X. If $E \in \mathcal{M}$ and

$$E = \bigcup_{i=1}^{\infty} E_i$$

where $(E_i) \subset \mathcal{M}$ and $E_i \cap E_j = \phi$, $i \neq j$, then we call (E_i) a partition of E

Definition: Suppose $\mathcal{M}$ is a σ-algebra of subsets of X. A complex measure is a function $\mu : \mathcal{M} \to \mathbb{C}$ which is countably additive, i.e. if (E_i) is a partition of $E \in \mathcal{M}$, then

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$$

Remark: Since $\sum_{i=1}^{\infty} \mu(E_i)$ is required to be independent of permutations of the sets E_i, we are in fact requiring $\sum \mu(E_i)$ to be absolutely convergent.

Definition: Define the total variation $|\mu|$ of μ to be

$$|\mu|(E) := \sup \left\{ \sum_{i=1}^{\infty} |\mu(E_i)| : (E_i) \text{ partition of } E \right\}$$

$\forall E \in \mathcal{M}$.

So $|\mu| : \mathfrak{M} \to [0, \infty]$.

PROPOSITION: $|\mu|$ is a positive measure on $\mathfrak{M}$.

4/3 MEASURE THEORY CH 5 #6 (without H-B), #13, #16 (4/12)

Remark: Suppose λ is a positive measure on $\mathfrak{M}$ s.t.

$$\lambda(E) \geq |\mu(E)| \quad \forall E \in \mathfrak{M}$$

Then $\lambda(E) \geq |\mu|(E) \quad \forall E \in \mathfrak{M}$. [[Suppose $E = \bigcup_{i=1}^{\infty} E_i$, E_i disjoint

$$\Longrightarrow \quad \lambda(E) = \sum \lambda(E_i) \geq \sum |\mu(E_i)|$$

$$\Longrightarrow \quad \lambda(E) \geq |\mu|(E)$$

(sup over all partitions)]]

THEOREM: If μ is a complex measure on $\mathfrak{M}$, then $|\mu|$ is a positive measure.

Proof. Suppose $E \in \mathfrak{M}$, (E_i) partition of E. Must show $|\mu|(E) = \sum |\mu|(E_i)$

Suppose $t_i < |\mu|(E_i)$. By definition of $|\mu|$, there is a partition $(A_{ij} : j \in \mathbb{N})$ of E_i s.t.

$$\sum_{j=1}^{\infty} |\mu(A_{ij})| > t_i$$

Then $(A_{ij} : i,j \in \mathbb{N})$ is a partition of E, and so

$$|\mu|(E) \geq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\mu(A_{ij})| \geq \sum_{i=1}^{\infty} t_i$$

Since $t_i < |\mu|(E_i)$ is arbitrary, we get

$$|\mu|(E) \geq \sum_{i=1}^{\infty} |\mu|(E_i)$$

Let (A_j) be any partition of E.

$$\sum_{j=1}^{\infty} |\mu(A_j)| = \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} \mu(A_j \cap E_i) \right| \leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\mu(A_j \cap E_i)|$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\mu(A_j \cap E_i)|$$

$$\leq \sum_{i=1}^{\infty} |\mu|(E_i)$$

$\uparrow$

($(A_j \cap E_i : j \in \mathbb{N})$ partition of E_i)

Now sup over all partitions (A_j) of E, we get

$$|\mu|(E) \leq \sum_{i=1}^{\infty} |\mu|(E_i)$$

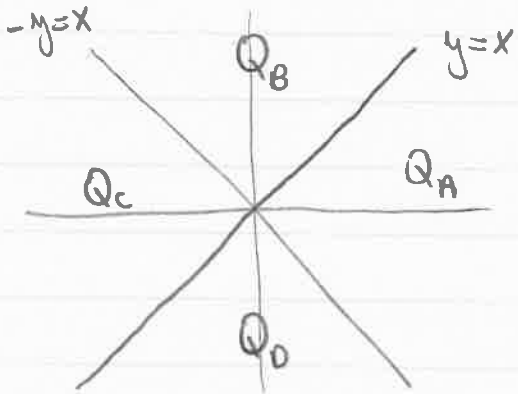
Since $|\mu|(\emptyset) = 0$, $|\mu|$ is not identically zero.

∎

LEMMA: Suppose $z_1, \dots, z_n$ are in $\mathbb{C}$. $\exists S \subset \{1, \dots, n\}$ s.t.

$$\left| \sum_{j \in S} z_j \right| \geq \frac{1}{6} \sum_{j=1}^{n} |z_j|$$

Proof. Let $W = \sum_{j=1}^{n} |z_j|$



WLOG : If we let $S = \{ j : 1 \le j \le n \text{ s.t. } z_j \in Q_A \}$, then

$$\sum_{j \in S} |z_j| \ge \frac{W}{4}$$

Then

$$\left| \sum_{j \in S} z_j \right| \ge \operatorname{Re} \sum_{j \in S} z_j \ge \frac{1}{\sqrt{2}} \sum_{j \in S} |z_j| > \frac{W}{\sqrt{2} \cdot 4} > \frac{W}{6}$$

$\boxtimes$

<u>PROPOSITION</u>: Suppose μ is a complex measure on a σ-algebra $\mathcal{M}$ of subsets of X. Then $|\mu|(X) < \infty$. (In particular, $\{\mu(E) : E \in \mathcal{M}\}$ is a bounded subset of $\mathbb{C}$)

Proof. Suppose $|\mu|(E) = \infty$ for some $E \in \mathcal{M}$. Then we claim $E = A \cup B$, where $A \cap B = \emptyset$, $|\mu|(A) = +\infty$ and $|\mu(B)| \ge 1$.

For every $t > 0$, there is a partition (E_i) of E s.t.

$$\sum_{j=1}^{\infty} |\mu(E_j)| > t$$

Apply with $t = 6(1+|\mu(E)|)$. By the lemma, there is a finite set S of integers s.t.

$$\left|\sum_{j\in S} \mu(E_j)\right| > t/6 \geq 1$$

Set $A = \bigcup_{j\in S} E_j$. Then $|\mu(A)| \geq 1$. Set $B = E - A$

Then $\mu(B) = \mu(E) - \mu(A)$, and so

$$|\mu(B)| = |\mu(A) - \mu(E)| \geq |\mu(A)| - |\mu(E)|$$

$$> \frac{t}{6} - |\mu(E)| \underset{\substack{\uparrow \\ \text{choice of } t}}{\geq} 1$$

Thus $|\mu(A)| \geq 1$, $|\mu(B)| \geq 1$.
Now

$$|\mu|(A) + |\mu|(B) = |\mu|(E) = \infty$$

so suppose WLOG that $|\mu|(A) = \infty$. This establishes the claim

Thus if $|\mu|(X) = \infty$, then $X = A_0 \cup B_0$, disjoint union, with $|\mu|(A_0) = \infty$ and $|\mu(B_0)| \geq 1$. Then $A_0 = A_1 \cup B_1$ disjoint with $|\mu|(A_1) = \infty$ and $|\mu(B_1)| \geq 1$. Continuing by

induction, $\exists$ disjoint $B_j \in \mathcal{M}$ s.t. $|\mu(B_j)| \geq 1 \ \forall j$
But

$$\mu\left(\bigcup_{j=1}^{\infty} B_j\right) \underset{(B_j) \text{ disjoint}}{\uparrow}{=} \sum_{j=1}^{\infty} \mu(B_j)$$

and the above shows that $\sum \mu(B_j)$ does not converge.
Thus $|\mu|(X) < \infty$.

$\boxtimes$

DEFINITION: Fix a σ-algebra $\mathcal{M}$ of subsets of X.
Suppose λ, μ are complex measures on $\mathcal{M}$, $c \in \mathbb{C}$.

$$\forall E \in \mathcal{M} \qquad \begin{aligned} (\mu+\lambda)(E) &:= \mu(E) + \lambda(E) \\ (c\mu)(E) &:= c\,\mu(E) \end{aligned}$$

(Then $\mu+\lambda$ and $c\mu$ are complex measures) Define

$$\|\mu\| := |\mu|(X)$$

Then the set of complex measures on $\mathcal{M}$ with this norm is a normed linear space

$$\|\mu_1+\mu_2\| = |\mu_1+\mu_2|(X) = \sup \sum |(\mu_1+\mu_2)(E_i)|$$

$$\leq \sup\left(\sum |\mu_1(E_i)| + |\mu_2(E_i)|\right)$$

$$\leq \sup \sum |\mu_1(E_i)| + \sup \sum |\mu_2(E_i)|$$

$$= |\mu_1|(X) + |\mu_2|(X) = \|\mu_1\| + \|\mu_2\|$$

$$\|\mu\| = 0 \iff |\mu|(X) = 0 \iff |\mu| = 0 \iff \mu = 0$$

$$\uparrow$$

$$|\mu(E)| \leq |\mu|(E)$$

<u>DEFINITION</u>: Let μ a complex measure taking only real values

$$\mu^+ := \frac{1}{2}(|\mu| + \mu)$$

$$\mu^- := \frac{1}{2}(|\mu| - \mu)$$

Then μ^+ and μ^- are positive measures (since $|\mu(E)| \leq |\mu|(E)$)

$$|\mu| = \mu^+ + \mu^- \ ; \quad \mu = \mu^+ - \mu^-$$

4/5 MEASURE THEORY

For this part μ will denote a positive measure on $\mathfrak{M}$ and λ a (complex or positive) measure on $\mathfrak{M}$

DEFINITION: λ is absolutely continuous w.r.t. μ ($\lambda \ll \mu$) if $\mu(E) = 0 \Rightarrow \lambda(E) = 0$

DEFINITION: If $A \in \mathfrak{M}$, we say λ is concentrated on A if

$$\lambda(E) = \lambda(E \cap A) \quad \forall E \in \mathfrak{M}$$

Remark. λ is concentrated on A iff $\lambda(E) = 0$ whenever $E \in \mathfrak{M}$ and $E \cap A = \emptyset$

Proof. Suppose $\lambda(B) = 0$ $\forall B$ s.t. $B \cap A = \emptyset$. Given any $E \in \mathfrak{M}$,

$$\lambda(E) = \lambda(E \cap A) + \lambda(E - A) = \lambda(E \cap A)$$

Conversely, if λ is concentrated on A and $E \subset X - A$, then

$$\lambda(E) = \lambda(E \cap A) = \lambda(\emptyset) = 0$$

DEFINITION: λ_1 and λ_2 are mutually singular ($\lambda_1 \perp \lambda_2$) if λ_1 and λ_2 are concentrated on disjoint sets.

PROPOSITION: $(X, \mathcal{M})$ σ-algebra. μ positive measure; $\lambda, \lambda_1, \lambda_2$ complex measures

★ (a) If λ concentrated on A, then $|\lambda|$ concentrated on A

Proof. $\lambda(B) = 0$ $\forall B \subset A^c$. If $B \subset A^c$ and (B_i) is a partition of B, then $B_i \subset B \subset A^c$. Hence

$$|\lambda|(B) = \sup_{\substack{\text{all} \\ \text{partitions}}} \sum |\lambda(B_i)| = 0 \qquad \forall B \subset A^c$$

★ (b) If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$

Proof. λ_1 concentrated on A_1, λ_2 concentrated on A_2, with $A_1 \cap A_2 = \emptyset$. Then (a) $\Rightarrow$ $|\lambda_1|$ concentrated on A_1 and $|\lambda_2|$ concentrated on A_2

★ (c) $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$ $\Rightarrow$ $\lambda_1 + \lambda_2 \ll \mu$

Proof. Suppose $\mu(E) = 0$. Then $\lambda_1(E) = \lambda_2(E) = 0$, and so $(\lambda_1 + \lambda_2)(E) = \lambda_1(E) + \lambda_2(E) = 0$.

★ (d) $\lambda_1 \perp \lambda$, $\lambda_2 \perp \lambda$ $\Rightarrow$ $\lambda_1 + \lambda_2 \perp \lambda$

Proof. λ_1 concentrated on A_1, λ concentrated on B_1 with $A_1 \cap B_1 = \emptyset$. λ_2 concentrated on A_2, λ concentrated on B_2 with $A_2 \cap B_2 = \emptyset$. Then $\lambda_1 + \lambda_2$ is concentrated on $A_1 \cup A_2$ and λ is concentrated on $B_1 \cap B_2$

$$\left(Y \subset X - (B_1 \cap B_2) = (X - B_1) \cup (X - B_2) \Rightarrow \right.$$

$$\lambda(Y) = \lambda(Y \cap (X - B_1)) + \lambda(Y \cap (X - B_2))$$

$$= 0 + 0 = 0$$

Note $(A_1 \cup A_2) \cap (B_1 \cap B_2) = \phi$.

★(e) If $\lambda \ll \mu$, then $|\lambda| \ll \mu$

Proof. If $\mu(E) = 0$ and (E_i) is a partition of E, $\lambda(E_i) = 0 \quad \forall i$. Hence $|\lambda|(E) = 0$

★(f) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.

Proof. λ_2 concentrated on A, μ concentrated on B, with $A \cap B = \phi$. Then $\mu(E) = 0 \quad \forall E \subset X - B \Rightarrow \lambda_1(E) = 0 \quad \forall E \subset X - B$. Hence λ_1 is concentrated on B.

★(g) If $\lambda \ll \mu$, $\lambda \perp \mu$, then $\lambda = 0$

Proof. By (f), $\lambda \perp \lambda$. So $\exists\ A, B$ with $A \cap B = \phi$ and λ concentrated on A and concentrated on B. Hence $\lambda(E) = 0$ for any $E \in \mathfrak{M}$

$$\lambda(E) = \lambda(\underbrace{E \cap A}_{\subset X-B}) + \lambda(\underbrace{E \cap B}_{\subset X-A}) = 0 + 0 = 0$$

$\boxtimes$

<u>LEMMA</u>: Suppose μ is a positive measure on $(X, \mathcal{M})$ and $\mu(X) < \infty$. If $f \in L^1(\mu)$ and S a closed set in $\mathbb{C}$ s.t. for every $E \in \mathcal{M}$ with $\mu(E) > 0$ and

$$\frac{1}{\mu(E)} \int_E f \, d\mu \in S$$

then $f(x) \in S$ for almost all x.

Proof. Let $\Delta := \{ z : |z - \alpha| \leq r \} \subset \mathbb{C} - S$. Sufficient to show $\mu(E) = 0$ where $E = f^{-1}(\bar{\Delta})$ since $\mathbb{C} - S$ is a countable union of such $\bar{\Delta}$'s

Suppose $\mu(E) > 0$

$$\left| \frac{1}{\mu(E)} \int_E f \, d\mu - \alpha \right| = \left| \frac{1}{\mu(E)} \int_E (f - \alpha) \, d\mu \right|$$

$$\leq \frac{1}{\mu(E)} r \cdot \mu(E) = r$$

$$\Rightarrow \frac{1}{\mu(E)} \int_E f \, d\mu \in \bar{\Delta} \subset \mathbb{C} - S \quad \lightning$$

Hence $\mu(E) = 0$.

$\blacksquare$

<u>RADON-NIKODYM THEOREM</u>: Suppose λ and μ are both positive bounded measures on a σ-algebra $\mathcal{M}$ in X. Then there exists a unique pair of measures λ_a and λ_s such that λ_a is absolutely continuous w.r.t. μ, λ_s is singular w.r.t. μ and $\lambda = \lambda_a + \lambda_s$. λ_a and λ_s are positive measures and $\lambda_a \perp \lambda_s$.

Moreover, there is a unique $h \in L^1(\mu)$ s.t.

$$(*) \qquad \lambda_a(E) = \int_E h \, d\mu \quad \forall E \in \mathcal{M}$$

Proof. Suppose $\lambda = \lambda_a' + \lambda_s'$ where $\lambda_a' \ll \mu$ and $\lambda_s' \perp \mu$. Then $\lambda_a - \lambda_a' = \lambda_s - \lambda_s'$. Thus $(\lambda_a - \lambda_a') \ll \mu$ and $(\lambda_a - \lambda_a') \perp \mu$ $\Rightarrow \lambda_a - \lambda_a' = 0$, so $\lambda_a = \lambda_a'$ and $\lambda_s = \lambda_s'$.

Recall by (5), $\lambda_a \ll \mu$ and $\lambda_s \perp \mu \Rightarrow \lambda_a \perp \lambda_s$

Suppose there were another $h_1 \in L^1(\mu)$ satisfying $(*)$. Then

$$\int_E (h_1 - h) \, d\mu = 0 \quad \forall E \in \mathcal{M}$$

and so $h = h_1$ a.e., i.e. $h = h_1$ in $L^1(\mu)$.

4/7 MEASURE THEORY

(writing λ as $\lambda_a + \lambda_s$ where $\lambda_a \ll \mu$ and $\lambda_s \perp \mu$ is called the Lebesgue decomposition of λ w.r.t. μ)

(Continuation of proof of R-N)

Set $\varphi = \lambda + \mu$. Note $\varphi(X) < \infty$. $E \in \mathfrak{M} \Rightarrow$ $\varphi(E) = \lambda(E) + \mu(E)$, so

$$(*) \qquad \int_X f\, d\varphi = \int_X f\, d\lambda + \int f\, d\mu$$

for $f = \chi_E$, $E \in \mathfrak{M}$. Hence $(*)$ holds for $f =$ simple function, and so for non-negative measurable functions by M.C.T. Therefore $(*)$ holds for all $f \in L^1(\varphi)$

If $f \in L^2(\varphi)$, then

$$\left|\int_X f\, d\lambda\right| \le \int_X |f|\, d\lambda \le \int_X |f|\, d\varphi \le \left(\int_X |f|^2\, d\varphi\right)^{1/2} \varphi(X)^{1/2}$$

Hence $f \to \int f\, d\lambda$ is a bounded linear functional on $L^2(\varphi)$, so there exists $g \in L^2(\varphi)$ s.t.

$$(**) \qquad \int_X f\, d\lambda = (f, \bar{g}) = \int_X fg\, d\varphi \quad \forall f \in L^2(\varphi)$$

Take $f = \chi_E$, $E \in \mathfrak{M}$ for which $\varphi(E) > 0$. By $(**)$

$$\lambda(E) = \int_E g \, d\varphi$$

$$\Rightarrow \frac{1}{\varphi(E)} \int_E g \, d\varphi = \frac{\lambda(E)}{\varphi(E)} \in [0,1]$$

Lemma from previous section $\Rightarrow$ $g(x) \in [0,1]$ a.e. $[\varphi]$. WLOG $g(x) \in [0,1]$ $\forall x \in X$.

$$(*) \text{ and } (**) \Rightarrow \int_X f(1-g)\, d\lambda = \int_X fg \, d\mu \quad \forall f \in L^2(\varphi) \quad (\dagger)$$

$f \in L^2(\varphi)$, $g \in L^2(\varphi) \Rightarrow fg \in L^1(\varphi)$ by Holder

Set

$$A := \{x \in X : 0 \leq g(x) < 1\}$$

$$B := \{x \in X : g(x) = 1\}$$

Define

$$\lambda_a(E) := \lambda(E \cap A) \quad \forall E \in \mathfrak{M}$$
$$\lambda_s(E) := \lambda(E \cap B) \quad \forall E \in \mathfrak{M}$$

Clearly λ_a and λ_s are positive measures on $\mathfrak{M}$ since λ is, and $\lambda = \lambda_a + \lambda_s$ (since $A \cap B = \emptyset$)

If $Y \cap B = \emptyset$, then $\lambda_s(Y) = \lambda(\emptyset) = 0$, whence λ_s is concentrated on B. Set $f = \chi_B$ in $(\dagger)$

$$0 = \int_B (1-g)\, d\lambda = \int_B g\, d\mu = \mu(B)$$

since $g=1$ on B

Therefore $\mu \perp \lambda_s$

In $(\dagger)$ set $f = (1+g+g^2+\dots+g^n)\chi_E$. Note $f \in L^2(\varphi)$

$$\int_E (1-g^{n+1})\, d\lambda = \int_E (g+g^2+\dots+g^{n+1})\, d\mu$$

On B, $1-g^{n+1} = 0$. On A, $(1-g^{n+1}) \uparrow 1$. Therefore $1-g^{n+1} \uparrow \chi_A$

$$\text{MCT} \Rightarrow \text{LHS} \longrightarrow \int_E \chi_A\, d\lambda = \lambda(E\cap A) = \lambda_a(E)$$

As $n \to \infty$, $g+g^2+\dots+g^{n+1} \longrightarrow g/1-g$. Set

$$h = \begin{cases} +\infty & \text{if } g(x)=1 \iff x \in B \\ g/1-g & \text{otherwise} \end{cases}$$

Then

$$\text{RHS} \longrightarrow \int_E h\, d\mu$$

Hence $\forall E \in \mathfrak{M}$

$$\lambda_a(E) = \int_E h\, d\mu$$

Set $E = X$

$$\infty > \lambda_a(X) = \int_X h\,d\mu \implies h \in L^1(\mu)$$

Also, if $\mu(E) = 0$

$$\lambda_a(E) = \int_E h\,d\mu = 0$$

and so $\lambda_a \ll \mu$.

$\blacksquare$

EXTENSIONS

Case I: $\lambda(X) < \infty$, X σ-finite w.r.t. μ OR
Case II: λ complex measure, X σ-finite w.r.t. μ OR

I $\Rightarrow$ II : Write $\lambda = \lambda_1 + i\lambda_2$ where λ_1, λ_2 are real-valued

$$\left.\begin{aligned} \lambda_1^+ &= \tfrac{1}{2}(|\lambda_1| + \lambda_1) \\ \lambda_1^- &= \tfrac{1}{2}(|\lambda_1| - \lambda_1) \end{aligned}\right\} \text{positive, bounded measures}$$

Then $\lambda_1^+ = (\lambda_1^+)_a + (\lambda_1^+)_s$ where $(\lambda_1^+)_a \ll \mu$, $(\lambda_1^+)_s \perp \mu$ and

$$(\lambda_1^+)_a = \int_E h_1\,d\mu$$

$h_1 \geq 0$, $h_1 \in L^1(\mu)$. Also, $\lambda_1^- = (\lambda_1^-)_a + (\lambda_1^-)_s$ etc.

$$\lambda_1 = \underbrace{[(\lambda_1^+)_a - (\lambda_1^-)_a]}_{\substack{\text{absolutely cont.} \\ \text{w.r.t. } \mu}} + \underbrace{[(\lambda_1^+)_s - (\lambda_1^-)_s]}_{\substack{\text{singular} \\ \text{w.r.t. } \mu}}$$

Similar for imaginary part

Sketch of proof for case I: WLOG, $X_n \cap X_m = \emptyset$. Define

$$\mu_n(E) := \mu(E \cap X_n)$$
$$\lambda_n(E) := \lambda(E \cap X_n)$$

Then μ_n and λ_n satisfy hypothesis of R-N, so

$$\lambda_n = (\lambda_n)_a + (\lambda_n)_s$$

where $(\lambda_n)_a \ll \mu_n$ and $(\lambda_n)_s \perp \mu$, and

$$(\lambda_n)_a(E) = \int_E h_n \, d\mu_n$$

WLOG $h_n = 0$ on $X - X_n$.

Note $\lambda = \sum \lambda_n$ (If $E \in \mathcal{M}$, $\lambda(E) = \lambda(\bigcup (E \cap X_n))$
$= \sum \lambda(E \cap X_n) = \sum \lambda_n(E)$). Let

$$\lambda_a := \sum_{n=1}^{\infty} (\lambda_n)_a$$
$$\lambda_s := \sum_{n=1}^{\infty} (\lambda_n)_s$$

Check λ_a, λ_s measures on $\mathcal{M}$; $\lambda_a \ll \mu$, $\lambda_s \perp \mu$; for $E \in \mathcal{M}$

$$\lambda_a(E) = \int_E h \, d\mu$$

where $h = \sum_{n=1}^{\infty} h_n \in L^1(\mu)$

4/10 MEASURE THEORY

PROPOSITION: Let μ be a positive measure and λ a complex measure. TFAE

(a) $\lambda \ll \mu$

(b) $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $E \in \mathcal{M}$ with $\mu(E) < \delta$, then $|\lambda|(E) < \varepsilon$

Proof. Suppose (b) holds. Suppose $E \in \mathcal{M}$, $\mu(E) = 0$. Let $\varepsilon > 0$. Then by (b) $|\lambda|(E) < \varepsilon$. Hence $|\lambda|(E) = 0$, so that $\lambda(E) = 0$

Suppose (b) doesn't hold. $\exists\ \varepsilon > 0$ and $(E_n) \subset \mathcal{M}$ s.t.

$$\mu(E_n) < 1/2^n$$
$$|\lambda|(E_n) \geq \varepsilon$$

Let $A_n = \bigcup_{j=n}^{\infty} E_j$ and $A = \bigcap_{n=1}^{\infty} A_n$. Then

$$\mu(A) = \lim_{n\to\infty} \mu(A_n) \leq \lim_{n\to\infty} 2^{1-n} = 0$$

However

$$|\lambda|(A_n) \geq |\lambda|(E_n) \geq \varepsilon \quad \forall n \in \mathbb{N}$$

and so

$$|\lambda|(A) = \lim_{n\to\infty} |\lambda|(A_n) \geq \varepsilon$$

Therefore $|\lambda|$ is not absolutely continuous w.r.t. μ, so that λ is not absolutely cont. w.r.t. μ

$\blacksquare$

(Can replace statement in (b) by $|\lambda(E)| < \varepsilon$)

<u>THEOREM</u>: Suppose λ is a complex measure on $(X, \mathcal{M})$. Then there exist a measurable $h : X \longrightarrow \{z \in \mathbb{C} : |z| = 1\}$ s.t.

$$\forall E \in \mathcal{M} \qquad \lambda(E) = \int_E h \, d|\lambda|$$

(also written $d\lambda = h \, d|\lambda|$)

Proof. Certainly $\lambda << |\lambda|$. Now

$$\lambda = \operatorname{Re}\lambda + i \operatorname{Im}\lambda$$

and so
$$(\operatorname{Re}\lambda)^+ << |\lambda|$$
$$(\operatorname{Im}\lambda)^+ << |\lambda|$$
$$(\operatorname{Re}\lambda)^- << |\lambda|$$
$$(\operatorname{Im}\lambda)^- << |\lambda|$$

Recall $|\lambda|(X) < \infty$. By the Radon-Nikodym Theorem

$$(\operatorname{Re}\lambda)^+(E) = \int_E h_1 \, d|\lambda|$$

for some $h_1 \geq 0$, $h_1 \in L^1[|\lambda|]$. If we do this for each part, we see that

$$\lambda(E) = \int_E h \, d|\lambda|$$

for some $h \in L^1[|\lambda|]$.

Must show h can be chosen so that $|h(x)| = 1$ everywhere. Select $r < 1$, and let

$$A_r = \{x : |h(x)| \leq r\}$$

Let $\{E_i\}$ be any partition of A_r.

$$\sum_{j=1}^{\infty} |\lambda(E_j)| = \sum_{j=1}^{n} \left| \int_{E_j} h \, d|\lambda| \right| \leq \sum_{j=1}^{\infty} \int_{E_j} |h| \, d|\lambda|$$

$$\leq r \sum_{j=1}^{\infty} \int_{E_j} d|\lambda| = r \, |\lambda|(A_r)$$

Sup over all partitions:

$$|\lambda|(A_r) \leq r \, |\lambda|(A_r)$$

But $r < 1$, so we must have $|\lambda|(A_r) = 0$. Therefore $|h(x)| \geq 1$ a.e.

Suppose $E \in \mathfrak{M}$, $|\lambda|(E) > 0$.

$$\left| \frac{1}{|\lambda|(E)} \int_E h \, d|\lambda| \right| = \frac{1}{|\lambda|(E)} |\lambda(E)| \leq 1$$

Hence all averages of h over E s.t. $|\lambda|(E) > 0$ lie in $\{|z| \leq 1\}$.
Therefore $|h(x)| \leq 1$ a.e.

Hence $|h(x)| = 1$ a.e. Redefine h as follows: if $|h(x)| \neq 1$, change so that $h(x) = 1$; Don't change h anywhere else.

$\boxed{/}$

(will use this ↑ to define complex integration)

<u>HAHN DECOMPOSITION THEOREM</u>: μ real measure on $(X, \mathcal{M})$.
Then there are sets $A, B \in \mathcal{M}$ such that

$$A \cap B = \emptyset$$
$$A \cup B = X$$

such that

$$\mu^+(E) = \mu(E \cap A)$$
$$\mu^-(E) = -\mu(E \cap B)$$

(Note: If $E \subset A$, then $\mu(E) = \mu^+(E) \geq 0$ and if $E \subset B$ then $\mu(E) = -\mu^-(E) \leq 0$.)

Proof $\exists\, h : X \longrightarrow T$ s.t. $\forall E \in \mathcal{M}$

$$\mu(E) = \int_E h \, d|\mu|$$

Let $E = \{x : \operatorname{Im} h(x) > 0\}$

$$\int_E \operatorname{Im} h(x)\, d|\mu| = \operatorname{Im} \int_E h(x)\, d|\mu| = 0$$

$\int h(x)\, d|\mu| = \mu(E) \in \mathbb{R}$

Hence $|\mu|(E) = 0$

Therefore $h(x) = \pm 1$ a.e. $[|\mu|]$. Modify h s.t. $h(x) = 1$ if previously $\operatorname{Im} h(x) \neq 0$. Then $h(x) = \pm 1$ everywhere.

$$\mu^+ = \frac{1}{2}(|\mu| + \mu)$$

$$E \in \mathfrak{M} \Rightarrow \mu^+(E) = \frac{1}{2}\int_E (1+h)\, d|\mu| = \int_{E\cap A} h\, d|\mu| = \mu(E\cap A)$$

$$\left(\text{Note} \quad \frac{1}{2}(1+h) = \begin{cases} h & x \in A := \{x : h(x) = 1\} \\ 0 & x \in B := \{x : h(x) = -1\} \end{cases} \right)$$

Now

$$\mu(E) = \mu^+(E) - \mu^-(E)$$

$$\mu(E) = \mu(A\cap E) + \mu(B\cap E)$$

and so $\mu^-(E) = -\mu(B\cap E)$.

$\blacksquare$

Corollary: If μ is a real measure on X and

$\mu = \lambda_1 - \lambda_2$, where λ_1, λ_2 are positive, then $\mu^+ \leq \lambda_1$ and $\mu^- \leq \lambda_2$

Proof. Let A be as in Hahn Decomposition Theorem. Let $E \in \mathcal{M}$.

$$\mu^+(E) = \mu(E \cap A) \underset{\lambda_2 \geq 0}{\leq} \lambda_1(E \cap A) \underset{\lambda_1 \geq 0}{\leq} \lambda_1(E)$$

Also

$$\mu^- = \mu^+ - \mu = \mu^+ - \lambda_1 + \lambda_2 \underset{\mu^+ - \lambda_1 \leq 0}{\leq} \lambda_2$$

∎

<u>**PROPOSITION:**</u> μ positive measure on $(X, \mathcal{M})$, $g \in L^1(\mu)$. Let

$$\lambda(E) := \int_E g \, d\mu \qquad \forall E \in \mathcal{M}$$

(Note λ is a complex measure by D.C.T.) Then

$$|\lambda|(E) = \int_E |g| \, d\mu \qquad \forall E \in \mathcal{M}$$

Proof. $\exists\, h : X \longrightarrow T$ s.t.

$$\lambda(E) = \int_E h \, d|\lambda|$$

Also

$$\lambda(E) = \int_E g \, d\mu$$

Therefore

$$\int_E h \, d|\lambda| = \int_E g \, d\mu \qquad \forall E \in \mathfrak{M}$$

Hence

$$\int_E f h \, d|\lambda| = \int_E f g \, d\mu$$

for $f = \chi_{E'}$ $\Rightarrow$ for $f =$ simple function $\Rightarrow$ for $f =$ unif. limit of simple functions. Now $\bar{h}$ can be unif. approx. by simple functions, so

$$E \in \mathfrak{M} \Rightarrow \int_E g\bar{h} \, d\mu = \int_E h\bar{h} \, d|\lambda| = \int_E d|\lambda| = |\lambda|(E)$$

Now left to show that $g\bar{h} \geq 0$ a.e. Hence $g\bar{h} = |g\bar{h}| = |g|$ a.e. Therefore

$$|\lambda|(E) = \int_E |g| \, d\mu$$

$\blacksquare$

4/12 MEASURE THEORY

THEOREM: μ positive σ-finite measure. $\Phi : L^p(\mu) \to \mathbb{C}$ bounded linear functional ($1 \le p < \infty$). Then there is a unique $g \in L^q(\mu)$ such that

$$(*) \qquad \Phi(f) = \int_X fg \, d\mu$$

where $1/p + 1/q = 1$. Furthermore, $\|g\|_q = \|\Phi\|$.

Proof: uniqueness

If $\int fg \, d\mu = \int f g' \, d\mu \quad \forall f \in L^p(\mu)$, then $\int_E (g - g') \, d\mu = 0$ for all $E \in \mathfrak{M}$ with $\mu(E) < \infty$. Hence $g = g'$ a.e. (need σ-finiteness here)

If $(*)$ holds, then $\|\Phi\| \le \|g\|_q$ by Hölder. So, we must produce $g \in L^q(\mu)$ and show $\|g\|_q \le \|\Phi\|$ and $(*)$.

First suppose $\mu(X) < \infty$. For $E \in \mathfrak{M}$, define

$$\lambda(E) := \Phi(\chi_E)$$

Finitely additive since

$$E_1 \cap E_2 = \emptyset \Rightarrow \lambda(E_1 \cup E_2) = \Phi(\chi_{E_1 \cup E_2}) = \Phi(\chi_{E_1} + \chi_{E_2})$$

$$= \Phi(\chi_{E_1}) + \Phi(\chi_{E_2}) = \lambda(E_1) + \lambda(E_2)$$

Now suppose (E_i) is a partition of $E \in \mathfrak{M}$. Let

$$A_k = \bigcup_{i=1}^{k} E_i$$

Then $E - A_k \supset E - A_{k-1}$ and $\bigcap (E - A_k) = \emptyset$, so

$$\| \chi_E - \chi_{A_k} \|_p = (\mu(E - A_k))^{1/p} \longrightarrow 0$$

Thus $\chi_{A_k} \longrightarrow \chi_E$ in L^p, so $\Phi(\chi_{A_k}) \longrightarrow \Phi(\chi_E)$, i.e.

$$\sum_{i=1}^{k} \lambda(E_i) \longrightarrow \lambda(E)$$

Therefore λ is a complex measure. Moreover, $\lambda \ll \mu$, for if $\mu(E) = 0$, then $\chi_E = 0$ in L^p, so $\lambda(E) = \Phi(0) = 0$.
By the Radon-Nikodym theorem, there is a $g \in L^1(\mu)$ such that

$$\lambda(E) = \int_E g \, d\mu$$

Hence

$$(\ast\ast) \qquad \Phi(f) = \int_X f g \, d\mu$$

if $f = \chi_E \Rightarrow$ if f = simple function. If $f = \lim f_n$, f_n simple and limit uniform, then $\Phi(f_n) \to \Phi(f)$ since $\mu(X) < \infty \Rightarrow$ (uniform convergence $\Rightarrow L^p$ convergence). Therefore

$$\Phi(f) = \lim \Phi(f_n) = \lim \int f_n g \, d\mu = \int_X f g \, d\mu$$

Hence (**) holds for all bounded measurable f.

Case I: $p=1$

Let $f = \chi_E$, $\mu(E) > 0$. Then

$$\left| \int_E g \, d\mu \right| = \left| \Phi(\chi_E) \right| \leq \|\Phi\| \, \mu(E)$$

and so

$$\left| \frac{1}{\mu(E)} \int_E g \, d\mu \right| \leq \|\Phi\|$$

Therefore $|g| \leq \|\Phi\|$ a.e., whence $\|g\|_\infty \leq \|\Phi\|$

Case II: $1 < p < \infty$

For $n \in \mathbb{N}$, let $E_n = \{x \in X : |g(x)| \leq n\}$. $\exists$ measurable α such that $\alpha(x) g(x) = |g(x)|$ $\forall x \in X$. Consider the bounded measurable f on X given by

$$\forall x \in X \qquad f(x) := |g(x)|^{q-1} \alpha(x) \chi_{E_n}(x)$$

Note $f(x) g(x) = \chi_{E_n}(x) \, |g(x)|^q$. Also

$$|f(x)|^p = |g(x)|^{(q-1)p} \chi_{E_n}(x) = |g(x)|^q \chi_{E_n}(x)$$

$$\int_{E_n} |g|^q \, d\mu = \int_X f(x) g(x) \, d\mu \underset{\substack{\uparrow \\ f \text{ bounded, meas}}}{=} \Phi(f) \leq \|\Phi\| \, \|f\|_p$$

$$= \|\Phi\| \left(\int_X |f|^p \, d\mu \right)^{1/p} = \|\Phi\| \left(\int_{E_n} |g|^q d\mu \right)^{1/p}$$

and so

$$\int_{E_n} |g|^q d\mu = \left(\int_{E_n} |g|^q \right)^{q - q/p} \leq \|\Phi\|^q$$

Let $n \to \infty$ MCT shows that

$$\int |g|^q \, d\mu \leq \|\Phi\|^q$$

so that $g \in L^q(\mu)$ and $\|g\|_q \leq \|\Phi\|$.

Recall the set of bounded measurable functions is dense in $L^p(\mu)$. Therefore, given $f \in L^p$, $\exists (f_n) \subset L^p$ s.t. $\|f_n - f\|_p \to 0$ and

$$\Phi(f) = \lim \Phi(f_n) = \lim \int g f_n \, d\mu \underset{\substack{\uparrow \\ \text{Holder since } g \in L^q}}{\longrightarrow} \int g f \, d\mu$$

Done for $\mu(X) < \infty$.

Now suppose $X = \bigcup_{n=1}^{\infty} X_n$ with $0 < \mu(X_n) < \infty$ and X_n disjoint

Define $h : X \to (0,\infty)$ by

$$h(x) := \frac{1}{n^2} \frac{1}{\mu(X_n)} \qquad x \in X_n$$

Then $h \in L^1(\mu)$.

For $E \in \mathcal{M}$, let

$$\tilde{\mu}(E) := \int_E h \, d\mu$$

$\tilde{\mu}$ is a finite, positive measure on X. Recall

$$\int r(x) \, d\tilde{\mu} = \int r(x) h(x) \, d\mu$$

if $r(x) \geq 0$ is measurable. This also holds for $r \in L^1(\tilde{\mu})$.
Consider the mapping $F \mapsto h^{1/p} F$ for $F \in L^p(\tilde{\mu})$. This maps $L^p(\tilde{\mu})$ onto $L^p(\mu)$ and is 1-1, linear, norm-preserving

$$\int |F|^p \, d\tilde{\mu} = \int |F|^p h \, d\mu = \int \left(|F| h^{1/p}\right)^p d\mu$$

If $k \in L^p(\mu)$, then $h^{-1/p} k(x) \in L^p(\tilde{\mu})$ since

$$\int_X h^{-1} |k|^p \, d\tilde{\mu} = \int h \, h^{-1} |k|^p \, d\mu = \int |k|^p \, d\mu < \infty$$

Define $\psi : L^p(\tilde{\mu}) \to \mathbb{C}$ by

$$\psi(F) = \Phi(h^{1/p}F)$$

ψ bounded linear functional on $L^p(\tilde{\mu})$ with $\|\psi\| = \|\Phi\|$

$$\|\psi\| = \sup_{F \in L^p(\tilde{\mu})} \frac{|\psi(F)|}{\left(\int |F|^p d\tilde{\mu}\right)^{1/p}} = \sup_{\substack{\uparrow \\ F \in L^p(\tilde{\mu})}} \frac{|\Phi(h^{1/p}F)|}{\left(\int |h^{1/p}F|^p d\mu\right)^{1/p}} = \sup_{\substack{\uparrow \\ K \in L^p(\mu)}} \frac{|\Phi(K)|}{\left(\int |K|^p d\mu\right)^{1/p}} = \|\Phi\|$$

4/14 MEASURE THEORY

For $F \in L^p(\tilde{\mu})$ let $\psi(F) = \Phi(h^{1/p} F)$. ψ is a bounded linear functional on $L^p(\tilde{\mu})$ with $\|\psi\| = \|\Phi\|$. By the first part of the proof, $\exists\, G \in L^q(\tilde{\mu})$ s.t.

$$\psi(F) = \int F G \, d\tilde{\mu} \qquad \forall F \in L^p(\tilde{\mu})$$

Also $\|\psi\| = \|G\|_q$

Case I: $p = 1$ Set $g = G$

$$\underbrace{\|g\|_\infty}_{\text{w.r.t. } \mu} \underbrace{=}_{\mu(E)>0 \iff \tilde{\mu}(E)>0} \|G\|_\infty \leftarrow \text{w.r.t. } \tilde{\mu}$$

Then $\|g\|_\infty = \|G\|_\infty = \|\psi\| = \|\Phi\|$. Hence $g \in L^\infty(\mu)$

Case II: $1 < p < \infty$. Set $g = h^{1/q} G$

$$\int_X |g|^q \, d\mu = \int_X h |G|^q \, d\tilde{\mu} = \int_X |G|^q \, d\tilde{\mu}$$

Hence $g \in L^q(\mu)$ and $\|g\|_q = \|G\|_q = \|\psi\| = \|\Phi\|$

Back to case I:

Suppose $f \in L^p(\mu)$

$$\Phi(f) = \psi(h^{-1}f) = \int_X h^{-1} f G \, d\tilde{\mu} = \int_X h(h^{-1} f G) \, d\mu$$

$h^{-1} f G \in L^1(\tilde{\mu})$

$$= \int_X f G \, d\mu = \int_X f g \, d\mu$$

For case II: $\forall f \in L^p(\mu)$

$$\Phi(f) = \psi(h^{-1/p} f) = \int_X h^{-1/p} f G \, d\tilde{\mu} = \int_X h(h^{-1/p} f G) \, d\mu$$

$$= \int_X h^{1/q} f G \, d\mu = \int_X f g \, d\mu$$

$\blacksquare$

<u>LEMMA</u>: Suppose $\Phi : C_0(X) \to \mathbb{C}$ is a bounded $(\|\Phi\| = 1)$ linear functional (X locally compact T_2 space) $\exists \Lambda : C_c(X) \to \mathbb{C}$ positive linear functional s.t.

$$|\Phi(f)| \leq \Lambda(|f|) \leq \|f\|_\infty$$

Proof. Let $C_c^+(X) = \{ f \in C_c(X) : f(x) \geq 0 \ \forall x \in X \}$

For $f \in C_c^+(X)$, define

$$\Lambda f := \sup\{|\Phi(h)| : h \in C_c(X),\ |h| \le f\} < \infty$$

First show if $f, g \in C_c^+(X)$, then $\Lambda(f+g) = \Lambda f + \Lambda g$.

Suppose $\varepsilon > 0$. There exists $h_1 \in C_c(X)$ s.t. $|h_1| \le f$ and

$$|\Phi(h_1)| + \varepsilon > \Lambda f$$

$\exists\, h_2 \in C_c(X)$ s.t. $|h_2| \le g$ and

$$|\Phi(h_2)| + \varepsilon > \Lambda g$$

$\exists\, |\alpha_1| = 1, |\alpha_2| = 1$ s.t. $\alpha_j \Phi(h_j) = |\Phi(h_j)|$ $j = 1, 2$. Then

$$\begin{aligned}\Lambda f + \Lambda g &< |\Phi(h_1)| + |\Phi(h_2)| + 2\varepsilon \\ &= \alpha_1 \Phi(h_1) + \alpha_2 \Phi(h_2) + 2\varepsilon \\ &= \Phi(\alpha_1 h_1 + \alpha_2 h_2) + 2\varepsilon\end{aligned}$$

Note that $|\alpha_1 h_1 + \alpha_2 h_2| \le f + g$, and so

$$\Lambda f + \Lambda g \le \Lambda(f+g) + 2\varepsilon$$

Therefore $\Lambda f + \Lambda g \le \Lambda(f+g)$

Suppose $h \in C_c(X)$ and $|h| \le f + g$. Let

$$V = \{x \in X : f(x) + g(x) > 0\}$$

Define
$$h_1 = \begin{cases} \dfrac{f}{f+g}\, h & \text{on } V \\ 0 & \text{off } V \end{cases} \qquad h_2 = \begin{cases} \dfrac{g}{f+g}\, h & \text{on } V \\ 0 & \text{off } V \end{cases}$$

Then $h_1 + h_2 = h$ on all of X. Also $|h_j| \le |h|$ on all of X, $j = 1, 2$. Moreover h_j is continuous on X: clear on V; off V $h_j = h = 0$; result follows from $|h_j| \le |h|$ and h continuous. Also $|h_j| \le |h|$ $\Rightarrow$ supp h_j compact.

$$\Phi(h) = \Phi(h_1 + h_2) \underset{\substack{\uparrow \\ h_1, h_2 \in C_c(X)}}{=} \Phi(h_1) + \Phi(h_2)$$

$$\Longrightarrow |\Phi(h)| \le |\Phi(h_1)| + |\Phi(h_2)| \le \Lambda f + \Lambda g$$

$|h_1| = \dfrac{|h|}{f+g} f \le f$ on V

$|h_1| = 0 \le f$ on $X - V$

same for h_2

Since h was arbitrary, by taking sups we have

$$\Lambda(f+g) \le \Lambda f + \Lambda g$$

If $f \in C_c(X)$ and f real, define

$$\Lambda f := \Lambda f^+ - \Lambda f^-$$

If $f \in C_c(X)$, let

$$\Lambda f := \Lambda(\operatorname{Re} f) + i\, \Lambda(\operatorname{Im} f)$$

Definition of Λ of positive function $\Rightarrow$ $|\Phi(f)| \le \Lambda(|f|)$

If $|h| \le |f|$ on X, $\|\Phi\| = 1$ $\Rightarrow$ $|\Phi(h)| \le 1 \cdot \|h\|_\infty \le \|f\|_\infty$

Sup over all $h \in C_c(X)$, $|h| \le |f|$ gives

$$\Lambda(|f|) \le \|f\|_\infty$$

4/17 MEASURE THEORY

Chapter 3 Urysohn $\Rightarrow$ $C_c(X)$ dense in $C_0(X)$

Remark: If $\Phi : C_c(X) \to \mathbb{C}$ is a bounded linear functional then $\exists$ extension $\tilde{\Phi} : C_0(X) \to \mathbb{C}$ of the same norm

Integration with respect to a complex measure

Suppose μ is a complex measure on $(X, \mathfrak{M})$. Then there is a measurable h with $|h| = 1$ everywhere s.t. $d\mu = h\, d|\mu|$, i.e.

$$(*) \qquad \mu(E) = \int_E h\, d|\mu|$$

Note, if h_1 also satisfies $(*)$ and $|h_1| = 1$, then

$$\int_E (h - h_1)\, d|\mu| = 0 \quad \forall E \in \mathfrak{M}$$

and so $h_1 = h$ a.e. $[|\mu|]$. Thus we can define unambiguously for $f \in L^1(\mu)$

$$\int_X f\, d\mu := \int_X f h\, d|\mu|$$

Set $f = \chi_E$ for $E \in \mathfrak{M}$. Then

$$\int_X \chi_E\, d\mu = \int_X \chi_E h\, d|\mu| = \int_E h\, d|\mu| = \mu(E)$$

$$\int_X \chi_E \, d(\mu+\lambda) = (\mu+\lambda)(E) = \mu(E)+\lambda(E)$$
$$= \int_X \chi_E \, d\mu + \int_X \chi_E \, d\lambda$$

Hence $\int f \, d(\mu+\lambda) = \int f \, d\mu + \int f \, d\lambda$ for f = simple function

$\Rightarrow$ for $f \in L^1(|\mu|+|\lambda|)$ (since simple functions dense in this space - given $f \in L^1(|\mu|+|\lambda|)$, $\exists (f_n)$ simple with $f_n \to f$ in $L^1(|\mu|+|\lambda|)$

$$\left| \int (f_n - f) \, d\mu \right| = \left| \int (f_n - f) h \, d|\mu| \right|$$
$$\leq \int |f_n - f| \, d|\mu| \to 0$$

so $\int f_n \, d\mu \to \int f \, d\mu$)

Now take $f \in L^1(|\mu|)$.

$$\mu = (\mathrm{Re}\,\mu)^+ - (\mathrm{Re}\,\mu)^- + i\{(\mathrm{Im}\,\mu)^+ - \mathrm{Im}(\mu)^-\}$$

so that the above shows that

$$\int f \, d\mu = \int f \, d((\mathrm{Re}\,\mu)^+) - \int f \, d(\mathrm{Re}\,\mu^-)$$
$$+ i\left(\int f \, d(\mathrm{Im}\,\mu^+) - \int f \, d(\mathrm{Im}\,\mu^-) \right)$$

Let X be a locally compact T_2-space and μ a positive measure on $(X, \mathfrak{M})$, where $\mathfrak{M} \supset$ Borel sets. Recall that μ is regular if for every Borel set E

$$\sup\{\mu(K) : K \subset E, K \text{ compact}\} = \mu(E) = \inf\{\mu(V) : E \subset V, V \text{ open}\}$$

DEFINITION: If μ is a complex measure, we call μ regular if $|\mu|$ is regular.

Suppose μ is a complex measure on X. Then

$$f \xmapsto{\Phi} \int_X f \, d\mu$$

is a linear functional on $C_0(X)$ and

$$\left| \int f \, d\mu \right| = \left| \int f h \, d|\mu| \right| \leq \|f\|_\infty \, |\mu|(X)$$

so $\Phi : C_0(X) \to \mathbb{C}$ is a bounded linear functional with norm $\leq |\mu|(X)$

RIESZ REPRESENTATION THEOREM (#2). Let X be a locally compact T_2-space, $\Phi : C_0(X) \to \mathbb{C}$ a bounded linear functional. Then there is a unique regular complex Borel measure μ s.t. $\|\Phi\| = |\mu|(X)$ and

$$(*) \qquad \Phi(f) = \int_X f \, d\mu$$

Proof. Uniqueness: First show if μ_1 and μ_2 are both regular Borel measures on X, then $\mu_1 - \mu_2$ is a regular Borel measure. Suppose E is a Borel set. Let $\varepsilon > 0$. μ_1 regular $\Rightarrow \exists$ open $V_1 \supset E$ s.t.

$$|\mu_1|(V_1 - E) < \varepsilon$$

μ_2 regular $\Rightarrow \exists$ open $V_2 \supset E$ s.t.

$$|\mu_2|(V_2 - E) < \varepsilon$$

Let $V = V_1 \cap V_2 \supset E$.

$$|\mu_1 - \mu_2|(V - E) \leq |\mu_1|(V-E) + |\mu_2|(V-E) < 2\varepsilon$$

Hence $\mu_1 - \mu_2$ is outer regular. Inner regularity works the same.

Suppose μ_1 and μ_2 are regular complex Borel measures satisfying $(*)$. Then

$$\int_X f \, d(\mu_1 - \mu_2) = 0$$

$\forall f \in C_0(X)$. Let $\mu = \mu_1 - \mu_2$ and write $d\mu = h \, d|\mu|$. Consider $(f_n) \subset C_0(X)$ and

$$\int_X (\bar{h} - f_n) h \, d|\mu| = \int_X d|\mu| - \int_X f_n h \, d|\mu| = |\mu|(X) - \int f_n d\mu$$

$$= |\mu|(X)$$

Hence

$$|\mu|(X) = \int (\bar{h} - f_n) h \, d|\mu| \leq \int_X |\bar{h} - f_n| \, d|\mu| \to 0$$

(By chapter 3, since $|\mu|$ is regular, there exist $(f_n) \subset C_c(X)$ s.t. $f_n \to \bar{h}$ in $L^1(|\mu|)$) Hence $|\mu|(X) = 0 \Rightarrow |\mu| = 0 \Rightarrow \mu = 0$

4/19 MEASURE THEORY

By the last lemma, $\exists\, \Lambda : C_c(X) \to \mathbb{C}$ positive linear functional s.t.

$$(*) \quad |\Phi(f)| \leq \Lambda(|f|) \leq \|f\|_\infty \qquad \forall f \in C_c(X)$$

By the first Riesz Representation theorem, there is a positive measure λ on the Borel sets of X s.t.

$$\Lambda(f) = \int f \, d\lambda \quad \forall f \in C_c(X)$$

Recall

$$\lambda(X) = \sup\{\Lambda f : f \prec X\} = \sup\{\Lambda f : f \in C_c(X),\ 0 \leq f \leq 1\}$$

So if $f \in C_c(X)$, $0 \leq f \leq 1$ on X, then $(*)$ gives

$$\Lambda f \leq \|f\|_\infty \leq 1$$

and thus $\lambda(X) \leq 1$. By part (d) of RRT #1, $\lambda(X)$ finite implies λ is regular. Moreover $\lambda(X) < \infty \Rightarrow C_c(X) \subset L^1(\lambda)$
For $f \in C_c(X)$

$$|\Phi(f)| \leq \Lambda(|f|) = \int_X |f| \, d\mu = \|f\|_1$$

Therefore $\Phi\big|_{C_c(X)}$ is a bounded linear functional of norm ≤ 1
(regarded as a subspace of $L^1(\lambda)$)

By the Hahn-Banach theorem, Φ extends to a bounded linear functional $\tilde{\Phi}$ on $L^1(\lambda)$ with $\|\tilde{\Phi}\| \leq 1$. Therefore $\exists\, g \in L^\infty(\lambda)$ with $\|g\|_\infty \leq 1$ (so can take $|g(x)| \leq 1$ everywhere) such that

$$\tilde{\Phi}(f) = \int_X fg\, d\lambda \quad \forall f \in L^1(\lambda)$$

Hence if $f \in C_c(X)$

$$\Phi(f) = \int_X fg\, d\lambda$$

Given $f \in C_0(X)$, take $f_n \in C_c(X)$ s.t. $\|f_n - f\|_\infty \to 0$. Then

$$\Phi(f_n) \longrightarrow \Phi(f)$$

$$\left| \int f_n g\, d\lambda - \int fg\, d\lambda \right| \leq \|f_n - f\|_\infty \lambda(X) \longrightarrow 0$$

Hence

$$\Phi(f) = \int_X fg\, d\lambda \quad \forall f \in C_0(X)$$

Define a measure μ by

$$\mu(E) := \int_E g\, d\lambda \quad (E \text{ Borel set})$$

Then

$$\int_X f\, d\mu = \int_X fg\, d\lambda$$

for $f = \chi_E$, E Borel set $\Rightarrow$ for f simple $\Rightarrow$ for f uniform limit of simple functions $\Rightarrow$ for f bounded measurable functions $\Rightarrow$ for $f \in C_0(X)$. Hence

$$\Phi(f) = \int_X fg\,d\lambda = \int_X f\,d\mu$$

for $f \in C_0(X)$

Recall if $\mu(E) := \int_E g\,d\lambda$, then $|\mu|(E) = \int |g|\,d\lambda$

We know λ is regular.

Given a Borel set A and $\varepsilon > 0$, $\exists$ open $V \supset A$ s.t. $\lambda(V-A) < \varepsilon$. By taking ε sufficiently small and setting $E = V - A$, we see that $|\mu|(V-A)$ can be made as small as we wish. Hence μ is regular

Now

$$\int |g|\,d\lambda \geq \sup\left\{ |\Phi(f)| : f \in C_c(X),\ \|f\|_\infty \leq 1 \right\} = \|\Phi\|$$

so that

$$1 \leq \int |g|\,d\lambda \leq \lambda(X) \leq 1$$

Hence $\lambda(X) = 1$ and $\int |g|\,d\lambda = 1$, so that

$$|\mu|(X) = \int |g|\,d\lambda = 1 = \|\Phi\|$$

$\blacksquare$

INTEGRATION ON PRODUCT SPACES

$(X, \mathcal{S})$, $(Y, \mathcal{T})$ measurable spaces

DEFINITION: $A \times B \subset X \times Y$ is a measurable rectangle if $A \in \mathcal{S}$ and $B \in \mathcal{T}$

DEFINATION: An elementary set is a finite, disjoint union of measurable rectangles

$\mathcal{E}$ = collection of all elementary sets
$\mathcal{S} \times \mathcal{T}$:= smallest σ-algebra containing the measurable rectangles

DEFINITION: A monotone class of subsets of a set Z is a collection Ω of subsets of Z satisfying

$$E_i \subset E_{i+1} \;,\; E_i \in \Omega \implies \bigcup_{i=1}^{\infty} E_i \in \Omega$$

$$A_{i+1} \subset A_i \;,\; A_i \in \Omega \implies \bigcap_{i=1}^{\infty} A_i \in \Omega$$

If $E \subset X \times Y$, and $x \in X$, $y \in Y$, then

$$E_x := \{y \in Y : (x, y) \in E\} \subset Y$$

$$E^y := \{x \in X : (x, y) \in E\} \subset X$$

PROPOSITION: $(X, \mathcal{S}), (Y, \mathcal{T})$ measurable spaces. If $E \in \mathcal{S} \times \mathcal{T}$, then $E_x \in \mathcal{T}$ $\forall x \in X$ and $E^y \in \mathcal{S}$ $\forall y \in Y$

Proof. Let Ω be the collection of all members E of $\mathcal{S} \times \mathcal{T}$ such that $E_x \in \mathcal{T}$ $\forall x \in X$. Sufficient to show Ω is a σ-algebra containing all the measurable rectangles. Then $\mathcal{S} \times \mathcal{T} \subset \Omega$.

Suppose $A \times B$ is a measurable rectangle. Then

$$(A \times B)_x = \begin{cases} B & x \in A \\ \phi & x \notin A \end{cases}$$

and so $A \times B \in \Omega$.

Ω is a σ-algebra:

i) $X \times Y \in \Omega$ $\quad$ $X \times Y$ measurable rectangles

ii) $E \subset X \times Y$, then

$$(E_x)^c = (E^c)_x$$

and so $E \in \Omega \Rightarrow E_x \in \mathcal{T} \Rightarrow (E_x)^c \in \mathcal{T} \Rightarrow E^c \in \Omega$

iii) $E_i \in X \times Y$, then

$$\left(\bigcup_{i=1}^{\infty} E_i\right)_x = \bigcup_{i=1}^{\infty} (E_i)_x$$

Hence $(E_i) \subset \Omega \Rightarrow \bigcup E_i \in \Omega$

Hence $\Omega = \mathcal{S} \times \mathcal{T}$. Do same thing for E^y's. ▨

4/21 MEASURE THEORY

DEFINITION: $f: X \times Y \to Z$ (top. space). Then $f_x : Y \to Z$ is given by

$$f_x(y) = f(x,y)$$

and $f^y : X \to Z$ is given by

$$f^y(x) = f(x,y)$$

PROPOSITION: If $f: X \times Y \to Z$ (top. space) is $\mathcal{S} \times \mathcal{T}$ - measurable, then f_x is $\mathcal{T}$- measure and f^y is $\mathcal{S}$- measurable

Proof. Take V open in Z

$$f_x^{-1}(V) = \{y \in Y : f_x(y) \in V\} = \{y \in Y : f(x,y) \in V\}$$

$$= (f^{-1}(V))_x$$

Now $f^{-1}(V) \in \mathcal{S} \times \mathcal{T}$, so that $(f^{-1}(V))_x \in \mathcal{T}$.
Same thing for f_y.

$\blacksquare$

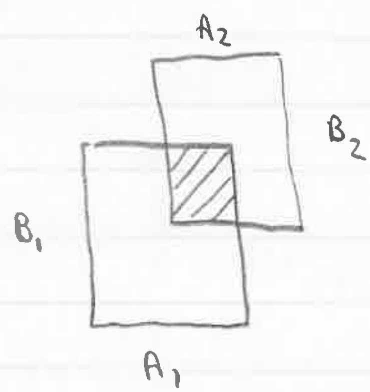
PROPOSITION: $\mathcal{S} \times \mathcal{T}$ is the smallest monotone class containing $\mathcal{E}$

Proof. Let $\mathcal{M}$ be the intersection of all monotone classes containing $\mathcal{E}$ (so $\mathcal{M}$ is the smallest monotone class containing $\mathcal{E}$) Also, $\mathcal{M} \subset \mathcal{S} \times \mathcal{T}$. To show $\mathcal{S} \times \mathcal{T} \subset \mathcal{M}$ it suffices to show that $\mathcal{M}$ is a σ-algebra.

First note if $A_1 \times B_1$ and $A_2 \times B_2$ are measurable rectangles, then

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$$

(measurable rectangles)



$$(A_1 \times B_1) - (A_2 \times B_2) = (A_1 - A_2) \times B_1 \cup (A_1 \cap A_2) \times (B_1 - B_2)$$

(elementary set)

Suppose $P \in \mathcal{E}$, $Q \in \mathcal{E}$. Claim $P \cap Q \in \mathcal{E}$.

$$P = \bigcup_{i=1}^{n} (A_i \times B_i)$$

(disjoint unions)

$$Q = \bigcup_{j=1}^{m} (C_j \times D_j)$$

Then

$$P \cap Q = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (A_i \times B_i) \cap (C_j \times D_j)$$

$$= \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (A_i \cap C_j) \times (B_i \cap D_j) \in \mathcal{E}$$

(disjoint union)

Claim: $P - Q \in \mathcal{E}$

$$P - Q = \bigcap_{j=1}^{m} \bigcup_{i=1}^{n} (A_i \times B_i) - (C_j \times D_j)$$

$A_i \times B_i$'s disjoint $\Rightarrow$ for each j

$$\bigcup_{i=1}^{n} \big((A_i \times B_i) - (C_j \times D_j) \big) \in \mathcal{E}$$

By first claim (extended by induction), $P - Q \in \mathcal{E}$

Claim: $P \cup Q \in \mathcal{E}$.

$$P \cup Q = (P - Q) \cup Q \in \mathcal{E}$$

since $Q \cap (P - Q) = \phi$

For $P \subset X \times Y$, let

$$\Omega(P) := \{ Q \subset X \times Y : P - Q \in \mathcal{M},\ Q - P \in \mathcal{M},\ P \cup Q \in \mathcal{M} \}$$

Remarks: a) $Q \in \Omega(P)$ iff $P \in \Omega(Q)$

b) $\forall P$, $\Omega(P)$ is a monotone class.

Proof. Suppose $Q_i \in \Omega(P)$, Q_i monotone $\uparrow$. $Q := \bigcup_{i=1}^{\infty} Q_i$

$$P - Q = P - \left(\bigcup Q_i\right) = \bigcap (P - Q_i) \in \mathcal{M}$$

$P - Q_i \in \mathcal{M}\ \forall i,\ (P - Q_i) \downarrow$

Similarly, $Q - P = \bigcup_i (Q_i - P) \in \mathcal{M}$

$Q_i - P \in \mathcal{M}\ \forall i,\ (Q_i - P) \uparrow$

Finally, $P \cup Q = \bigcup_i (P \cup Q_i) \in \mathcal{M}$

$P \cup Q_i \in \mathcal{M}\ \forall i,\ (P \cup Q_i) \uparrow$

Suppose $P \in \mathcal{E}$. If $Q \in \mathcal{E}$, we know $Q \in \Omega(P)$ and so $\mathcal{E} \subset \Omega(P)$. Definition of $\mathcal{M} \Rightarrow \mathcal{M} \subset \Omega(P)$. Now suppose $Q \in \mathcal{M}$. If $P \in \mathcal{E}$, then $Q \in \Omega(P) \Rightarrow P \in \Omega(Q)$. Hence $\mathcal{E} \subset \Omega(Q) \Rightarrow \mathcal{M} \subset \Omega(Q)$

If $P \in \mathcal{M}, Q \in \mathcal{M}$, then $P \in \Omega(Q) \Rightarrow P \cup Q \in \mathcal{M}$ and $\underset{(*)}{P - Q} \in \mathcal{M}$.

Claim: $\mathcal{M}$ is a σ-algebra.

(a) $X \times Y \in \mathcal{M}$ (since $X \times Y \in \mathcal{E}$)

(b) $\mathcal{M}$ closed under complementation by (a) and (*)

(c) $(Q_i) \subset \mathcal{M}$, $Q = \bigcup Q_i$. Let $P_N = \bigcup_{i=1}^{N} Q_i \in \mathcal{M}$
$P_N \uparrow Q \Rightarrow Q \in \mathcal{M}$ ($\mathcal{M}$ monotone class) ▨

PROPOSITION: $(X, \mathcal{S}, \mu)$, $(Y, \mathcal{T}, \lambda)$ positive σ-finite measure spaces. Given $Q \in \mathcal{S} \times \mathcal{T}$, define $\varphi : X \to [0, \infty]$ and $\psi : Y \to [0, \infty]$ by

$$\varphi(x) := \lambda(Q_x)$$

$$\psi(y) := \mu(Q^y)$$

Then φ is $\mathcal{S}$-measure, ψ is $\mathcal{T}$-measurable and

$$(*) \qquad \int_X \varphi(x)\, d\mu(x) = \int_Y \psi(y)\, d\lambda(y)$$

Remark: $\varphi(x) = \lambda(Q_x) = \int_Y \chi_{Q_x}(y)\, d\lambda(y) = \int_Y \chi_Q(x,y)\, d\lambda(y)$

LHS of $(*)$ is thus

$$\int_X \left(\int_Y \chi_Q(x,y)\, d\lambda(y) \right) d\mu(x)$$

But notice that $\psi(y) = \mu(Q^y) = \int_X \chi_{Q^y}(x)\, d\mu(x) = \int_X \chi_Q(x,y)\, d\mu(x)$

so that the RHS is then

$$\int_Y \left(\int_X \chi_Q(x,y)\, d\mu(x) \right) d\lambda(y)$$

Proof. Let Ω be the collection of all $Q \in \mathcal{S} \times \mathcal{T}$ for which the conclusion holds. Show

(a) Ω contains all measurable rectangles

Let $Q = A \times B$ (measurable rectangle). Then

$$Q_x = \begin{cases} B & \text{if } x \in A \\ \phi & \text{if } x \notin A \end{cases}$$

and so

$$\varphi(x) = \lambda(Q_x) = \lambda(B)\, \chi_A(x)$$

Hence φ is $\mathcal{S}$-measurable. Similarly

$$\lambda(y) = \mu(Q^y) = \mu(A)\, \chi_B(y)$$

which is $\mathcal{T}$-measurable. Also

$$\int \varphi(x)\, d\mu(x) = \lambda(B)\mu(A)$$

$$\int \psi(y)\, d\lambda(y) = \mu(A)\,\lambda(B)$$

(b) Suppose $Q_i \in \Omega$ with $Q_{i+1} \supset Q_i$. Let $Q = \cup Q_i$. Then $Q \in \Omega$

Associated with each Q_i are function φ_i ($\mathcal{S}$-measurable) and ψ_i ($\mathcal{T}$ measurable) with

$$\int_X \varphi_i(x)\, d\mu(x) = \int_Y \psi_i(y)\, d\lambda(y)$$

Now $Q_i \uparrow Q \Rightarrow (Q_i)_x \uparrow Q_x \Rightarrow$

$$\varphi(x) = \lambda(Q_x) = \lim \lambda((Q_i)_x) = \lim \varphi_i(x)$$

Hence $\varphi_i(x) \uparrow \varphi(x)$ $\forall x \in X$, so that φ is $\mathcal{S}$-measurable and

$$\text{MCT} \Rightarrow \lim_{i \to \infty} \int \varphi_i(x)\, d\mu(x) = \int \varphi(x)\, d\mu(x)$$

Similarly, $\psi(y)$ is $\mathcal{T}$-measurable and

$$\lim_{i \to \infty} \int \psi_i(y)\, d\lambda(y) = \int \psi(y)\, d\lambda(y)$$

4/24 MEASURE THEORY

(Proof continued)

Claim: $(Q_i) \subseteq \Omega$, Q_i disjoint $\Longrightarrow \bigcup_{i=1}^{\infty} Q_i \in \Omega$

Let $P = \bigcup_{i=1}^{N} Q_i$. For $x \in X$,

$$P_x = \bigcup_{i=1}^{N} (Q_i)_x$$

(disjoint union) Therefore

$$\lambda(P_x) = \sum_{i=1}^{N} \lambda((Q_i)_x)$$

If $\varphi_i(x) = \lambda((Q_i)_x)$ and $\psi_i(y) = \mu((Q_i)^y)$, then φ_i is $\mathcal{S}$-measurable, ψ_i is $\mathcal{T}$-measurable and

$$\int_X \varphi_i \, d\mu = \int_Y \psi_i \, d\lambda$$

Then $\varphi(x) = \lambda(P_x) = \sum_{i=1}^{N} \varphi_i(x)$, so φ is $\mathcal{S}$-measurable and

$$\int_X \varphi \, d\mu = \sum_{i=1}^{N} \int_X \varphi_i \, d\mu$$

Similarly $\psi(y) = \mu(P^y) = \sum_{i=1}^{N} \psi_i(y)$, so ψ is $\mathcal{T}$-measurable and

$$\int \psi(y)\,d\lambda(y) = \sum_{i=1}^{N} \int \psi_i \, d\lambda$$

Hence $P \in \Omega$.

Now $\cup Q_i$ is an increasing sequence of sets of the form P, so $\cup Q_i \in \Omega$.

CLAIM: If $\mu(A) < \infty$ and $\lambda(B) < \infty$, and if

$$A \times B \supset Q_1 \supset Q_2 \supset Q_3 \supset \dots$$

Then $\bigcap_{i=1}^{\infty} Q_i \in \Omega$.

Proof. Let $\varphi_i(x) = \lambda((Q_i)_x)$, $\psi_i(y) = \mu((Q_i)^y)$, $Q = \cap Q_i$. Note

$$Q_x = \bigcap_{i=1}^{\infty} (Q_i)_x$$

$\lambda(B) < \infty$ implies $\lambda(Q_x) = \lim_{i \to \infty} \lambda((Q_i)_x)$. Let $\varphi(x) = \lambda(Q_x)$. Then $\varphi(x) = \lim \varphi_i(x)$, so φ is $\mathcal{S}$-measurable. Similarly, $\psi(y) = \lim \psi_i(y)$ is $\mathcal{T}$-measurable.

Note also

$$\int_X \varphi_i \, d\mu = \int_Y \psi_i \, d\lambda$$

Now $\varphi_i(x) = \lambda((Q_i)_x) \leq \lambda((A \times B)_x) = \lambda(B)\, \chi_A(x) \in L^1(\mu)$

and $\psi_i(y) = \mu((Q_i)^y) \leq \mu((A\times B)^y) = \mu(A)\chi_B(y) \in L^1(\lambda)$

By the Dominated Convergence theorem

$$\int_X \varphi_i \, d\mu \longrightarrow \int_X \varphi \, d\mu$$

$$\int_Y \psi_i \, d\lambda \longrightarrow \int_Y \psi \, d\lambda$$

Hence $Q \in \Omega$

Write $X = \overset{\infty}{\underset{n=1}{\bigcup}} X_n$ (disjoint) and $Y = \overset{\infty}{\underset{m=1}{\bigcup}} Y_m$ (disjoint), where $\mu(X_n) < \infty$, $\lambda(Y_m) < \infty$. If $Q \in \mathcal{S}\times\mathcal{T}$, let

$$Q_{mn} := Q \cap (X_n \times Y_m)$$

Let $\mathcal{M}$ be the collection of all $\underline{Q \in \mathcal{S}\times\mathcal{T}}$ s.t. $Q_{mn} \in \Omega$ $\forall n,m$

i) Every measurable rectangle in in $\mathcal{M}$

ii) Every elementary set is in $\mathcal{M}$

iii) $\mathcal{M}$ is a monotone class

Hence $\mathcal{S}\times\mathcal{T} \subset \mathcal{M}$ ($\mathcal{S}\times\mathcal{T}$ smallest monotone class containing $\mathcal{E}$). But $\mathcal{M} \subset \mathcal{S}\times\mathcal{T}$, so $\mathcal{M} = \mathcal{S}\times\mathcal{T}$

Now the second to last claim $\Rightarrow$ every $Q \in \mathcal{S}\times\mathcal{T}$ belongs to Ω

$\blacksquare$

<u>DEFINITION:</u> $(X, \mathcal{S}, \mu)$, $(Y, \mathcal{T}, \lambda)$ σ-finite. For $Q \in \mathcal{S} \times \mathcal{T}$, let

$$(\mu \times \lambda)(Q) := \int_X \lambda(Q_x)\, d\mu(x)$$

$$= \int_Y \mu(Q^y)\, d\lambda(y)$$

<u>PROPOSITION:</u> $\mu \times \lambda$ is a σ-finite measure.

Proof: Take $Q = \bigcup_{i=1}^{\infty} Q_i$ (disjoint). $\forall x \in X$

$$Q_x = \bigcup_{i=1}^{\infty} (Q_i)_x \quad \text{(disjoint)}$$

$$\Longrightarrow \lambda(Q_x) = \sum_{i=1}^{\infty} \lambda((Q_i)_x)$$

Therefore

$$(\mu \times \lambda)(Q) = \int_X \lambda(Q_x)\, d\mu(x) = \int_X \sum_{i=1}^{\infty} \lambda((Q_i)_x)\, d\mu(x)$$

$$\overset{\text{MCT}}{=} \sum_{i=1}^{\infty} \int_X \lambda((Q_i)_x)\, d\mu(x)$$

$$= \sum_{i=1}^{\infty} (\mu \times \lambda)(Q_i)$$

Consider $A\times B$, where $\mu(A)<\infty$, $\mu(B)<\infty$.

$$(\mu\times\lambda)(A\times B) = \int_X \lambda((A\times B)_x)\,d\mu(x) = \int_X \lambda(B)\chi_A(x)\,d\mu(x)$$

$$= \lambda(B)\mu(A) < \infty$$

$\square$

FUBINI'S THEOREM $(X,\mathcal{S},\mu)$, $(Y,\mathcal{T},\lambda)$ σ-finite measure spaces. Let $f(x,y)$ be measurable w.r.t. $\mathcal{S}\times\mathcal{T}$.

(a) Suppose $f\geq 0$. Let

$$\varphi(x) := \int_Y f_x(y)\,d\lambda(y)$$

(1)

$$\psi(y) := \int_X f^y(x)\,d\mu(x)$$

Then

$$(*)\quad \int_X \varphi(x)\,d\mu(x) = \int_{X\times Y} f(x,y)\,d(\mu\times\lambda) = \int_Y \psi(y)\,d\lambda(y)$$

(b) Let

$$\varphi^*(x) := \int_Y |f_x(y)|\,d\lambda(y)$$

Then if $\int \varphi^*(x)\, d\mu(x) < \infty$, we have $f \in L^1(\mu \times \lambda)$

(c) If $f \in L^1(\mu \times \lambda)$, then $f_x \in L^1(\lambda)$ for almost all x $[\mu]$ and $f^y \in L^1(\mu)$ for almost all y $[\lambda]$, and the functions φ and ψ defined a.e. by equations (1) are in $L^1(\mu)$ and $L^1(\lambda)$ respectively. Furthermore, $(*)$ holds

Remark about (a):

$$\int_X\int_Y f(x,y)\,d\lambda(y)\,d\mu(x) = \int_{X\times Y} f(x,y)\,d(\mu\times\lambda) = \int_Y\int_X f(x,y)\,d\mu(x)\,d\lambda(y)$$

Proof of (a). We know this holds if $f = \chi_Q$ where $Q \in \mathcal{S}\times\mathcal{T}$ so it holds for simple functions. Given $f \geq 0$, there exist simple $s_n \uparrow f$ on $X\times Y$ with each s_n $\mathcal{S}\times\mathcal{T}$-measurable. Let

$$\varphi_n(x) = \int_Y s_n(x,y)\,d\lambda(y)$$

$$\psi_n(y) = \int_X s_n(x,y)\,d\mu(x)$$

Let

$$\varphi(x) = \int_Y f(x,y)\,d\lambda(y)$$

$$\psi(y) = \int_X f(x,y)\,d\mu(x)$$

By the Monotone Convergence theorem, $\varphi_n(x)\uparrow\varphi(x)$ and $\psi_n(y)\uparrow\psi(y)$. Since (a) holds for each s_n,

$$\int \varphi_n(x)\,d\mu(x) = \int s_n(x,y)\,d(\mu\times\lambda) = \int \psi_n(y)\,d\lambda(y)$$

MCT

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\int \varphi(x)\,d\mu(x) \qquad \int f(x,y)\,d(\mu\times\lambda) \qquad \int \psi(y)\,d\lambda(y)$$

4/26 MEASURE THEORY

Remarks about Fubini theorem: Summary - a) iterated integrals are equal if $f \geq 0$

(b),(c): If one of the iterated integrals of $|f|$ is finite then the two iterated integrals of f are equal

Proof of (b) Apply a to $|f|$. (*) becomes

$$\int |f(x,y)|\, d(\mu\times\lambda) = \int_X \varphi^*(x)\, d\mu(x) < \infty$$

(c) First assume f is real. Write $f = f^+ - f^-$. Apply (a) to f^+, f^-. Let

$$\varphi_1(x) := \int_Y f^+_x(y)\, d\lambda(y)$$

$$\varphi_2(x) := \int_Y f^-_x(y)\, d\lambda(y)$$

φ_j is $\mathcal{S}$-measurable and

$$\int_X \varphi_1(x)\, d\mu(x) = \int_{X\times Y} f^+(x,y)\, d(\mu\times\lambda) < \infty$$

Hence $\varphi_1(x) < \infty$ a.e. $[\mu]$. Similarly $\varphi_2(x) < \infty$ a.e. $[\mu]$

$\Rightarrow (f^+)_x \in L^1[\lambda]$ $\qquad\qquad$ $\Rightarrow (f^-)_x \in L^1[\lambda]$

But $f_x = (f^+)_x - (f^-)_x$, so $f_x \in L^1(\mu)$ for almost all x $[\mu]$.

For x s.t. $\varphi_j(x) < \infty$

$$\varphi(x) = \int_Y f_x(y)\, d\lambda(y) = \varphi_1(x) - \varphi_2(x)$$

so $\varphi(x) = \varphi_1(x) - \varphi_2(x)$ a.e. $[\mu]$. Since $\varphi_j \in L^1(\mu)$, we get $\varphi \in L^1(\mu)$.

$$\int_X \varphi(x)\, d\mu(x) = \int_X \varphi_1(x)\, d\mu(x) - \int \varphi_2(x)\, d\mu(x)$$

$$= \int_{X \times Y} f(x,y)\, d(\mu \times \lambda)$$

(Part of proof for ψ is done in the same way)

Now consider $f = u + iv$, so $u \in L^1(\mu \times \lambda)$ and $v \in L^1(\mu \times \lambda)$. Then $u_x \in L^1(\lambda)$ a.e. $[\mu]$ and $v_x \in L^1(\lambda)$ a.e. $[\mu]$ whence $f_x \in L^1(\lambda)$ a.e. $[\mu]$. We also have

$$\int_Y u_x(y)\, d\lambda(y) \in L^1(\mu)$$

$$\int_Y v_x(y)\, d\lambda(y) \in L^1(\mu)$$

$\Longrightarrow \int_Y f_x(y)\, d\lambda(y) \in L^1(\mu)$, i.e. $\varphi(x) \in L^1(\mu)$

$$\int_X \varphi(x)\, d\mu(x) = \int_X (u_x + iv_x)\, d\mu(x) = \int_{X \times Y} u(x,y) + iv(x,y)\, d(\mu \times \lambda)$$

$\boxed{///}$

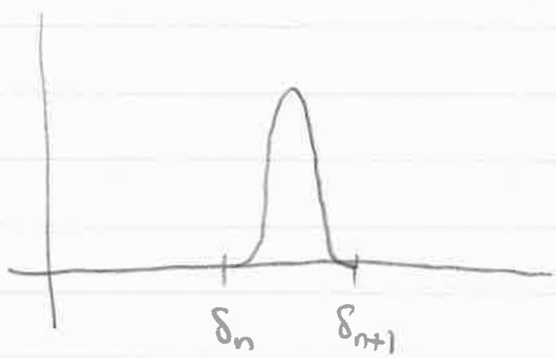
EXAMPLES

I. $X = Y = [0,1]$, Lebesgue measure. Take

$$0 = \delta_1 < \delta_2 < \dots < \delta_n \longrightarrow 1$$

Define $g_n : [0,1] \to [0,\infty)$ s.t. $\operatorname{supp} g_n \subset (\delta_n, \delta_{n+1})$ and
(continuous)

$$\int_0^1 g_n(t)\,dt = 1$$



Define

$$f(x,y) := \sum_{n=1}^{\infty} \left[g_n(x) - g_{n+1}(x) \right] g_n(y)$$

	δ_1 – δ_2	δ_2 – δ_3	δ_3 – δ_4	δ_4 –
δ_3 – δ_4	0	0	$g_3(x)\,g_3(y)$	$-g_4(x)\,g_3(y)$
δ_2 – δ_3	0	$g_2(x)\,g_2(y)$	$-g_3(x)\,g_2(y)$	0
δ_1 – δ_2	$g_1(x)\,g_1(y)$	$-g_2(x)\,g_1(y)$	0	0

For $\delta_N < y \leq \delta_{N+1}$

$$\int_0^1 f(x,y)\,dx = g_N(y)\int_0^1 [g_N(x) - g_{N+1}(x)]\,dx = 0$$

$$\Longrightarrow \int_0^1\int_0^1 f(x,y)\,dx\,dy = 0$$

For $N \geq 2$, $\delta_N < x \leq \delta_{N+1}$

$$\int_0^1 f(x,y)\,dy = g_N(x)\int_0^1 [g_N(y) - g_{N-1}(y)]\,dy = 0$$

For $\delta_1 < x \leq \delta_2$

$$\int_0^1 f(x,y)\,dy = g_1(x)\int_0^1 g_1(y)\,dy = g_1(x)$$

$$\Longrightarrow \int_0^1\int_0^1 f(x,y)\,dy\,dx = \int_0^1 g_1(x)\,dx = 1$$

<u>NOTE</u>

$$\int_0^1 |f(x,y)|\,dx = 2\,g_N(y)$$

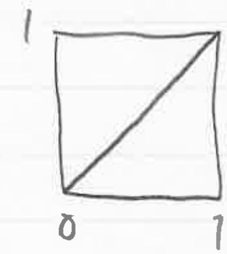
for $\delta_N < y \leq \delta_{N+1}$, and so

$$\int_0^1 \int_0^1 |f(x,y)|\, dx\, dy = \infty$$

so that $f \notin L^1(\mu \times \lambda)$

II: $X = Y = [0,1]$ Lebesgue measure on X μ
Counting measure on Y λ

$$f := \chi_D$$



D = diagonal of square
$\mathcal{S} \times \mathcal{T}$ measurable

$$\int_0^1 \chi_D(x,y)\, d\lambda(y) = 1$$

$$\Rightarrow \int_0^1 \int_0^1 f(x,y)\, d\lambda(y)\, d\mu(x) = 1$$

But

$$\int_0^1 \chi_D(x,y)\, d\mu(x) = 0$$

$$\Rightarrow \int_0^1 \int_0^1 f(x,y)\, d\mu(x)\, d\lambda(y) = 0$$

NOTE: Y, λ is not σ-finite

III. $X = Y = [0,1]$ Lebesgue measure

Continuum hypothesis $\Rightarrow \exists\ j : [0,1] \xrightarrow{1-1} W$ (well-ordered)
s.t. $\forall x \in [0,1]$, $j(x)$ has at most countably many predecessors
Define

$$Q = \{(x,y) : j(x) \text{ precedes } j(y) \text{ in } W\}$$

$$\int_0^1 \chi_Q(x,y)\,dy = 1$$

$\uparrow$

$=1$ except on a countable set

$$\Rightarrow \int_0^1\int_0^1 \chi_Q(x,y)\,dy\,dx = 1$$

But
$$\int_0^1 \chi_Q(x,y)\,dx = 0$$

$\uparrow$

$=0$ except on a countable set

$$\Rightarrow \int_0^1\int_0^1 \chi_Q(x,y)\,dx\,dy = 0$$

Note. χ_Q is not $\mathcal{S}\times\mathcal{T}$ measurable

4/28 MEASURE THEORY

THEOREM: If $f, g \in L^1(\mathbb{R})$, then $|f(x-y)\,g(y)| \in L^1(\mathbb{R})$ for almost every x, i.e.

$$\int_{\mathbb{R}} |f(x-y)\,g(y)|\,dy < \infty \quad (\text{almost all } x)$$

For such x, define

$$h(x) := \int_{\mathbb{R}} f(x-y)\,g(y)\,dy \quad (\text{convolution})$$

Then $\|h\|_1 \le \|f\|_1 \|g\|_1$.

Proof. WLOG f and g are Borel measurable and finite everywhere.

[[Lusin's theorem $\Rightarrow \exists$ continuous f_n s.t. $f_n \to f$ a.e. Let

$$F := \overline{\lim}\,(\operatorname{Re} f_n) + i\,\overline{\lim}\,(\operatorname{Im} f_n)$$

Then F is Borel measurable and $F = f$ a.e. If either $\overline{\lim}\,(\operatorname{Re} f_n(x)) = \pm\infty$ or $\overline{\lim}\,(\operatorname{Im} f_n(x)) = \pm\infty$, modify F at that x to be 0. We still have $F = f$ a.e., and F is Borel measurable. Moreover, now F is finite everywhere.

Note that the integrands in the theorem are changed only on sets of measure 0.]]

Let $F(x,y) := f(x-y)\, g(y)$. F is measurable w.r.t. B_2, the σ-algebra of Borel sets in $\mathbb{R}^2$. For let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$\varphi(x,y) := x - y$$

Then φ is Borel measurable. Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$\psi(x,y) := y$$

Then ψ is Borel measurable. Then $f \circ \varphi, g \circ \psi$ Borel measurable and

$$F = (f \circ \varphi)(g \circ \psi)$$

so F is Borel measurable
Let B_1 be the Borel sets in $\mathbb{R}$.

Exercise : $B_2 = B_1 \times B_1$

Therefore F is $B_1 \times B_1$ measurable. Now notice that

$$\int_{\mathbb{R}} |F(x,y)|\, dx = |g(y)| \int_{\mathbb{R}} |f(x-y)|\, dx = |g(y)|\, \|f\|_1$$

translation invariance of Lebesgue measure

$$\int \int |F(x,y)|\, dx\, dy = \|f\|_1 \int |g(y)|\, dy = \|f\|_1 \|g\|_1$$

Therefore by Fubini (b), $F \in L^1(m_1 \times m_1)$, and from (c), for almost every x, $F_x(y) \in L^1(\mathbb{R})$, i.e.

$$h(x) = \int f(x-y)\, g(y)\, dy \quad \text{exists}$$

and $h \in L^1(\mathbb{R})$. Note

$$\|h\|_1 = \int_{\mathbb{R}} |h(x)|\, dx \le \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y)\, g(y)|\, dy\, dx$$

$$\underset{\text{Fubini}}{=} \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-y)\, g(y)|\, dx\, dy = \|f\|_1 \|g\|_1$$

$\blacksquare$

EXAMPLE: $(X, \mathcal{S}, \mu)$ $(Y, \mathcal{T}, \lambda)$

$$\left.\begin{array}{l}\text{Suppose } \exists\, A \in \mathcal{S} \text{ such that } \mu(A) = 0 \text{ and } A \neq \emptyset. \\ \text{Suppose } \exists\, B \subset Y \text{ such that } B \notin \mathcal{T}\end{array}\right\} \text{very weak hypotheses}$$

Claim: $\mu \times \lambda$ is not complete, i.e. $(X \times Y, \mathcal{S} \times \mathcal{T}, \mu \times \lambda)$ is not a complete measure space

$A \times B \subset A \times Y$ and

$$(\mu \times \lambda)(A \times Y) = \int \chi_{A \times Y}(x,y)\, d(\mu \times \lambda)$$

$$= \int_X \int_Y \chi_{A\times Y}(x,y)\, d\lambda(y)\, d\mu(x)$$

$$= \int_X \lambda(Y)\chi_A(x)\, d\mu(x) = \lambda(Y)\mu(A) = 0$$

If $A\times B \in \mathcal{S}\times\mathcal{T}$, then $(A\times B)_x \in \mathcal{T}$ $\forall x\in X$. But

$$(A\times B)_x = \begin{cases} \emptyset & x\notin A \\ B & x\in A \end{cases}$$

Since $A\neq\emptyset$, $\exists\, x_0\in A$, whence $B = (A\times B)_{x_0} \in \mathcal{T}$ $\lightning$.
Therefore $A\times B \notin \mathcal{S}\times\mathcal{T}$.

THEOREM: m_p be Lebesgue measure on $\mathbb{R}^p$. Then the completion of $m_r\times m_s$ is m_k, where $k=r+s$.

(Recall: $(X,\mathcal{M},\mu)$ $\mathcal{M}^* := \{E : \exists\, A\subset E\subset B,\ A\in\mathcal{M}, B\in\mathcal{M}, \mu(B-A)=0\}$. For $E\in\mathcal{M}^*$, let $\mu^*(E)=\mu(A)$. $(X,\mathcal{M}^*,\mu^*)$ is the completion of $(X,\mathcal{M},\mu)$.)

Proof. Let $\mathcal{B}_k$ be the Borel sets of $\mathbb{R}^k$ and $\mathcal{M}_k$ the Lebesgue measurable sets. First note

$$\mathcal{B}_k \subset \mathcal{M}_r\times\mathcal{M}_s \subset \mathcal{M}_k$$

Every Euclidean rectangle in $\mathbb{R}^k$ is a measurable rectangle, hence in $\mathcal{M}_r \times \mathcal{M}_s$. Hence $\mathcal{M}_r \times \mathcal{M}_s$ contains all open sets in $\mathbb{R}_k$ and hence all Borel sets.

Suppose $E \in \mathcal{M}_r$. Claim: $E \times \mathbb{R}_s \in \mathcal{M}_k$. Recall $E \in \mathcal{M}_p$ iff $\exists$ F_σ set A, G_δ set B s.t.

$$A \subset E \subset B$$
$$m_p(B-A) = 0$$

Hence $\exists$ F_σ set A in $\mathbb{R}^r$ and a G_δ set B in $\mathbb{R}^r$ s.t. $A \subset E \subset B$ and $m_r(B-A) = 0$. Then

$$\underset{G_\delta}{B \times \mathbb{R}^s} \supset E \times \mathbb{R}^s \supset \underset{F_\sigma}{A \times \mathbb{R}^s}$$

$$(B \times \mathbb{R}^s) - (A \times \mathbb{R}^s) = (B-A) \times \mathbb{R}^s.$$

THM 2.20 $\quad m_r \times r_s$ $\left\{ \begin{array}{l} \text{translation inv.} \\ \text{finite compact sets} \\ \text{defined on} \\ \text{Borel sets} \end{array} \right.$

$\Downarrow$

$m_r \times m_s$ multiple of m_k on Borel sets

$$m_k((B-A) \times \mathbb{R}^s) = (m_r \times m_s)((B-A) \times \mathbb{R}^s)$$

$$= m_r(B-A)\, m_s(\mathbb{R}^s)$$

$$= 0 \cdot \infty = 0$$

Hence $E \times \mathbb{R}^s \in \mathcal{M}_k$. Same argument shows $\mathbb{R}^r \times F \in \mathcal{M}_k$ if $F \in \mathcal{M}_s$. Therefore

$$E \times F = (E \times \mathbb{R}^s) \cap (\mathbb{R}^r \cap F) \in \mathcal{M}_k$$

Hence $\mathfrak{M}_r \times \mathfrak{M}_s \subset \mathfrak{M}_k$

5/1 MEASURE THEORY

COMPLETION OF PROOF

(Have shown $\mathcal{B}_k \subset \mathfrak{M}_r \times \mathfrak{M}_s \subset \mathfrak{M}_k$)

CLAIM: $m_r \times m_s$ coincides with m_k on $\mathfrak{M}_r \times \mathfrak{M}_s$

Suppose $Q \in \mathfrak{M}_r \times \mathfrak{M}_s$. Then $Q \in \mathfrak{M}_k$, so there are F_σ-set A and G_δ-set B s.t.

$$m_k(B-A) = 0$$
$$A \subset Q \subset B$$

Then

$$(m_r \times m_s)(Q-A) \le (m_r \times m_s)(B-A) \underset{\substack{\uparrow \\ \text{Thm 2.20}}}{=} m_k(B-A) = 0$$

and so

$$(m_r \times m_s)(Q) = (m_r \times m_s)(A) = m_k(A) = m_k(Q)$$

We want to show $(\mathbb{R}^k, (\mathfrak{M}_r \times \mathfrak{M}_s)^*, (m_r \times m_s)^*) = (\mathbb{R}^k, \mathfrak{M}_k, m_k)$
Suppose $Q \in (\mathfrak{M}_r \times \mathfrak{M}_s)^*$. By definition $\exists\, A \subset Q \subset B$ where $A, B \in \mathfrak{M}_r \times \mathfrak{M}_s$ and $m_r \times m_s (B-A) = 0$. Therefore $m_k(B-A) = 0$
$A \in \mathfrak{M}_k$, $Q - A \in \mathfrak{M}_k \Rightarrow Q \in \mathfrak{M}_k$ and $m_k(Q) = m_k(A) = (m_r \times m_s)^*(Q)$

Suppose $Q \in \mathcal{M}_k$. $\exists$ Borel sets A, B s.t. $A \subset Q \subset B$ and $m_k(B-A)=0$. But $m_r \times m_s\,(B-A) = m_k(B-A) = 0$, so $Q \in (\mathcal{M}_r \times \mathcal{M}_s)^*$. Moreover

$$(m_r \times m_s)^*(Q) = (m_r \times m_s)(A) = m_k(A) = m_k(Q)$$

$\boxtimes$

Since $B_2 \subset \mathcal{M}_1 \times \mathcal{M}_1$, to show $F(x,y)$ is measurable, it suffices to show F is Borel measurable (recall composition of Borel measurable functions is measurable)

DIFFERENTIATION OF MEASURES

Let $m = m_k$ on $\mathbb{R}^k$

<u>DEFINITION</u>: If E_i is a sequence of Borel sets in $\mathbb{R}^k$, $x \in \mathbb{R}^k$, we say E_i shrinks to x nicely if $\exists\ r_i \downarrow 0$, $\alpha > 0$ s.t.

$$E_i \subset B(x; r_i)$$
$$m(E_i) > \alpha\, m(B(x; r_i))$$

DEFINITION: Suppose μ is a complex Borel measure on $\mathbb{R}^k$. Suppose $x \in \mathbb{R}^k$. If

$$\lim_{i\to\infty} \frac{\mu(E_i)}{m(E_i)} = A$$

for every sequence of Borel sets E_i which shrinks to x nicely, we say the derivative of μ w.r.t. m at x is A, and write

$$D\mu(x) = A$$

PROPOSITION: Suppose Ω is a collection of open balls in $\mathbb{R}_k$. Suppose $t < m(\bigcup_{\Omega} B)$. Then there is a disjoint subcollection $\{B_1, \ldots, B_N\} \subset \Omega$ s.t.

$$\sum_{i=1}^{N} m(B_i) > 3^{-k} t$$

Proof. Since m is regular, there is a compact K s.t. $t < m(K)$ and $K \subset \bigcup_{\Omega} B$. By compactness

$$K = S_1 \cup \ldots \cup S_M$$

where $S_i \in \Omega$ and radius $S_j \geq$ radius S_{j+1}. Let $B_1 = S_1$. Discard all S_j s.t. $S_j \cap S_1 \neq \phi$. Let $B_2 = 1^{st}$ surviving S. Discard all S_j s.t. $S_j \cap S_2 \neq \phi$. Continue until process stops. Arrive at a disjoint collection $B_1, B_2, \ldots, B_N$. The union of all the S_j's $\subset$ the union of balls β_i, where center β_i = center B_i, radius $\beta_i = 3$ radius B_i

$$t < m(K) \leq \sum_{i=1}^{N} m(\beta_i) = 3^k \sum_{i=1}^{N} m(B_i)$$ ▨

LEMMA: μ = positive Borel measure on $\mathbb{R}^k$, finite on compact sets (Recall this implies μ is regular). If $\mu(A) = 0$, then $\exists\, A' \subset A$, A' Lebesgue measurable s.t. A Borel measurable

(1) $m(A - A') = 0$
(2) $D\mu(x) = 0 \quad \forall x \in A'$

Proof: Since μ is regular, if $\varepsilon > 0$ $\exists$ open $V \supset A$ s.t. $\mu(V) < \varepsilon$. Let

$$A' := \left\{ x \in A : \lim_{r \to 0} \frac{\mu(B(x,r))}{m(B(x,r))} = 0 \right\}$$

Let

$$P_j := \left\{ x \in A : \overline{\lim_{r \to 0}} \frac{\mu(B(x,r))}{m(B(x,r))} \geq 1/j \right\}$$

CLAIM: $m(P_j) = 0$ and $\displaystyle\bigcup_{j=1}^{\infty} P_j = A - A'$

(This proves (1))

If $x \in P_j$, $\exists\, r = r(x)$ s.t. $B(x; r(x)) \subset V$ and $m(B(x, r(x))) \leq j\, \mu(B(x, r(x)))$. Then

$$P_j \subset \bigcup_{x \in P_j} B(x; r(x))$$

By the proposition, if we could find $t < m\left(\bigcup_{x \in P_j} B(x, r(x))\right)$, then

$\exists \{B_1, \dots, B_N\}$ s.t
(disjoint)

$$t < 3^{-k} \sum_{k=1}^{N} m(B(x_i ; r_i(x_i))) < j\, 3^{-k} \sum_{k=1}^{N} \mu(B(x_i ; r_i(x_i)))$$

$$< j\, 3^{-k} \mu(V) < j\, 3^{-k} \varepsilon$$

By ε was arbitrary, no such t exists, so $m\left(\bigcup_{x \in P_j} B(x, r(x))\right) = 0$
Therefore $m(P_j) = 0$.

If $x \in A'$ and (E_j) shrinks to x nicely, then

$$\frac{\mu(E_i)}{m(E_i)} \leq \frac{\mu(B(x, r_i))}{\alpha m(B(x, r_i))} \to 0$$

$\blacksquare$

5/3 MEASURE THEORY

<u>THEOREM</u>: Suppose μ is a complex Borel measure on $\mathbb{R}^k$.
(a) $D\mu(x)$ exists a.e. $[m]$
(b) $D\mu(x) \in L^1(\mathbb{R}^k, m)$
(c) $\exists$ complex μ_s with $\mu_s \perp m$ and $D\mu_s(x) = 0$ a.e. $[m]$.

and moreover

$$(*) \qquad \mu(E) = \mu_s(E) + \int_E D\mu(x)\, dm(x)$$

for every Borel set E. (This gives the Lebesgue decomposition of μ w.r.t. m and shows that the Radon-Nykodym derivative of μ is $D\mu$)

<u>COROLLARY</u>: μ complex Borel measure on $\mathbb{R}^k$
(i) $\mu \perp m$ iff $D\mu(x) = 0$ a.e. $[m]$
(ii) $\mu \ll m$ iff $\mu(E) = \int_E D\mu(x)\, dm(x)$ $\forall$ Borel set E

Proof of Corollary: Recall $\mu = \mu_1 + \mu_2$ (uniquely) where $\mu_1 \perp m$ and $\mu_2 \ll m$.

(i) If $\mu \perp m$, then $\mu = \mu_s$, and so $D\mu(x) = D\mu_s(x) = 0$ a.e. $[m]$. On the other hand, if $D\mu = 0$ a.e., then from $(*)$ $\mu = \mu_s$ and so $\mu \perp m$.

(ii) If $\mu(E) = \int_E D\mu(x)\, dm(x)$, then certainly $\mu \ll m$.
If $\mu \ll m$, then by uniqueness $\mu_s = 0$ and so $\mu(E) = \int_E D\mu(x)\, dm(x)$

Proof of theorem: It is sufficient to prove separately for the cases $\mu \perp m$ and $\mu \ll m$. For in general, $\mu = \mu_1 + \mu_2$

where $\mu_1 \perp m$ and $\mu_2 << m$. Suppose theorem holds for μ_1 and μ_2. Then we know $D\mu_1$ exists a.e. and $D\mu_1 \in L^1(\mathbb{R}^k, m)$. Moreover (c) says $D\mu_1 = D\mu_s = 0$ a.e. $[m]$. Also $D\mu_2$ exists a.e. and $D\mu_2 \in L^1(\mathbb{R}^k, m)$. Then $D\mu = D\mu_1 + D\mu_2 \in L^1(\mathbb{R}^k, m)$ and

$$\mu(E) = \mu_1(E) + \mu_2(E) = \mu_1(E) + \int_E D\mu_2(x)\, dm(x)$$

$$= \mu_1(E) + \int_E D\mu(x)\, dm(x) \quad [[D\mu = D\mu_2 \text{ a.e.}]]$$

It is also sufficient to prove for the real and imaginary parts of μ separately

CASE I: μ real, $\mu \perp m$

$\mu^+ = \frac{1}{2}(|\mu| + \mu) \perp m$, so $\exists$ Borel set A s.t. m is concentrated on A and μ^+ is concentrated on $\mathbb{R}^k - A$

$$m(\mathbb{R}^k - A) = 0 = \mu^+(A)$$

The previous lemma $\Rightarrow \exists A' \subset A$ s.t. $m(A - A') = 0$ and $D\mu^+(x) = 0$ everywhere on A'. Hence $D\mu^+ = 0$ a.e. $[m]$. Similarly $D\mu^- = 0$ a.e. $[m]$, so $D\mu = 0$ a.e. $[m]$. Then (a), (b), (c) are satisfied

CASE II: μ real, $\mu << m$

Radon-Nikodym theorem $\Rightarrow \exists$ Borel measurable

$f \in L^1(\mathbb{R}^k, m)$ s.t.

$$\mu(E) = \int_E f\, dm \quad \forall \text{Borel } E$$

It is sufficient to show $f(x) = D\mu(x)$ a.e.
For $r \in \mathbb{Q}$, let

rationals in $\mathbb{C}$

$$A_r := \{ x : f(x) < r \}$$
$$B_r := \{ x : f(x) \geq r \}$$
(Borel sets)

For $r \in \mathbb{Q}$, define a positive measure λ_r on the Borel sets by

$$\lambda_r(E) := \int_{E \cap B_r} (f(x) - r)\, dm(x)$$

Note that $\lambda_r(A_r) = 0$ since $A_r \cap B_r = \phi$. By the lemma $\exists\, A_r' \subset A_r$ s.t. $m(A_r - A_r') = 0$ and $D\lambda_r(x) = 0$ on A_r'.
Let

$$Y = \bigcup_{r \in \mathbb{Q}} (A_r - A_r')$$

Then Y is Lebesgue measurable with $m(Y) = 0$. Suppose $x \notin Y$. Sufficient to show $D\mu(x) = f(x)$. Consider a sequence of Borel sets E_i shrinking to x nicely. Consider $r \in \mathbb{Q}$ with $r > f(x)$. Then $x \in A_r$. But $x \notin Y$, so we must have $x \in A_r'$. Therefore $D\lambda_r(x) = 0$.

$$\mu(E_i) - r\,m(E_i) = \int_{E_i} (f(x) - r)\, dm(x)$$

$$\Longrightarrow \quad \frac{\mu(E_i)}{m(E_i)} - r = \frac{1}{m(E_i)} \int_{E_i} (f(t) - r)\, dm(t)$$

$$\leq \frac{1}{m(E_i)} \int_{E_i \cap B_r} (f(t) - r)\, dm(t)$$

$$= \frac{\lambda_r(E_i)}{m(E_i)} \xrightarrow[i\to\infty]{} D\lambda_r(x) = 0$$

Hence $\overline{\lim} \dfrac{\mu(E_i)}{m(E_i)} \leq r \implies \overline{\lim} \dfrac{\mu(E_i)}{m(E_i)} \leq f(x)$

Now consider $-\mu$. Its R-N derivative is $-f$. Applying result just obtained, we get

$$\overline{\lim} \frac{-\mu(E_i)}{m(E_i)} \leq -f(x)$$

$$\shortparallel$$

$$-\underline{\lim} \frac{\mu(E_i)}{m(E_i)}$$

Hence $\underline{\lim} \dfrac{\mu(E_i)}{m(E_i)} \geq f(x)$. Therefore $f(x) = \lim\limits_{i\to\infty} \dfrac{\mu(E_i)}{m(E_i)} = D\mu(x)$. $\square$

5/5 ANALYSIS

Remark: Suppose $f \in L^1(\mathbb{R}^k, m)$. Define

$$\mu(E) := \int_E f(x)\, dm(x) \quad \forall \text{Borel } E$$

D.C.T. $\Rightarrow$ μ complex Borel measure. Moreover, $\mu \ll m$. By (c) of the last theorem,

$$\mu(E) = \int_E D\mu(x)\, dm(x) \ \forall \text{Borel } E$$

Therefore $D\mu(x) = f(x)$ a.e. on $[m]$. Suppose x_0 is such that $f(x_0) = D\mu(x_0)$. Consider a seq. of Borel sets E_i shrinking nicely to x_0.

$$\frac{\mu(E_i)}{m(E_i)} - f(x_0) = \frac{1}{m(E_i)} \int_{E_i} [f(x) - f(x_0)]\, dm(x)$$

As $i \to \infty$, LHS tends to $D\mu(x_0) - f(x_0) = 0$. Hence

$$\lim_{i \to \infty} \frac{1}{m(E_i)} \int_{E_i} [f(x) - f(x_0)]\, dm(x) = 0$$

Specifically, take the case $f = \chi_Q$, where $m(Q) < \infty$. Then $\mu(E) = m(E \cap Q)$. For almost every x_0, for every (E_i) Borel sets shrinking to x_0 nicely, we have

$$\frac{m(Q\cap E_i)}{m(E_i)} \longrightarrow \chi_Q(x_0)$$

(density of $Q = \chi_Q$ a.e.)

<u>THEOREM</u>: Suppose $f \in L^1(\mathbb{R}^k)$. Let L_f (the Lebesgue set of f) be the set of all $x_0 \in \mathbb{R}^k$ such that

$$\lim_{i\to\infty} \frac{1}{m(E_i)} \int_{E_i} |f(x)-f(x_0)|\, dm(x) = 0$$

for every sequence of Borel sets E_i shrinking nicely to x_0. Then

$$m(\mathbb{R}^k - L_f) = 0$$

Proof. Sufficient to show $m(B(0,1) - L_f) = 0$. For $r \in \mathbb{Q}$, define for E Borel

$$\mu_r(E) = \int_E |f(x)-r|\, \chi_{B(0,2)}(x)\, dm(x)$$

Then μ_r is a complex Borel measure on $\mathbb{R}^k$. As before

$$D\mu_r(x) = |f(x)-r|\, \chi_{B(0,2)}(x) \quad \text{a.e. } [m]$$

Let $Y_r = \{x \in B(0;1) : D\mu_r(x) \neq |f(x)-r|\}$. Then $m(Y_r) = 0$.
Let $Y = \bigcup_{r\in\mathbb{Q}} Y_r$. Then $m(Y) = 0$.

If $x_0 \in B(0,1) - Y$, we will show $x_0 \in L_f$. Given $\varepsilon > 0$, $\exists \tilde{r} \in \mathbb{Q}^2$ s.t. $|f(x_0) - \tilde{r}| < \varepsilon$. Then if (E_i) shrinks nicely to x_0

$$\frac{1}{m(E_i)} \int_{E_i} |f(x) - f(x_0)| \, dm(x)$$

$$\leq \frac{1}{m(E_i)} \int_{E_i} \left(|f(x) - \tilde{r}| + |\tilde{r} - f(x_0)| \right) dm(x)$$

$$\leq \varepsilon + \frac{1}{m(E_i)} \int_{E_i} |f(x) - \tilde{r}| \, dm(x)$$

$$= \varepsilon + \frac{\mu_{\tilde{r}}(E_i)}{m(E_i)} < 2\varepsilon \quad \text{if } i \text{ large}$$

$\downarrow$

$|f(x_0) - \tilde{r}|$ since $x_0 \notin Y_{\tilde{r}}$

$\blacksquare$

Functions of Bounded Variation

Definition: $f : \mathbb{R} \to \mathbb{C}$. $x_0 < x_1 < \dots < x_N = x$. (partition)

$$T_f(x) := \sup_{\substack{\text{all such} \\ \text{partitions}}} \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})|$$

If $\lim_{x \to \infty} T_f(x) < \infty$, say $f \in BV$

DEFINITION: $f \in NBV$ (normalized) if

a) $f \in BV$

b) $\lim_{x \to -\infty} f(x) = 0$

c) f is left continuous everywhere (i.e. $\forall x_0$, $\lim_{x \uparrow x_0} f(x) = f(x_0)$)

PROPOSITION: $f \in NBV \Rightarrow T_f \in NBV$

Proof: $f \in BV \Rightarrow T_f$ is bounded and non-decreasing, so $T_f \in BV$

Select $x, \varepsilon > 0$. $\exists$ $x_0 < x_1 < \dots < x_n = x$ s.t.

$$\sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| \geq T_f(x) - \varepsilon$$

Suppose $t_0 < t_1 < \dots < t_M = x_0$.

$$\sum_{j=1}^{M} |f(t_j) - f(t_{j-1})| + \underbrace{\sum_{i=1}^{N} |f(x_i) - f(x_{i-1})|}_{\geq T_f(x) - \varepsilon} \leq T_f(x)$$

Hence $\sum_{j=1}^{M} |f(t_j) - f(t_{j-1})| \leq \varepsilon$, and so $T_f(x_0) \leq \varepsilon$

Therefore $\lim_{x \to -\infty} T_f(x) = 0$

Consider the same x_i's

$$x_0 < x_1 < \dots < x_{N-1} < t < x_N = x$$

$$\sum_{i=1}^{N-1} |f(x_i) - f(x_{i-1})| + |f(t) - f(x_{N-1})|$$

$$\leq T_f(t) \leq T_f(x-) \leq T_f(x)$$

Let $t \uparrow x$. Since f is left continuous

$$T_f(x) - \varepsilon \underset{\substack{\uparrow \\ \text{choice of } x_i\text{'s}}}{\leq} \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| \leq T_f(x-) \leq T_f(x)$$

Therefore $T_f(x-) = T_f(x) \Rightarrow T_f$ is left continuous

$\boxtimes$

<u>THEOREM</u>: (a) Suppose μ is a complex Borel measure on $\mathbb{R}$. Then $\exists\, f : \mathbb{R} \to \mathbb{C}$ s.t. $f(x) = \mu(-\infty, x)$ and $f \in NBV$

(b) If $f \in NBV$, then $\exists!$ complex Borel measure μ on $\mathbb{R}$ s.t. $f(x) = \mu(-\infty, x)$ and $|\mu|(-\infty, x) = T_f(x)$ $\forall x \in \mathbb{R}$

Proof. (a) Show $f \in BV$. Consider $x_0 < x_1 < \dots < x_N = x$

$$\sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{N} |\mu([x_{i-1}, x_i))|$$

$$\leq |\mu|(-\infty, x) \leq |\mu|(\mathbb{R}) < \infty$$

Therefore $T_f(x) \leq |\mu|(\mathbb{R}) \ \forall x$, so $f \in BV$.

5/8 MEASURE THEORY (241 10 A.M. FRI)

Proof of previous theorem

(a) Showed $T_f(x) \le |\mu|(-\infty, x)$.

f is left continuous: Suppose $x_n \uparrow x$. Then

$$f(x_n) = \mu(-\infty, x_n) \longrightarrow \mu(-\infty, x) = f(x)$$

write $\mu = Re\mu^+ - Re\mu^- + i(Im\mu^+ - Im\mu^-)$
and use results on positive measures

Now suppose $x_n \downarrow -\infty$. Then $\bigcap_{n=1}^{\infty}(-\infty, x_n) = \phi$, and so

$$0 = |\mu|(\phi) = \lim_{n\to\infty} |\mu|(-\infty, x_n) \Rightarrow$$

$$|f(x)| = |\mu(-\infty, x)| \le |\mu|(-\infty, x) \xrightarrow[x\to-\infty]{} 0$$

(b) Suppose f real. Write $f = f_1 - f_2$ where f_j is strictly increasing, positive and bounded. WARNING: Also assume f_j continuous. For E Borel, define

$$\mu_j(E) = m(f_j(E))$$

(f_j is a homeomorphism of R onto $(0, \alpha)$, so $f_j(E)$ is Borel)
Then f_j 1-1 $\Rightarrow$ μ_j is a Borel measure. Define

$$\mu = \mu_1 - \mu_2$$

Note that $\mu_j(-\infty, x) = m(f_j(-\infty, x)) = m(0, f_j(x)) = f_j(x)$
Then

$$\mu(-\infty, x) = f_1(x) - f_2(x) = f(x)$$

Now for f complex, work with real and imaginary parts separately.
uniqueness: Suppose λ is a complex Borel measure s.t.

$$\lambda(-\infty, x) = f(x)$$

We know λ, μ are regular (by Th^m 2.18). Since

$$\lambda(-\infty, x) = \mu(-\infty, x) \quad \forall x$$

Then $\lambda[\alpha, \beta) = \mu[\alpha, \beta)$ $\forall \alpha < \beta$, and so μ and λ agree on all open intervals $\Rightarrow$ on all open sets. Now suppose E is Borel. By regularity, $\exists$ open $(V_n) \supset E$ s.t. $V_{n+1} \subset V_n$

$$|\mu|(V_n) < |\mu|(E) + 1/n$$
$$|\lambda|(V_n) < |\lambda|(E) + 1/n$$

Let $V = \bigcap V_n \supset E$. Then $|\mu|(V - E) = 0 = |\lambda|(V - E)$
Hence

$$\lambda(E) = \lambda(V) \ ; \quad \mu(E) = \mu(V)$$

But

$$\mu(V) = \lim_{n\to\infty}\mu(V_n) \overset{V_n \text{ open}}{=} \lim_{n\to\infty}\lambda(V_n) = \lambda(V)$$

and so $\lambda(E) = \mu(E)$. Hence $\lambda = \mu$

From (a), $T_f(x) \le |\mu|(-\infty, x)$. $f \in NBV \Rightarrow T_f \in NBV$ (last time) Hence there is a complex Borel measure λ such that for every x

$$\lambda(-\infty, x) = T_f(x)$$

Since $|f(\alpha) - f(\beta)| \le T_f(\beta) - T_f(\alpha)$ for $\alpha < \beta$, we have

$$|\mu[\alpha, \beta)| \le \lambda[\alpha, \beta)$$

Therefore $|\mu(E)| \le \lambda(E)$ for all Borel sets E. Hence $|\mu|(E) \le \lambda(E)$ and so

$$|\mu|(-\infty, x) \le \lambda(-\infty, x) = T_f(x)$$

Therefore $T_f(x) = |\mu|(-\infty, x)$

$\boxed{}$

DEFINITION: $f: \mathbb{R} \to \mathbb{C}$ is absolutely continuous if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if the intervals (a_i, b_i), $1 \le i \le N$, are disjoint and $\sum_{i=1}^{N} (b_i - a_i) < \delta$, then

$$\sum_{i=1}^{N} |f(b_i) - f(a_i)| < \varepsilon$$

THEOREM: Suppose $f \in NBV$. Then f is absolutely continuous if and only if the unique complex Borel measure μ associated with f is absolutely continuous w.r.t. Lebesgue measure.

Proof. Suppose $\mu \ll m$. Given $\varepsilon > 0$ $\exists \delta > 0$ s.t. if $m(E) < \delta$, then $|\mu|(E) < \varepsilon$. Suppose (a_i, b_i), $1 \le i \le N$ are disjoint and $\sum (b_i - a_i) < \delta$. Let

$$E = \bigcup_{i=1}^{N} [a_i, b_i)$$

(disjoint union). Then

$$\sum_{i=1}^{N} |f(b_i) - f(a_i)| = \sum_{i=1}^{N} |\mu[a_i, b_i)|$$

$$\le |\mu|(E) < \varepsilon$$

since $m(E) < \delta$.

Now suppose f is absolutely continuous. Suppose E is Borel and $m(E) = 0$. Given $\varepsilon > 0$ $\exists \delta > 0$ s.t. definition of δ is satisfied. Will show $|\mu(E)| \leq \varepsilon$. m regular $\Rightarrow \exists$ open $\mathcal{O} \supset E$ s.t. $m(\mathcal{O}) < \delta$. Since μ is regular, $\exists$ open $V_n \supset E$ s.t. $|\mu|(V_n) < |\mu|(E) + 1/n$. Let $W_n = \mathcal{O} \cap V_n$. Then $m(W_n) < \delta$. WLOG $W_{n+1} \subset W_n \ \forall n$. Let

$$W := \bigcap_{n=1}^{\infty} W_n$$

Then $|\mu|(W-E) = 0 \Rightarrow \mu(W) = \mu(E)$, and so $\mu(W_n) \to \mu(E)$. W_n open, so we can write

$$W_n = \bigcup_k I_{nk} \quad \text{(disjoint, closed on left, open on right)}$$

Sufficient to show $|\mu(W_n)| \leq \varepsilon$. But

$$|\mu(W_k)| \leq \sum_k^{\infty} |\mu(I_{nk})| = \lim_{N\to\infty} \sum_{k=1}^{N} |\mu(I_{nk})|$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} |f(b_{kn}) - f(a_{kn})| \qquad [[I_{nk} = [a_{kn}, b_{kn})]]$$

$$\leq \varepsilon$$

each of these $\leq \varepsilon$

since $m(\sum b_k - a_k) \leq m(\bigcup_{k=1}^{\infty} I_{nk}) < \delta$

5/10 MEASURE THEORY

REVIEW

THEOREM I: μ complex Borel measure on $\mathbb{R}^k$

1) $D\mu$ exists a.e. $[m]$
2) $D\mu \in L^1(\mathbb{R}^k, m)$
3) $\exists$ complex Borel measure $\mu_s \perp m$, $D\mu_s' = 0$ a.e. $[m]$ s.t.

$\forall$ Borel E

$$\mu(E) = \mu_s(E) + \int_E D\mu(x)\, dm(x)$$

THEOREM II:

a) μ complex Borel measure on $\mathbb{R} \Rightarrow f(x) := \mu(-\infty, x) \in NBV$
b) $f \in NBV \Rightarrow \exists!$ complex Borel measure μ s.t.

$$\mu(-\infty, x) = f(x) \quad \forall x$$

Also $|\mu|(-\infty, x) = T_f(x)$

THEOREM III: Suppose $f \in NBV$. f is absolutely continuous iff ~~the~~ unique μ from theorem IIb is such that $\mu \ll m$

<u>THEOREM</u>: Suppose $g \in L^1(\mathbb{R})$. Then

$$F(x) := \int_{-\infty}^{x} g(t)\,dt$$

satisfies $F \in NBV$, F is absolutely continuous, and $F'(x) = g(x)$ a.e. $[m]$

Proof. Define μ complex Borel measure μ by

$$\mu(E) := \int_E g(t)\,dt$$

for every Borel E

Then by IIa, $F(x) = \mu(-\infty, x)$ is in NBV. Clearly $\mu \ll m$, so III $\Rightarrow$ F is absolutely continuous

By Theorem I and the uniqueness of the Lebesgue decomposition

$$\int g(t)\,dt = \mu(E) = \int_E (D\mu)(t)\,dm(t)$$

and so $g(x) = D\mu(x)$ a.e $[m]$. Select x_0 s.t. $D\mu(x_0)$ exists.

Claim: $F'(x_0) = D\mu(x_0)$. Take $h > 0$.

$$\frac{F(x_0+h) - F(x_0)}{h} = \frac{\mu([x_0, x_0+h))}{m([x_0, x_0+h))} \to D\mu(x_0)$$

(As $h \to 0$, $[x_0, x_0+h)$ shrinks nicely to x_0). This shows claim.

Hence $F' = g$ a.e. $[m]$.

$\boxed{/\!/}$

THEOREM: Suppose $f \in NBV$

1) f' exists a.e.
2) $f' \in L^1(\mathbb{R})$
3) $\exists f_s$ s.t. $f_s' = 0$ a.e. and

$$f(x) = f_s(x) + \int_{-\infty}^{x} f'(t)\,dt$$

$\forall x \in \mathbb{R}$. Furthermore, $f_s = 0$ if and only if f is absolutely continuous. If f is real and non-decreasing, then f_s is real and non-decreasing.

Proof. Apply Theorem IIb to get a complex Borel measure μ s.t. $\mu(-\infty, x) = f(x)$ $\forall x$. By theorem I, $D\mu$ exists a.e. and $D\mu \in L^1(\mathbb{R})$. The claim of the previous proof $\Rightarrow f' = D\mu$ wherever $D\mu$ exists. Hence f' exists a.e. and $f' \in L^1(\mathbb{R})$

By theorem I, $\exists \mu_s \perp m$ with $D\mu_s' = 0$ a.e. and

$$\mu(E) = \mu_s(E) + \int_E D\mu(x)\,dm(x)$$

Define

$$f_s(x) := \mu_s(-\infty, x)$$

As before, $f_s' = D\mu_s' = 0$ a.e. $[m]$. Moreover

$$f(x) = \mu(-\infty, x) = \mu_s(-\infty, x) + \int_{-\infty}^{x} D\mu(t)\,dt = f_s(x) + \int_{-\infty}^{x} f'(t)\,dt$$

By Theorem III, f is absolutely continuous $\Leftrightarrow$ $\mu \ll m \Leftrightarrow \mu_s = 0$
But $\mu_s = 0 \Rightarrow f_s = 0$. If $f_s = 0$, then uniqueness of Theorem IIb $\Rightarrow \mu_s = 0$

uniqueness of Lebesgue decomposition

Suppose f real, non-decreasing (Recall –

$$\lim_{x \to -\infty} f(x) = 0 \;)$$

So $f \geq 0$. Claim: $\mu \geq 0$. $f \uparrow \Rightarrow$

$$|\mu|(-\infty, x) = T_f(x) = f(x) = \mu(-\infty, x)$$

Suppose $E \subset (-\infty, x)$, $\mu(E) \not\geq 0$. Then

$$\mu(-\infty, x) = \mu(E) + \mu((-\infty, x) - E)$$

and so

$$\mu(-\infty, x) < |\mu(E)| + |\mu((-\infty, x) - E)| \leq |\mu|(-\infty, x) \quad \lightning$$

Hence $\mu(E) \geq 0$.

Claim. $\mu_s \geq 0$. Suppose E Borel set with $\mu_s(E) < 0$
$\mu_s \perp m \Rightarrow \exists$ Borel set A s.t. $\mu_s(E) = \mu_s(E \cap A)$ and $m(A) = 0$
Then

$$0 \leq \mu(E \cap A) = \mu_s(E \cap A) + \int_{E \cap A} D\mu(t)\, dm(t) \underset{\substack{\uparrow \\ m(E \cap A) = 0}}{=} \mu_s(E \cap A) = \mu_s(E)$$

But now $\mu_s \geq 0 \Rightarrow f_s \geq 0$ and if $a<b$

$$f_s(b) - f_s(a) = \mu_s([a,b)) \geq 0$$

$$\blacksquare$$

Suppose $a<b$. Then

$$f(b) = f_s(b) + \int_{-\infty}^{b} f'(t)\,dt$$

$$f(a) = f_s(a) + \int_{-\infty}^{a} f'(t)\,dt$$

$$\Rightarrow \int_a^b f' = [f(b)-f(a)] - [f_s(b) - f_s(a)] \leq f(b)-f(a)$$

<u>THEOREM</u>: If f' exists everywhere on $[a,b]$ and is integrable, then

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

Proof uses Vitali - Carathéodory

5/12 MEASURE THEORY

X compact T_2-space
$C_{\mathbb{R}}(X)$ continuous real functions on X } Banach spaces with sup norm
$C(X)$ continuous complex-valued functions on X

<u>STONE-WEIERSTRASS THEOREM</u>: A subspace A of $C_{\mathbb{R}}(X)$ is dense in $C_{\mathbb{R}}(X)$ if
a) A is an algebra (i.e. $f_1, f_2 \in A \Rightarrow f_1 f_2 \in A$)
b) A contains (real) constants
c) A separates points of X, i.e. if $x \neq y$ in X, then $\exists f \in A$ s.t. $f(x) \neq f(y)$)

<u>COROLLARY</u>: A subspace A of $C(X)$ is dense in $C(X)$ if
a) A is an algebra
b) A contains complex constants
c) A separates points of X
d) A is closed under conjugation (i.e. $f \in A \Rightarrow \bar{f} \in A$)

Remark – ① Recall from 441 that Weierstrass' theorem says that the real polynomials are dense in $C_{\mathbb{R}}(X)$, where $X = [a,b]$. This is a special case of the S-W theorem. Note that polynomials with complex coefficients are dense in $C[a,b]$.
② The trigonometric polynomials are dense in $C(T)$ (consequence of Féjer's theorem) This is also a trivial consequence of S-W. Note that the real-valued trigonometric polynomials are dense in $C_{\mathbb{R}}(T)$ (The real part of a

trig. polynomial is a trig. polynomial

$$\mathrm{Re}\, a_n e^{in\theta} = \frac{1}{2} a_n e^{in\theta} + \frac{1}{2}\overline{a_n} e^{-in\theta} \Big)$$

Examples: 1) $X = [-1,1]$, $A =$ even real polynomials. A is not dense (can't approx odd polynomials) Note c) fails

2) $X = [-1,1]$ $A =$ real polynomials with $P(0) = 0$. A is not dense. Note b) fails

3) $X = \mathbb{R}$, $A =$ real polynomials. $\|P(x) - e^x\|_\infty = \infty$ $\forall$ real polynomials

only locally compact, not compact

Notation: $f_1, f_2 \in C_{\mathbb{R}}(X)$, let

$$\left.\begin{array}{l} f_1 \wedge f_2 := \min(f_1, f_2) \\ f_1 \vee f_2 := \max(f_1, f_2) \end{array}\right\} \in C_{\mathbb{R}}(X)$$

__DEFINITION:__ $L \subset C_{\mathbb{R}}(X)$ is a lattice if $f_1, f_2 \in L \Longrightarrow$ $f_1 \wedge f_2 \in L$ and $f_1 \vee f_2 \in L$

Proof of Theorem: Throughout X is a compact T_2-space

__LEMMA 1:__ Suppose $L \subset C_{\mathbb{R}}(X)$ is a lattice. Let $g := \inf_{f \in L} f$. If g is continuous, then $\forall \varepsilon > 0$ $\exists f \in L$ s.t.

$$0 \leq f - g < \varepsilon \quad \text{everywhere on } X$$

[g need not always be continuous: $L = \{x^n : n \in \mathbb{N}\}$, $X = [0,1]$

$$g(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$]

Proof. $\forall x \in X$, $\exists f_x \in L$ such that

$$0 \leqslant f_x(x) - g(x) < \varepsilon/3$$

Since both f_x and g are continuous, $\exists$ open O_x containing x such that

$$y \in O_x \Longrightarrow \begin{cases} |f_x(x) - f_x(y)| < \varepsilon/3 \\ |g(x) - g(y)| < \varepsilon/3 \end{cases}$$

Then $y \in O_x \Rightarrow |f_x(y) - g(y)| < \varepsilon$

Since X is compact, there is a finite subset $F \subset X$ such that

$$X = \bigcup_{x \in F} O_x$$

Let $f := \bigwedge_{x \in F} f_x \in L$ (since Lattice). If $y \in X$, then $y \in O_x$ for some $x \in F$, and so

$$0 \leq f(y) - g(y) \leq f_x(y) - g(y) \overset{\text{since } y \in O_x}{<} \varepsilon \qquad \blacksquare$$

<u>LEMMA 2</u>: If $\mathcal{F} \subset C_{\mathbb{R}}(X)$ satisfies

(i) $\mathcal{F}$ separates points

(ii) $f \in \mathcal{F}, c \in \mathbb{R} \Rightarrow cf \in \mathcal{F}$ and $c+f \in \mathcal{F}$

then if $x \neq y \in X$ and $a, b \in \mathbb{R}$, then $\exists f \in \mathcal{F}$ such that

$$f(x) = a, \quad f(y) = b$$

Proof: Suppose $x \neq y$. $\exists g \in \mathcal{F}$ s.t. $g(x) \neq g(y)$ (by (i))

Define

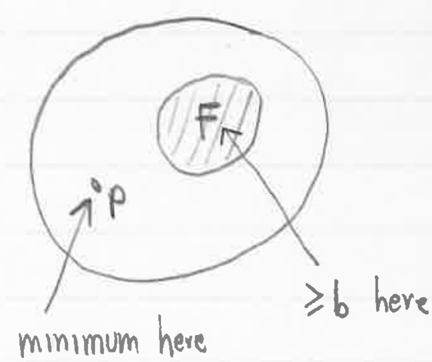
$$f := \frac{a-b}{g(x)-g(y)} g + \frac{b\,g(x) - a\,g(y)}{g(x)-g(y)}$$

Then $f \in \mathcal{F}$ by (ii).

$\blacksquare$

<u>LEMMA 3</u>: Suppose $L \subset C_{\mathbb{R}}(X)$ is a lattice which has properties (i) and (ii) above. Suppose F is a closed subset of X, and $p \in X - F$. If $a < b$ in $\mathbb{R}$, then $\exists f \in L$ s.t.

$$f(x) \geq a \quad \forall x \in X$$
$$f(x) \geq b \quad \forall x \in F$$
$$f(p) = a$$



Proof. Note lemma 2 applies to L. So $\forall x \in F$, $\exists f_x \in L$ s.t. $f_x(p) = a$ and $f_x(x) = b+1$. Let

$$\mathcal{O}_x := \{ y \in X : f(y) > b \}$$

Then $\mathcal{O}_x$ is open and $x \in \mathcal{O}_x$. F compact $\Rightarrow$

$$F \subset \bigcup_{x \in A} \mathcal{O}_x$$

where $A \subset F$ is finite. Let

$$g := \bigvee_{x \in A} f_x \in L$$

Then $g(p) = a$ and $g(x) > b$ $\forall x \in F$. Now let

$$f = g \vee a \in L$$

L contains 0 $\Rightarrow$ contains all constants (property (ii))

$\square$

<u>LEMMA 4</u>: Suppose L is a lattice which separates points and has property $c \in \mathbb{R}$, $f \in L \Rightarrow cf \in L$ and $c + f \in L$. $\forall g \in C_{\mathbb{R}}(X)$ and $\forall \varepsilon > 0$ $\exists f \in L$ s.t.

$$0 \le f(x) - g(x) < \varepsilon \quad \forall x \in X$$

Proof. Let $L' \subset L$ be given by

$$L' = \{ f \in L : f(x) \geq g(x) \ \forall x \in X \}$$

Then L' is a lattice. It is sufficient to show $g = \inf_{f \in L'} f$ by lemma 1.

Select $p \in X$ and $\eta > 0$. The set

$$F := \{ x \in X : g(x) \geq g(p) + \eta \}$$

is closed. Certainly $\exists M > g(p) + \eta$ s.t. $g(x) \leq M \ \forall x \in X$.
By lemma 3, $\exists f_0 \in L$ s.t.

$$f_0(p) = g(p) + \eta$$
$$f_0(x) \geq M \quad \forall x \in F$$
$$f_0(x) \geq g(p) + \eta \quad \forall x \in X$$

Then $f_0 \in L'$, and so

$$\inf_{f \in L'} f(p) \leq f_0(p) = g(p) + \eta$$

But clearly $g(p) \leq \inf_{f \in L'} f(p)$ by definition of L'. Hence $g(p) = \inf_{p \in L'} f(p)$

Recall: Weierstrass's Thm $\Rightarrow$ $\forall \varepsilon > 0$ $\exists$ real poly. P s.t.
$|P(x) - |x|| < \varepsilon$ for $-1 \leq x \leq 1$

Proof of theorem: Note by lemma 4 that it is sufficient to show $\overline{A}$ is a lattice (closure in sup topology)

It is clear that $\overline{A}$ is an algebra. Suppose $f \in \overline{A}$ and $\|f\|_\infty \leq 1$. Then

$$|P(f(x)) - |f(x)|| < \varepsilon \quad \forall x \in X$$

(P from above remark) $\overline{A}$ an algebra $\Rightarrow$ $P(f) \in \overline{A}$. Also $\overline{A}$ closed $\Rightarrow$ $|f| \in \overline{A}$

Suppose $f \in \overline{A}$. Then $f / \|f\|_\infty \in \overline{A}$, and so by above paragraph, $|f| / \|f\|_\infty \in \overline{A} \Rightarrow |f| \in \overline{A}$.

Suppose $f, g \in \overline{A}$; then

$$f \wedge g = \frac{1}{2}\left((f+g) - |f-g|\right) \in \overline{A}$$

$$f \vee g = \frac{1}{2}\left((f+g) + |f-g|\right) \in \overline{A}$$

Hence $\overline{A}$ is a lattice.

Now lemma 4 says $\overline{A}$ is dense in $C_{\mathbb{R}}(X)$, and so A is dense ($\overline{A}$ is closed, so actually $\overline{A}$ dense $\Rightarrow$ $\overline{A} = C_{\mathbb{R}}(X)$)

$\boxtimes$

Proof of corollary:

$$f \in A \Rightarrow \overline{f} \in A \Rightarrow \operatorname{Re} f = \frac{1}{2}(f + \overline{f}) \in A$$

Let $A' = \{ \operatorname{Re} f : f \in A \} \subset C_{\mathbb{R}}(X)$. Then A' satisfies (a), (b), and (c) of S-W theorem, and so A' is dense in $C_{\mathbb{R}}(X)$. But $A' \subset A$, so given $g \in C(X)$ we can approximate $\operatorname{Re} g$ and $\operatorname{Im} g$ by members of A, and thus can approximate g by a member of A.

$\blacksquare$

Fourier Analysis

$$H := L^2\left([-\pi,\pi], \frac{d\theta}{2\pi}\right)$$

normalized Lebesgue measure

$$T := \{z \in \mathbb{C} : |z| = 1\} \quad \text{(circle group)}$$

$C(T) :=$ continuous complex-valued functions on T

(The elements of $C(T)$ can be identified with the continuous periodic complex-valued functions on $\mathbb{R}$ with period 2π)

Proposition 1 (p105): $\{e^{int} : n \in \mathbb{Z}\}$ is an orthonormal family in H.

Proof.

$$(e^{int} | e^{imt}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)t}\,dt = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

$\blacksquare$

Remember that $L^2([-\pi,\pi])$ is actually a space of equivalence classes. If $f \in L^2([-\pi,\pi])$, we can define

$$f_0(t) := \begin{cases} f(t) & t \neq \pi \\ f(-\pi) & t = \pi \end{cases}$$

Then

$$\int_{-\pi}^{\pi} (f-f_0)^2(t)\,dt = 0$$

so f and f_0 both represent the same "element" in $L^2([-\pi,\pi])$. Therefore we may consider the functions in $L^2([-\pi,\pi])$ as periodic functions on $\mathbb{R}$ with period 2π, or equivalently, as elements of $L^2(\mathbb{T})$.

DEFINITION: If $f \in L^2([-\pi,\pi])$, its n^{th} Fourier coefficient is

$$\hat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)\,dt$$

for every $n \in \mathbb{Z}$. The Fourier series of f is

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{int}$$

DEFINITION: For $f \in L^2([-\pi,\pi])$ and $N \in \mathbb{N}$, define

$$S_N(x,f) = S_N(x) := \sum_{k=-N}^{N} \hat{f}(k) e^{ikt}$$

Then S_N is the N^{th} partial sum of the Fourier series for f.

<u>FEJÉR'S THEOREM</u> (p110) Suppose $f \in C(T)$. Let

$$\sigma_N(x,f) = \sigma_N(x) := \frac{1}{N+1} \sum_{k=0}^{N} S_k(x)$$

Then $\sigma_N \to f$ uniformly on $[-\pi,\pi]$.

Proof later

Note that

$$\sigma_N(x) = \sum_{k=-N}^{N} c_k e^{ikx}$$

for some choice of c_k; σ_N is a trigonometric polynomial of degree N.

⌈ If (X,M,μ) is a measure space where μ has the properties of the conclusion of the Riesz Representation theorem, then $C_c(X)$ is dense in $L^p(\mu)$ for $1 \le p < \infty$. (p 84) ⌋

We know $C(T)$ is dense in $L^2([-\pi,\pi])$. By Fejér's theorem the set of finite linear combinations of $\{e^{ikx} : k \in \mathbb{Z}\}$ (i.e. the trigonometric polynomials) is dense in $C(T)$, and therefore are dense in $L^2([-\pi,\pi])$. Hence $\{e^{inx} : n \in \mathbb{Z}\}$ is a maximal orthonormal family

⌈ Suppose H is a Hilbert space and $(u_\alpha : \alpha \in A)$ is an orthonormal family in H. TFAE

i) $(u_\alpha : \alpha \in A)$ is a maximal orthonormal family

ii) The set of finite linear combinations of members of this family is dense in H

iii) PARSEVAL'S THEOREM: $\forall x \in H$

$$\|x\|^2 = \sum_{\alpha \in A} |\hat{x}(\alpha)|^2$$

iv) $$(x|y) = \sum_{\alpha \in A} \hat{x}(\alpha)\, \overline{\hat{y}(\alpha)} \qquad \forall x, y \in H$$

(p 103)

Suppose $f \in L^2([-\pi,\pi])$. By Parseval's theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\,dt = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2$$

Also, if $f, g \in L^2([-\pi,\pi])$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\overline{g(t)}\,dt = \sum_{n \in \mathbb{Z}} \hat{f}(n)\,\overline{\hat{g}(n)}$$

Suppose $f \in L^2([-\pi,\pi])$. Then for each $N \in \mathbb{N}$

$$\widehat{f - S_N}(k) = \begin{cases} \hat{f}(k) & |k| > N \\ 0 & |k| \leq N \end{cases}$$

Therefore, by Parseval's theorem

$$\| f - S_N \|_2^2 = \sum_{|k| > N} |\hat{f}(k)|^2 \longrightarrow 0 \quad \text{as } N \to \infty$$

Hence S_N converges to f in the L^2-norm, and so there is a <u>subsequence</u> (S_{N_j}) of (S_N) such that $S_{N_j}(x) \to f(x)$ almost everywhere.

Note that S_N is the trigonometric polynomial of degree N which best approximates f in the L^2 sense

⟦ Suppose F is a finite orthonormal family in H. For every $x \in H$

$$\| x - \sum_{u \in F} (x|u) u \| \leq \| x - \sum_{u \in F} \lambda_u u \|$$

for any family $(\lambda_u : u \in F)$ of scalers. Equality holds if and only if $\lambda_u = (x|u) \ \forall u \in F$ (p98) ⟧

Thus

$$\| f - S_N \|_2 \leq \| f - \sum_{k=-N}^{N} c_k e^{ikx} \|_2$$

for any family $(c_k : -N \leq k \leq N)$ of scalers.

Definition: Dirichlet Kernel

$$D_m(x) := \sum_{k=-m}^{m} e^{ikx} \qquad m \in \mathbb{N}$$

Fejér Kernel

$$K_n(x) := \frac{1}{n+1} \sum_{m=0}^{n} D_m(x) \qquad n \in \mathbb{N}$$

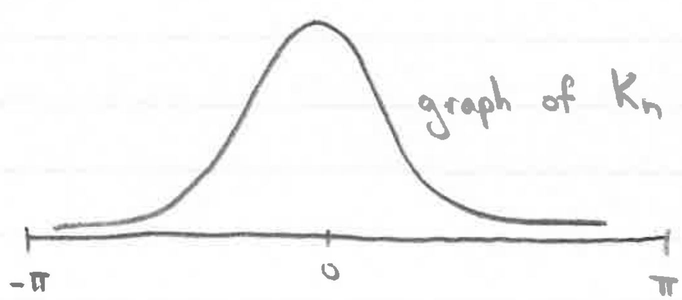
Proposition:

(1) $$D_m(x) = \frac{\sin(m+\frac{1}{2})x}{\sin \frac{1}{2}x}$$

(2) $$K_n(x) = \frac{1}{n+1} \frac{1-\cos(n+1)x}{1-\cos x}$$

(3) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_m(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1$$

(4) $0 \le K_n(x)$ $\forall x$ and $K_n(x) \le \frac{2}{n+1} \frac{1}{1-\cos\delta}$ for $\delta \le |x| \le \pi$



Proof. (1) Observe that

$$(*) \quad (e^{ix}-1)\,D_m(x) = \sum_{k=-m}^{m} e^{i(k+1)x} - \sum_{k=-m}^{m} e^{ikx} = e^{i(m+1)x} - e^{-imx}$$

Multiplying by $e^{-ix/2}$

$$(e^{ix/2} - e^{-ix/2})\,D_m(x) = e^{i(m+1/2)x} - e^{-i(m+1/2)x}$$

$$(2i \sin \tfrac{1}{2}x)\,D_m(x) = 2i \sin (m+1/2)x$$

$$D_m(x) = \frac{\sin (m+1/2)x}{\sin 1/2\, x}$$

(2) (*) also implies that

$$(n+1)(e^{ix}-1)\,K_n(x) = \sum_{m=0}^{n}\left(e^{i(m+1)x} - e^{-imx}\right) = \sum_{k=-n}^{n+1} c_k e^{ikx}$$

where

$$c_k = \begin{cases} 1 & 1 \le k \le n+1 \\ -1 & -n \le k \le 0 \end{cases}$$

Hence

$$(n+1)(e^{ix}-1)(e^{-ix}-1)\,K_n(x) = \sum_{k=-n}^{n+1} c_k e^{i(k-1)x} - \sum_{k=-n}^{n+1} c_k e^{ikx}$$

$$= -e^{-i(n+1)x} - e^{i(n+1)x} + 2$$

$$= 2 - 2\cos (n+1)x$$

Therefore

$$(n+1)K_n(x) = \frac{2-2\cos(n+1)x}{2-2\cos x} = \frac{1-\cos(n+1)x}{1-\cos x}$$

(3)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_m(x)\,dx = \frac{1}{2\pi}\sum_{k=-m}^{m}\int_{-\pi}^{\pi} e^{ikx}\,dx$$

$$= \frac{1}{2\pi}\sum_{\substack{k=-m\\k\neq 0}}^{m}\frac{1}{ik}\left(e^{ik\pi}-e^{-ik\pi}\right)+1$$

$$= 0+1 = 1$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = \frac{1}{n+1}\left(\sum_{m=0}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi} D_m(x)\,dx\right)$$

$$= \frac{1}{n+1}\sum_{m=0}^{n} 1 = 1$$

(4) It is clear from (2) that $K_n(x) \geq 0$ $\forall x$. Also, if $\delta \leq |x| \leq \pi$, then $1-\cos x \geq 1-\cos\delta$, so that

$$K_n(x) \leq \frac{2}{n+1}\frac{1}{1-\cos x} \leq \frac{2}{n+1}\frac{1}{1-\cos\delta}$$

$\blacksquare$

FEJÉR'S THEOREM: Suppose $f \in C(\mathbb{T})$. Let

$$S_N(x) := \sum_{k=-N}^{N} \hat{f}(k)\, e^{ikx}$$

and

$$\sigma_N(x) := \frac{1}{N+1} \sum_{k=0}^{N} S_k(x)$$

Then $\sigma_N \to f$ uniformly on $[-\pi,\pi]$.

Proof. Observe that

$$S_N(x) = \sum_{k=-N}^{N} \hat{f}(k)\, e^{ikx} = \sum_{k=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt}\, dt \right) e^{ikx}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{k=-N}^{N} e^{ik(x-t)}\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, D_N(x-t)\, dt$$

$$= \frac{1}{2\pi} \int_{x+\pi}^{x-\pi} f(x-u)\, D_N(u)\, (-du)$$

($f(x-u) D_N(u)$ has period 2π, so we may replace x by 0 in limits)

$$= \frac{1}{2\pi} \int_{\pi}^{-\pi} f(x-u)\, D_N(u)\, (-du)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)\, D_N(t)\, dt$$

Then

$$\sigma_n(x) = \frac{1}{n+1} \sum_{N=0}^{n} S_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) \left[\frac{1}{n+1} \sum_{N=0}^{n} D_N(t) \right] dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)\, K_n(t)\, dt$$

Because $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(t)\, dt = 1$, we have

$$\sigma_n(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[f(x-t) - f(t) \right] K_n(t)\, dt$$

and so

$$| \sigma_n(x) - f(x) | \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} | f(x-t) - f(t) |\, K_n(t)\, dt$$

(since $K_n(t) \geq 0$!) Now f is continuous, and so uniformly continuous. Therefore $\exists\, M > 0$ such that

$$|f(y)| < M \quad \forall y \in [-\pi, \pi]$$

and, given $\varepsilon > 0$, $\exists\, 0 < \delta < \pi$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon/2$$

For $\delta \leq |t| \leq \pi$, we have

$$K_n(t) \leq \frac{2}{n+1} \frac{1}{1-\cos\delta}$$

and so we can find $L \in \mathbb{N}$ such that $\forall n \geq L$

$$\delta \leq |t| \leq \pi \Rightarrow K_n(t) \leq \frac{\varepsilon}{4M}$$

Therefore, $\forall n \geq L$

$$|\sigma_n(x) - f(x)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-t) - f(x)| K_n(t)\, dt$$

$$= \frac{1}{2\pi} \left(\int_{-\pi}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\pi} \right) |f(x-t) - f(x)| K_n(t)\, dt$$

$$\leq \frac{1}{2\pi} \left(2M \cdot \frac{\varepsilon}{4M} \cdot 2\pi + 2\pi \cdot \frac{\varepsilon}{2} \right) = \varepsilon$$

$\blacksquare$

Note that we must use $K_n(x)$ instead of $D_m(x)$ since

$$|S_N(x) - f(x)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-t) - f(x)|\, |D_N(t)|\, dt$$

and

$$\int_{-\pi}^{\pi} |D_N(t)|\, dt > c \log N$$

so we can not get a good estimate on $|S_N(x) - f(x)|$.

RIESZ-FISCHER THEOREM: Let H be a Hilbert Space and $(u_\alpha : \alpha \in A)$ an orthonormal family. Given $\varphi \in \ell^2(A)$, there exists $x \in H$ such that $\hat{x} = \varphi$ (p101)

PROPOSITION: If $(c_n : n \in \mathbb{Z})$ satisfies

$$\sum_{n\in\mathbb{Z}} |c_n|^2 < \infty$$

then there is an $f \in L^2[-\pi,\pi]$ such that $\forall n \in \mathbb{Z}$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int}\,dt$$

Recall that $S_n \longrightarrow f$ in the L^2-norm (for $f \in C(\mathbb{T})$) and so there is some subsequence S_{n_k} which converges to f a.e.

QUESTION: If $f \in C(\mathbb{T})$, does $S_n(x,f) \to f(x)$ for every $x \in [-\pi,\pi]$?

Define $\Lambda_n : C(\mathbb{T}) \longrightarrow \mathbb{C}$ by

$$\Lambda_n(f) := S_n(0,f)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, D_n(t)\,dt$$

By Hölder's inequality

$$|\Lambda_n f| \leq \|f\|_\infty \|D_n\|_1$$

and so $\|\Lambda_n\| \leq \|D_n\|_1$.

Define for each $n \in \mathbb{N}$

$$g_n(t) := \begin{cases} +1 & \text{if } D_n(t) \geq 0 \\ -1 & \text{if } D_n(t) < 0 \end{cases}$$

Then g_n is a step function, and we can find a sequence $(f_j) \subset C(T)$ with $\|f_j\|_\infty = 1$ and

$$\lim_{j\to\infty} f_j(t) = g_n(t) \quad \forall t \in [-\pi,\pi] \text{ a.e.}$$

By the Dominated Convergence theorem

$$\lim \Lambda_n f_j = \lim \frac{1}{2\pi}\int_{-\pi}^{\pi} f_j(t)\, D_n(t)\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(t)\, D_n(t)\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\, dt$$

$$= \|D_n\|_1$$

Because $\|f_j\| = 1$ $\forall j \in \mathbb{N}$, we have $\|\Lambda_n\| \geq \|D_n\|_1$. Therefore $\|\Lambda_n\| = \|D_n\|_1$ $\forall n \in \mathbb{N}$.

Claim: $\|D_n\|_1 \longrightarrow \infty$

$$\|D_n\|_1 = \frac{1}{\pi}\int_0^{\pi} \frac{|\sin(n+1/2)t|}{\sin t/2}\,dt$$

$$\geq \frac{2}{\pi}\int_0^{\pi} \frac{|\sin(n+1/2)t|}{t}\,dt \qquad [\![\sin x \leq x \ \ \forall x \geq 0]\!]$$

$$= \frac{2}{\pi}\int_0^{(n+1/2)\pi} \frac{|\sin u|}{u}\,du$$

$$\geq \frac{2}{\pi}\sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi} \frac{|\sin u|}{u}\,du$$

$$\geq \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi} |\sin u|\,du$$

$$= \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k} \longrightarrow \infty \quad \text{as } n\to\infty$$

Hence $\|\Lambda_n\| \to \infty$ as $n \to \infty$

[[UNIFORM BOUNDEDNESS THEOREM: Suppose X is a Banach space, and Y is a normed linear space. Suppose $\{\Lambda_\alpha : \alpha\in A\} \subset \mathcal{B}(X,Y)$. Then one of the following alternatives must occur:

(1) $\exists M>0$ s.t. $\|\Lambda_\alpha\| \leq M \ \ \forall \alpha\in A$

(2) $\sup_{\alpha\in A} \|\Lambda_\alpha x\| = \infty$ for a dense G_δ-set in X

(p114)]]

Since $\|\Lambda_n\| \to \infty$, the Uniform Boundedness principle says there is a dense G_δ-set $E \subset C(T)$ such that

$$\sup_{n\in\mathbb{N}} |S_n(f,0)| = +\infty \quad \forall f \in E$$

and so $S_n(f,0)$ does not converge.

There is nothing special about 0. For every $x \in [-\pi,\pi]$ there exists a dense G_δ-set $E_x \subset C(T)$ such that

$$\sup_{n\in\mathbb{N}} |S_n(f,x)| = \infty \quad \forall f \in E_x$$

Remarks: ① Note RHS is a measure absolutely continuous w.r.t. μ
- use Dominated Convergence Theorem -

DEFINITION: X set, $\mathcal{M} \subset \mathcal{P}(X)$. We say $\mathcal{M}$ is a σ-algebra if

i) $X \in \mathcal{M}$

ii) $A \in \mathcal{M} \Rightarrow X \setminus A \in \mathcal{M}$

iii) $(A_n) \subset \mathcal{M} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$

Remarks: a) $\phi \in \mathcal{M}$

b) $(A_n) \in \mathcal{M} \Rightarrow \bigcup_{n=1}^{N} A_n \in \mathcal{M} \quad \forall N \in \mathbb{N}$

c) finite and countable intersections are in $\mathcal{M}$

d) $A \in \mathcal{M}, B \in \mathcal{M} \Rightarrow A \setminus B = A \cap X \setminus B \in \mathcal{M}$

DEFINITION: X set, $\mathcal{M}$ a σ-algebra of subsets of X. We say $(X, \mathcal{M})$ is a measurable space (or X if $\mathcal{M}$ is understood)

DEFINITION: Suppose $f: X \longrightarrow Y$, where X is a measurable space, Y a topological space. We say f is measurable if $f^{-1}(V) \in \mathcal{M}$ for all open V in Y.

PROPOSITION: Suppose $X \xrightarrow[\text{meas.}]{f} Y \xrightarrow[\text{cont.}]{g} Z$. Then $g \circ f$ is measurable

Proof. If V is open in Z,

$$(g \circ f)^{-1}(V) = f^{-1}(\underbrace{g^{-1}(V)}_{\text{open in } Y}) \in \mathcal{M}$$

▨

PROPOSITION: Suppose X is a measurable space and $u : X \to \mathbb{R}$, $v : X \to \mathbb{R}$ are measurable. Suppose $\Phi : \mathbb{R}^2 \to Y$ (top. space) is continuous. Let

$$f(x) := \Phi(u(x), v(x))$$

Then $f : X \to Y$ is measurable.

Proof. Let $h : X \to \mathbb{R}^2$ be $h(x) = (u(x), v(x))$. So $f = \Phi \circ h$. Last proposition $\Rightarrow$ sufficient to show h measurable. Let I_a be open interval in $\mathbb{R}$, I_b open interval in $\mathbb{R}$.
$S = I_a \times I_b$

$$h^{-1}(S) = u^{-1}(I_a) \cap v^{-1}(I_b) \in \mathcal{M}$$

If $V \subset \mathbb{R}^2$ is open, then $V = \bigcup S_i$ (countable union) where S_i is a rectangle. Then

$$h^{-1}(V) = \bigcup h^{-1}(S_i) \in \mathcal{M}$$

so h is measurable. ▨

PROPOSITION: X measurable space.

a) Suppose $f : X \to \mathbb{C}$, $f(x) = u(x) + i\,v(x)$ where u and v are measurable real-valued functions. Then f is measurable.

Proof. $\Phi(s,t) := s + it$ $\quad \Phi : \mathbb{R}^2 \to \mathbb{C}$ continuous. Then $f = \Phi \circ (u, v)$, so is measurable

b) Suppose $f(x) = u(x) + i\,v(x)$ is measurable. Then $u(x)$, $v(x)$, and $|f(x)|$ are measurable.

Proof. $u(x) = \mathrm{Re}\,(f(x))$ is composition of a measurable function followed by a continuous function. Similarly for others.

c) If $f : X \to \mathbb{C}$ and $g : X \to \mathbb{C}$ are measurable, then $f+g$ and fg are measurable.

Proof: Case I. f, g are real-valued. Let $\Phi(s,t) = s+t$ or st. Previous proposition $\Rightarrow \Phi(f, g)$ is measurable.

Case II: $f = u_1 + i v_1$, $g = u_2 + i v_2$. Case I $\Rightarrow$ $u_1 + u_2$ measurable and $v_1 + v_2$ measurable. a) $\Rightarrow$ $f+g$ measurable
Also $u_1 u_2$, $v_1 v_2$, $u_1 v_2$, $u_2 v_1$ measurable $\Rightarrow$
$fg = (u_1 u_2 - v_1 v_2) + i(u_1 v_2 + u_2 v_1)$ measurable

d) If $E \in \mathfrak{M}$, then χ_E is measurable

Proof: $\chi_E^{-1}(V)$ is either $\emptyset$, X, E, or $X \setminus E$ (all measurable)

e) $f : X \to \mathbb{C}$ measurable. Then $\exists\, \alpha : X \to \mathbb{C}$ α measurable and $|\alpha(x)| = 1$ $\forall x \in X$ s.t.

$$f(x) = \alpha(x)\,|f(x)| \quad \forall x \in X$$

Proof. Let $E = \{x \in X : f(x) = 0\} = f^{-1}(\underbrace{\mathbb{C} \setminus \{0\}}_{\text{open}}) \in \mathfrak{M}$

Let $Y = \mathbb{C} \setminus \{0\}$. Define $\varphi : Y \to$ unit circle by

$$\varphi(z) = \frac{z}{|z|}$$

φ is continuous on Y. $\forall x \in X$

$$f(x) + \chi_E(x) \in Y$$

Now $f + \chi_E : X \to Y$ is measurable. Hence

$$\alpha := \varphi \circ (f + \chi_E) : X \to \text{unit circle}$$

is measurable. Suppose $f(x) = 0$. Nothing to check. If $f(x) \neq 0$, $\chi_E(x) = 0$, so

$$\alpha(x) = \varphi(f(x)) = \frac{f(x)}{|f(x)|} \qquad \blacksquare$$

1/25 MEASURE THEORY

PROPOSITION: X set, $\mathcal{F} \subset \mathcal{P}(X)$, then there is a smallest σ-algebra of subsets containing $\mathcal{F}$

Proof. Let

$$\mathfrak{M}^* := \bigcap_{\mathfrak{M} \in \Omega} \mathfrak{M}$$

where Ω is the collection of all σ-algebras of subsets of X containing $\mathcal{F}$. Clearly $\mathcal{F} \subset \mathfrak{M}^*$. It is easy to check that $\mathfrak{M}^*$ is a σ-algebra.

DEFINITION: If X is a topological space, let $\mathcal{F}$ be the collection of open subsets of X. The smallest σ-algebra containing $\mathcal{F}$ is the collection $\mathcal{B}$ of Borel sets.

DEFINITION: $f : X^{\text{Top}} \longrightarrow Y^{\text{Top}}$ is Borel measurable (Borel function) if f is measurable w.r.t. $\mathcal{B}$

PROPOSITION: X set, $\mathfrak{M}$ σ-algebra in X, $f: X \to Y$ (top.)

a) $\{E \subset Y : f^{-1}(E) \in \mathfrak{M}\}$ is a σ-algebra in Y.

b) If f is measurable (w.r.t. $\mathfrak{M}$) then $f^{-1}(B) \in \mathfrak{M}$ for every Borel set B in Y.

Proof of b). f measurable $\Rightarrow f^{-1}(V) \in \mathfrak{M}$ $\forall$ open V in Y
Thus $\{E : f^{-1}(E) \in \mathfrak{M}\}$ is a σ-algebra (by (a)) containing the open sets of Y, and hence contains the Borel sets of Y. ▨

c) If $Y = \mathbb{R}_e$, then if $\forall \alpha \in \mathbb{R}$, $f^{-1}(\alpha, +\infty] \in \mathfrak{M}$, we have that f is measurable.

Proof of c). $\forall \alpha \in \mathbb{R}$, $f^{-1}[\alpha, +\infty] \in \mathfrak{M} \Rightarrow f^{-1}([-\infty, \alpha)) \in \mathfrak{M}$
$\Rightarrow f^{-1}(a,b) \in \mathfrak{M}$ $\forall a<b \Rightarrow f^{-1}(V) \in \mathfrak{M}$ $\forall$ open V ▨

d) $\underset{\text{measu.}}{X} \xrightarrow{f} \underset{\text{top.}}{Y} \xrightarrow{g} \underset{\text{top.}}{Z}$. If g is Borel measurable and f is measurable then $g \circ f : X \to Z$ is measurable.

Proof of d) V open in Z.

$$(g \circ f)^{-1}(V) = f^{-1}(\underbrace{g^{-1}(V)}_{\text{Borel set}}) \in \mathfrak{M}$$

▨

NOTE: d) is not true if we assume g is Lebesgue measurable. Recall example from 441.

Suppose $f_n : X \to \mathbb{R}_e$, X measurable space. If each f_n is measurable, then $\sup f_n$ is measurable:

$$\{x \in X : \sup f_n(x) > \alpha\} = \bigcup_n \{x \in X : f_n(x) > \alpha\}$$

Therefore $\overline{\lim}\, f_n$, $\underline{\lim}\, f_n$, and $\lim f_n$ (if it exists) are all measurable if each f_n is.

DEFINITION: $f : X \to \mathbb{R}_e$

$$f^+ := \max(f, 0)$$
$$f^{-1} := \max(-f, 0)$$

Note f^+ and f^- are measurable if f is. Certainly

$$f = f^+ - f^-$$
$$|f| = f^+ + f^-$$

Remark: If $f = g - h$, where $g \geq 0$ and $h \geq 0$, then $g \geq f^+$ and $h \geq f^-$, since

$$\left.\begin{array}{c} g \geq 0 \\ g = f + h \geq f \end{array}\right\} \Rightarrow g \geq f^+$$

and $h = g - f \geq f^+ - f = f^-$.

DEFINITION: X measurable. $s : X \to [0, \infty)$ is simple if the range of s is a finite set.

A simple function s has a canonical representation

$$s = \sum_{\alpha \in F} \alpha \chi_{A_\alpha}$$

where F is a finite set and $A_\alpha \cap A_\beta = \emptyset$ $\forall \alpha \neq \beta$.

PROPOSITION: X measurable space. $f : X \longrightarrow [0, \infty]$ measurable. Then $\exists$ simple functions s_n s.t.

(i) $0 \leq s_n \leq s_{n+1} \leq f$

(ii) $s_n(x) \longrightarrow f(x)$ $\forall x \in X$

(iii) s_n measurable

Proof. For $n \in \mathbb{N}^*$, let

$$E_{n,i} := \left\{ x \in X : \frac{i-1}{2^n} \leqslant f(x) < \frac{i}{2^n} \right\} \quad 1 \leq i \leq n2^n$$

$$F_n := \{ x \in X : f(x) \geqslant n \}$$

Now set

$$s_n := \sum_{i=2}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}} + n \chi_{F_n} \qquad \blacksquare$$

DEFINITION: Suppose $(X, \mathcal{M})$ is a measurable space. A positive measure μ on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \longrightarrow [0, \infty]$ s.t. $\mu(A) < \infty$ for some $A \in \mathcal{M}$ and μ is countably additive. $(X, \mathcal{M}, \mu)$ is called a measure space.

Elementary consequences:

1. $\mu(\emptyset) = 0$.

Take A s.t. $\mu(A) < \infty$. Then $\mu(A) = \mu(A \cup \bigcup_{n=1}^{\infty} \emptyset) = \mu(A) + \sum_{n=1}^{\infty} \mu(\emptyset)$ and so $\mu(\emptyset) = 0$.

2. μ is finitely additive

3. If $A \subset B$, $A, B \in \mathcal{M}$, then $\mu(A) \leq \mu(B)$

4. If $(A_n) \subset \mathcal{M}$, $A_n \subset A_{n+1}$, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$

Set $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. Then the B_n's are disjoint elements of $\mathcal{M}$ and

$$\bigcup_{n=1}^{N} B_n = A_N$$

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

Hence $\mu(\bigcup_1^{\infty} A_n) = \mu(\bigcup_1^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(B_n)$

$$= \lim_{N\to\infty} \mu(\bigcup_{n=1}^{N} B_n) = \lim_{N\to\infty} \mu(A_N)$$

5. If $(A_n) \subset \mathcal{M}$, $A_{n+1} \subset A_n$, and $\mu(A_1) < \infty$, then $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$

1/27 MEASURE THEORY

$(X, \mathcal{M}, \mu)$ measure space

DEFINITION: Suppose $s: X \to [0,\infty)$ is a measurable simple function with canonical representation

$$s = \sum_{\alpha \in F}' \alpha \chi_{E_\alpha}$$

Suppose $E \in \mathcal{M}$, Then

$$\int_E s \, d\mu := \sum_{\alpha \in F}' \alpha \, \mu(E \cap E_\alpha)$$

(If $s = 0$, $\int_E s \, d\mu := 0$).

Proposition: Suppose $s_1 \le s_2$ are measurable simple functions on X. Then for $E \in \mathcal{M}$

$$\int_E s_1 \, d\mu \le \int_E s_2 \, d\mu$$

Proof. WLOG neither $s_j \equiv 0$. Suppose

$$s_1 = \sum_{i=1}^{N} \alpha_i \chi_{A_i}$$

$$s_2 = \sum_{j=1}^{M} \beta_j \chi_{B_j}$$

For $1 \le i \le N$, let $T_i = \{ j : 1 \le j \le M \text{ and } B_j \cap A_i \neq \emptyset \}$

Claim: $s_1 \le s_2 \Rightarrow$ (a) $A_i \subset \bigcup_{j \in T_i} B_j \qquad 1 \le i \le N$

(b) $\alpha_i \le \beta_j \quad \forall j \in T_i$

Therefore $A_i \cap E = \bigcup_{j \in T_i} (A_i \cap B_j \cap E)$, so

$$\mu(A_i \cap E) = \sum_{j \in T_i} \mu(A_i \cap B_j \cap E)$$

$$\Rightarrow \alpha_i \mu(A_i \cap E) \le \sum_{j \in T_i} \beta_j \mu(A_i \cap B_j \cap E)$$

$$\Rightarrow \sum_{i=1}^{N} \alpha_i \mu(A_i \cap E) \le \sum_{i=1}^{N} \sum_{j \in T_i} \beta_j \mu(A_i \cap B_j \cap E)$$

$$\le \sum_{j=1}^{M} \beta_j \sum_{i=1}^{N} \mu(A_i \cap B_j \cap E)$$

$$\le \sum_{j=1}^{M} \beta_j \mu(B_j \cap E) = \int_E s_2 \, d\mu$$

and so $\int_E s_1 \, d\mu \le \int_E s_2 \, d\mu$. $\boxtimes$

DEFINITION: Suppose $f: X \to [0,\infty]$ is measurable. Then if $E \in \mathcal{M}$

$$\int_E f \, d\mu := \sup \left\{ \int_E s \, d\mu : s \le f, \ s \text{ measurable simple} \right\}$$

Remarks: ① well-defined by the last proposition
② agrees with 441 definition of $\int f$
"ASIDE" ③ Take case where $\mathcal{M}$ = Lebesgue measure, $X = \mathbb{R}$. Recall if $E \subset \mathbb{R}$, $m_*(E) := \sup\{m(\tilde{E}) : \tilde{E} \subset E, \tilde{E} \in \mathcal{M}\}$. Then $m_*(E) = m^*(E)$ iff $E \in \mathcal{M}$. Take $E \subset [0,1]$. If $A \subset [0,1]$ and $B = [0,1] \setminus A$, then $m_*(A) = 1 - m^*(B)$. Suppose we deleted requirement "f measurable" from last definition. What would happen? Consider $A \subset [0,1]$, $A \notin \mathcal{M}$. Let $f = \chi_A$, $g = \chi_B$. Under our "new" definition, $\int_{[0,1]} f = m_*(A)$ and $\int_{[0,1]} g = m_*(B)$. But

$f + g = 1$ yet $\int f + \int g = m_*(A) + m_*(B) < m_*(A) + m^*(B) = 1 = \int (f+g)$
so $\int f + \int g < \int (f+g)$ THIS IS UNACCEPTABLE!

PROPERTIES: $f, g \geq 0$ on X, measurable
(1) $f \leq g \Rightarrow \int_E f\,d\mu \leq \int_E g\,d\mu \quad \forall E \in \mathcal{M}$

(2) $E \in \mathcal{M}$, $\int_E f\,d\mu = \int_X f\chi_E\,d\mu$.

Proof. Suppose $s \leq f\chi_E$, $s = \sum \alpha \chi_{E_\alpha}$. Note $E_\alpha \subset E$. Then

$$\int_X s\,d\mu = \sum \alpha\mu(E_\alpha) = \sum \alpha\mu(E_\alpha \cap E) = \int_E s\,d\mu$$

and so $\int_E f\,d\mu \geq \int_X f\chi_E\,d\mu$.

Suppose $t \leq f$, $t = \sum \beta \chi_{E_\beta}$. Let

$$t' = \sum \beta \chi_{E_\beta \cap E}$$

Then $t' \le f\chi_E$ and

$$\int_E t \, d\mu = \sum \beta \mu(E_\beta \cap E) = \int_X t' \, d\mu$$

and so $\displaystyle\int_E f \, d\mu \le \int_X f \chi_E \, d\mu$. ▨

(13) $A \subset B \Longrightarrow \displaystyle\int_A f \, d\mu \le \int_B f \, d\mu$
$A, B \in \mathcal{M}$

(14) If $f = 0$ on $E \in \mathcal{M}$, then $\displaystyle\int_E f \, d\mu = 0$

(15) $\mu(E) = 0 \Longrightarrow \displaystyle\int_E f \, d\mu = 0$

(16) $\displaystyle\int_E cf \, d\mu = c \int_E f \, d\mu$, $c \ge 0$.

PROPOSITION: Suppose s, t are simple, measurable on X. For $E \in \mathcal{M}$, let

$$\varphi(E) := \int_E s \, d\mu$$

Then φ is a measure on $\mathcal{M}$. Also

$$\int_X (s+t) \, d\mu = \int_X s \, d\mu + \int_X t \, d\mu$$

Proof: Suppose $E_n \in \mathcal{M}$, E_n disjoint. Let

$$E = \bigcup_{n=1}^{\infty} E_n$$

Suppose $s = \sum_{j=1}^{M} \beta_j \chi_{B_j}$. Then

$$\varphi(E) = \sum_{j=1}^{M} \beta_j \mu(B_j \cap E) = \sum_{j=1}^{M} \beta_j \sum_{n=1}^{\infty} \mu(E_n \cap B_j)$$

$$= \sum_{n=1}^{\infty} \sum_{j=1}^{M} \beta_j \mu(E_n \cap B_j) = \sum_{n=1}^{\infty} \int_{E_n} s \, d\mu$$

$$= \sum_{n=1}^{\infty} \varphi(E_n)$$

Notice that $\varphi(\emptyset) = 0$, so $\varphi \neq \infty$ identically.

Let $\beta_0 := 0$ and $B_0 := s^{-1}(0)$, so $X = \bigcup_0^M B_j$. Suppose

$$t = \sum_{i=1}^{N} \alpha_i \chi_{A_i}$$

and $\alpha_0 := 0$, $A_0 := t^{-1}(0)$, so $X = \bigcup_0^N A_i$.

For $0 \leq i \leq N$, $0 \leq j \leq M$, let

$$E_{ij} = A_i \cap B_j$$

E_{ij} disjoint with union X. Then

$$\int_{E_{ij}} (s+t) \, d\mu = \int_{E_{ij}} s \, d\mu + \int_{E_{ij}} t \, d\mu$$

(all functions constant on E_{ij}). Now add over all i, j.

$$\sum_{j=0}^{M} \sum_{l=0}^{N} \int_{E_{ij}} s \, d\mu = \int_X s \, d\mu \quad \text{by } 1^{st} \text{ part}$$

Similarly for other parts. ▨

MONOTONE CONVERGENCE THEOREM: Given $(X, \mathfrak{M}, \mu)$. Suppose $f_n : X \to [0, \infty]$ are measurable and $f_n \leq f_{n+1}$. If $f := \lim f_n$, then f is measurable and

$$\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu$$

Proof. f is measurable since each f_n is measurable. By previous proposition

$$\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu$$

so $\lim \int_X f_n \, d\mu$ exists. If $\alpha = \lim \int_X f_n \, d\mu$, then $\alpha \leq \int_X f \, d\mu$

since $\int f_n \, d\mu \leq \int f \, d\mu \ \forall n \in \mathbb{N}$. Suppose $s \leq f$, s simple and measurable. Take $0 < c < 1$, and let

$$E_n = \{ x \in X : f_n(x) > c\, s(x) \}$$

Then $E_n \subset E_{n+1}$ and each E_n is measurable. Also

$$\bigcup_{n=1}^{\infty} E_n = X$$

1/30 MEASURE THEORY

(PROOF OF MCT, continued)

Let s be a simple measurable function, $s \le f$. For $n \in \mathbb{N}$ let

$$E_n := \{ x \in X : f_n(x) \geq c\, s(x) \}$$

where $0 < c < 1$. Then each E_n is measurable and $E_n \subset E_{n+1}$.
Claim: $X = \bigcup E_n$. Suppose $f(x) = 0$. Then $x \in E_1$ since $s(x) = 0$.
Suppose $f(x) > 0$. Then $f(x) \geq s(x) > c\, s(x)$. Hence $\exists\, n$ with $f_n(x) > c\, s(x)$, so $x \in E_n$.

For any n,

$$(*) \qquad \alpha \geq \int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq c \int_{E_n} s \, d\mu = c\varphi(E_n)$$

[RECALL: $\varphi(E) := \int_E s \, d\mu$, $E \in \mathfrak{M}$, is a measure]. Now

$$\varphi(X) = \lim_{n \to \infty} \varphi(E_n)$$

and so from $(*)$ we have

$$\alpha \geq c\varphi(X) = c \int_X s \, d\mu$$

Let $c \uparrow 1$. Then $\alpha \geq \int_X s \, d\mu$. Hence $\alpha \geq \int_X f \, d\mu$ by definition ▨

COROLLARY: (FATOU'S LEMMA) Given $(X, \mathcal{M}, \mu)$ and $f_n : X \to [0,\infty]$ measurable. Then

$$\int_X \underline{\lim}\, f_n \, d\mu \leq \underline{\lim} \int_X f_n \, d\mu$$

Proof. Set $g_k := \inf_{n \geq k} f_n$. Then

a) g_k measurable
b) $g_k \leq g_{k+1}$
c) $g_k \uparrow \underline{\lim}\, f_n$

Then M.C.T. applied to (g_k) says

$$\int_X \underline{\lim}_{n\to\infty} f_n \, d\mu = \lim_{k\to\infty} \int g_k \, d\mu \leq \underline{\lim}_{n\to\infty} \int_X f_n \, d\mu$$

$\blacksquare$

(Aside: Fatou $\Rightarrow$ MCT.

Given $0 \leq f_n \leq f_{n+1} \cdots$, $f_n \uparrow f$. Then

(PRELIM)

$$\overline{\lim} \int f_n \, d\mu \underset{\substack{\uparrow \\ f_n \uparrow f}}{\leq} \int f \, d\mu \underset{\substack{\uparrow \\ \text{FATOU}}}{\leq} \underline{\lim} \int f_n \, d\mu$$

Hence $\lim \int f_n = \int f$.)

PROPOSITION: Given $(X, \mathfrak{M}, \mu)$, $f_n : X \to [0,\infty]$, measurable. Set $f = \sum_{n=1}^{\infty} f_n$. Then

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$$

Proof. Suppose $h_1, h_2 \geq 0$ measurable. We know $\exists\, s_n \uparrow h_1$, $t_n \uparrow h_2$, where s_n, t_n measurable simple functions. Then $(s_n + t_n) \uparrow (h_1 + h_2)$. We know

$$\forall n \in \mathbb{N} \qquad \int_X (s_n + t_n)\, d\mu = \int_X s_n \, d\mu + \int_X t_n \, d\mu$$

By MCT

$$\int_X (s_n + t_n)\, d\mu \to \int_X (h_1 + h_2)\, d\mu, \qquad \int_X s_n \, d\mu \to \int_X h_1 \, d\mu, \qquad \int_X t_n \, d\mu \to \int_X h_2 \, d\mu$$

Hence

$$\int_X (h_1 + h_2)\, d\mu = \int_X h_1 \, d\mu + \int_X h_2 \, d\mu$$

Set

$$g_N := \sum_{n=1}^{N} f_n$$

Then

$$\int g_N \, d\mu = \sum_{n=1}^{N} \int f_n \, d\mu$$

Since $g_N \uparrow f$, MCT $\Rightarrow$

$$\int f\,d\mu = \lim_N \int g_N\,d\mu = \lim_N \sum_{n=1}^{N} \int f_n\,d\mu$$
$$= \sum_{n=1}^{\infty} \int f_n\,d\mu$$

$\blacksquare$

<u>PROPOSITION</u>: Given $(X, \mathcal{M}, \mu)$. Let $f: X \to [0,\infty]$ be measurable. For $E \in \mathcal{M}$, define

$$\varphi(E) = \int_E f\,d\mu$$

Then φ is a measure on $\mathcal{M}$ and furthermore, if $g: X \to [0,\infty]$ is measurable

$$(*) \qquad \int_X g\,d\varphi = \int_X gf\,d\mu$$

Proof. Given $(E_n) \subset \mathcal{M}$ disjoint, let $E = \bigcup_{n=1}^{\infty} E_n$. Must show $\varphi(E) = \sum \varphi(E_n)$. Clearly,

$$f\chi_E = \sum_{n=1}^{\infty} f\chi_{E_n}$$

so by the last proposition

$$\int_E f\,d\mu = \int f\chi_E\,d\mu = \sum_{n=1}^{\infty} \int f\chi_{E_n}\,d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\,d\mu$$

$$\Rightarrow \varphi(E) = \sum_{n=1}^{\infty} \varphi(E_n)$$

First suppose $g = \chi_E$ for some $E \in \mathfrak{M}$.

$$\int_X g\,d\varphi = \int_X \chi_E\,d\varphi = \varphi(E) = \int_E f\,d\mu$$

$$= \int_X f\chi_E\,d\mu = \int_X fg\,d\mu$$

Therefore $(*)$ holds for any simple function. Given a general g, $\exists\, s_n \uparrow g$, s_n simple, measurable.

$$\int_X g\,d\varphi = \lim_n \int_X s_n\,d\varphi = \lim_n \int_X f s_n\,d\mu = \int_X fg\,d\mu$$

$\blacksquare$

DEFINITION: Given $(X, \mathfrak{M}, \mu)$, $f : X \longrightarrow \mathbb{C}$ measurable. Then $f \in L^1(\mu)$ if

$$\int_X |f|\,d\mu < \infty$$

DEFINITION: For $f \in L^1(\mu)$, with $f = u + iv$, define

$$\int_X f\,d\mu := \int_X u^+\,d\mu - \int_X u^-\,d\mu + i\left[\int_X v^+\,d\mu - \int_X v^-\,d\mu\right]$$

REMARK: $u^+ \le |u| \le |f|$ and u^+ is measurable. Similarly for others. $\int f\,d\mu \in \mathbb{C}$

<u>PROPOSITION</u>: If $f, g \in L^1(\mu)$, then $\alpha f + \beta g \in L^1(\mu)$ $\forall \alpha, \beta \in \mathbb{C}$ and

$$\int_X (\alpha f + \beta g)\,d\mu = \alpha \int_X f\,d\mu + \beta \int_X g\,d\mu$$

Proof. $\alpha f + \beta g$ is measurable.

$$|\alpha f + \beta g| \le |\alpha||f| + |\beta||g|$$

$$\Rightarrow \int |\alpha f + \beta g|\,d\mu < \infty$$

Show $\int (f+g) = \int f + \int g$. Sufficient to show for f, g real.

Let $h = f+g$. $f = f^+ - f^-$, $g = g^+ - g^-$, $h = h^+ - h^-$

Then

$$f^+ + g^+ - f^- - g^- = h^+ - h^-$$
$$f^+ + g^+ + h^- = h^+ + f^- + g^-$$

$$\int f^+ d\mu + \int g^+ d\mu + \int h^- d\mu = \int h^+ d\mu + \int f^- d\mu + \int g^- d\mu$$

$$\int f^+ d\mu - \int f^- d\mu + \int g^+ d\mu - \int g^- d\mu = \int h^+ d\mu - \int h^- d\mu$$

2/1 MEASURE THEORY

HOMEWORK: Chap 1 #1, 9, 12 Due Monday, Feb. 13. Look at 7, 8

PROPOSITION: $(X, \mathfrak{M}, \mu)$. Suppose $f \in L^1(\mu)$. Then

$$\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu$$

Proof. Set $z = \int_X f \, d\mu$. $\exists\, \alpha \in \mathbb{C}$ with $|\alpha| = 1$ s.t. $\alpha z = |z|$. Then if $u = \operatorname{Re} \alpha f$, $|u| \leq |f|$, and so

$$\left| \int_X f \, d\mu \right| = \alpha \int_X f \, d\mu = \int_X \alpha f \, d\mu = \int_X u \, d\mu$$

$$\leq \int_X |f| \, d\mu$$

because $\int \alpha f \, d\mu$ is real!

DOMINATED CONVERGENCE THEOREM: Given $(X, \mathfrak{M}, \mu)$.

f_n measurable

Suppose $f_n : X \to \mathbb{C}$ are such that $|f_n(x)| \leq g(x)$ $\forall x \in X$ $\forall n$ for some $g \in L^1(\mu)$. Suppose $f_n(x) \to f(x)$ $\forall x \in X$. Then $f \in L^1(\mu)$ and

$$\int_X f_n \, d\mu \to \int_X f \, d\mu$$

Moreover, $\int |f_n - f| \, d\mu \to 0$

Proof. Note $|f_n| \leq g$, f_n measurable, $g \in L^1(\mu) \Rightarrow f_n \in L^1(\mu)$
Also note that $|f(x)| \leq g(x)$. Thus $f \in L^1(\mu)$ and

$$2g - |f_n - f| \geq 0$$

and is measurable, so Fatou's lemma $\Rightarrow$

$$\int 2g \, d\mu \leq \underline{\lim} \int (2g - |f_n - f|) \, d\mu$$

Finite

$$= \int 2g - \overline{\lim} \int |f_n - f| \, d\mu$$

$$\Rightarrow \overline{\lim} \int |f_n - f| \, d\mu = 0$$

$$\Rightarrow \lim_{n \to \infty} \int |f_n - f| \, d\mu = 0$$

But

$$\left| \int_X f_n \, d\mu - \int_X f \, d\mu \right| = \left| \int (f_n - f) \, d\mu \right| \leq \int |f_n - f| \, d\mu \to 0$$

$\blacksquare$

Sets of Measure Zero

<u>Theorem</u>: Given $(X, \mathfrak{M}, \mu)$. Let $\mathfrak{M}^*$ be the collection of all sets $E \subset X$ s.t. $\exists\, A \subset E \subset B$ with $A, B \in \mathfrak{M}$ and $\mu(B \setminus A) = 0$. For $E \in \mathfrak{M}^*$, let $\mu(E) := \mu(A)$. Then $\mathfrak{M}^*$ is a σ-algebra

containing $\mathcal{M}$ and μ (extended to $\mathcal{M}^*$) is a measure on $\mathcal{M}^*$.

<u>DEFINITION</u>: $\mathcal{M}^*$ is called the <u>μ-completion</u> of $\mathcal{M}$ and μ (on $\mathcal{M}^*$) is said to be <u>complete</u> (i.e. if $E \in \mathcal{M}^*$, $Y \subset E$, and $\mu(E) = 0$, then $Y \in \mathcal{M}^*$)

[[If $E \in \mathcal{M}$, $\mu(E) = 0$, $Y \subset E$, consider $\emptyset \subset Y \subset E$. Shows $Y \in \mathcal{M}^*$. If $E \notin \mathcal{M}$, $\exists\, B \in \mathcal{M}$, $E \subset B$, $\mu(B) = 0$ Consider $\emptyset \subset Y \subset B$. Shows $Y \in \mathcal{M}^*$]]

Proof of Theorem: $\mathcal{M}^*$ is a σ-algebra. $X \in \mathcal{M}^*$ since $\mathcal{M} \subset \mathcal{M}^*$. Suppose $E \in \mathcal{M}^*$. $\exists\, A \subset E \subset B$ with $A, B \in \mathcal{M}$, $\mu(B \setminus A) = 0$. Then $X \setminus B \subset X \setminus E \subset X \setminus A$ and $X \setminus B, X \setminus A \in \mathcal{M}$, $\mu((X \setminus B) \setminus (X \setminus A)) = \mu(B \setminus A) = 0$. Suppose $(E_n) \in \mathcal{M}^*$. $\exists\, A_n \subset E_n \subset B_n$, $A_n, B_n \in \mathcal{M}$, $\mu(B_n - A_n) = 0$. Then

$$\underbrace{\bigcup^{\infty} A_n}_{\in \mathcal{M}} \subset \bigcup E_n \subset \underbrace{\bigcup^{\infty} B_n}_{\in \mathcal{M}}$$

and

$$\mu\left(\bigcup B_n - \bigcup A_n\right) \leq \mu\left(\bigcup (B_n - A_n)\right) \leq \sum \mu(B_n - A_n) = 0$$

Hence $\bigcup E_n \in \mathcal{M}^*$.

NOTE: $\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$.

Next show μ is well-defined on $\mathfrak{M}^*$. Suppose $E \in \mathfrak{M}^*$ and

$$A_1 \subset E \subset B_1 \qquad \mu(B_1 - A_1) = 0$$
$$A_2 \subset E \subset B_2 \qquad \mu(B_2 - A_2) = 0$$

Must show $\mu(A_1) = \mu(A_2)$.

$$A_1 - A_2 \subset E_1 - A_2 \subset B_2 - A_2$$

$$\Rightarrow \mu(A_1 - A_2) = 0$$

Then $\mu(A_1) = \mu(A_1 - A_2) + \mu(A_1 \cap A_2) = \mu(A_1 \cap A_2) \underset{\uparrow \text{ symmetry}}{=} \mu(A_2)$

Left to show μ is countably additive on $\mathfrak{M}^*$. Suppose $E_n \in \mathfrak{M}^*$, E_n disjoint

$$A_n \subset E_n \subset B_n \qquad A_n, B_n \in \mathfrak{M} \, ; \, \mu(B_n - A_n) = 0$$

$$\mu(\cup E_n) = \mu(\cup A_n) = \sum \mu(A_n) = \sum \mu(E_n)$$

$\blacksquare$

<u>Observations</u>: If $f: X \to \mathbb{C}$, $g: X \to \mathbb{C}$ and $f, g \in L^1(\mu)$ and $f = g$ a.e. $[\mu]$, then

$$\int_X f \, d\mu = \int_X g \, d\mu$$

Proof. Show $\int_X (f-g)\,d\mu = 0$.

Let $\mathrm{Re}(f-g) = u$. Then $u^+ = 0$ a.e., so $\int_X u^+ d\mu = 0$. Similarly for others. $\blacksquare$

Suppose μ is complete, $S \subset X$ with $\mu(X-S) = 0$. Then $f : X \to Y$ (top. space) is measurable iff $\forall$ open $V \subset Y$, $f^{-1}(V) \cap S \in \mathcal{M}$, since

$$f^{-1}(V) = (f^{-1}(V) \cap S) \cup (\underbrace{f^{-1}(V) \cap (X-S)}_{\in \mathcal{M} \text{ since subset of set of measure } 0})$$

$(X, \mathcal{M}, \mu)$ complete

PROPOSITION: Suppose f_n are complex-valued measurable functions defined a.e. on X. Suppose

$$\sum_{n=1}^{\infty} \int_X |f_n|\, d\mu < \infty$$

Then $\sum f_n$ converges a.e. on X to some $f \in L^1(\mu)$ and

$$\int_X f\, d\mu = \sum_{n=1}^{\infty} \int f_n\, d\mu$$

Proof: Let $S_n \subset X$ be domain of f_n. So $\mu(X - S_n) = 0$
Let $S = \bigcap S_n$. Then $\mu(X-S) = 0$. Let

$$\varphi(x) := \sum_{n=1}^{\infty} |f_n(x)| \quad \forall x \in S$$

Corollary of MCT $\Rightarrow$

$$\rightarrow \int_S \varphi \, d\mu = \sum \int_S |f_n| \, d\mu = \sum \int_X |f_n| \, d\mu < \infty$$

so $\varphi \in L^1$

Definition of $\int \varphi \, d\mu \Rightarrow \varphi < +\infty$ a.e. on S. Hence $\sum f_n(x)$ converges a.e. on S to $f(x)$. Certainly

$$|f| \leq \varphi(x) \in L^1(\mu)$$

and so $f \in L^1(\mu)$. Let $g_N = \sum_{n=1}^{N} f_n$. On S, $|g_n| \leq \varphi$

DCT $\Rightarrow$

$$\int_S f \, d\mu = \int_S \sum^{\infty} f_n = \int_S \lim g_N \, d\mu = \lim \int_S g_N \, d\mu$$
$$= \lim \sum^{N} \int f_n$$
$$= \sum_1^{\infty} \int_S f_n \, d\mu$$

OR

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$$

2/3 MEASURE THEORY

Suppose $(X, \mathcal{M}, \mu)$ is a measure space with completion $(X, \mathcal{M}^*, \mu^*)$. Suppose $f: X \to [0,\infty]$ is $\mathcal{M}$-measurable. Then f is also $\mathcal{M}^*$-measurable.

QUESTION: Is

$$\int_X f \, d\mu = \int_X f \, d\mu^* \; ?$$

ANSWER - Yes

Suppose $s \le f$, s simple and $\mathcal{M}$-measurable. Then s is also $\mathcal{M}^*$-measurable. Also

$$\int_X s \, d\mu = \int_X s \, d\mu^*$$

Hence $\int_X f \, d\mu \le \int_X f \, d\mu^*$

Now suppose $\tilde{s}$ is simple and $\mathcal{M}^*$-measurable. Say

$$\tilde{s} = \sum_{i=1}^{N} \alpha_i \chi_{E_i}$$

where $\alpha_i > 0$ and $E_i \in \mathcal{M}^*$. But $\exists A_i \in \mathcal{M}$, $A_i \subset E_i$ and $\mu(A_i) = \mu^*(E_i)$. Let

$$s_1 = \sum \alpha_i \chi_{A_i}$$

Note that $s_1 \leq \tilde{s} \leq f$ and s_1 is $\mathfrak{M}$-measurable. Moreover,

$$\int_X s_1 \, d\mu = \int_X \tilde{s} \, d\mu^*$$

Hence $\int_X f \, d\mu^* \leq \int_X f \, d\mu$.

PROPOSITION: (1) $f : X \to [0,\infty]$, f measurable. If $E \in \mathfrak{M}$ such that

$$(*) \qquad \int_E f \, d\mu = 0$$

Then $f = 0$ μ-a.e. on E.

Proof. Let $\Delta_n = \{x \in E : f(x) > 1/n\}$ $\forall n \in \mathbb{N}$. Then $\mu(\Delta_n) = 0$ by $(*)$, ~~so~~: $\mu\{x \in E : f(x) \neq 0\} = \mu(\bigcup \Delta_n) = 0$

(2) Suppose $f \in L^1(\mu)$, $f : X \to \mathbb{C}$. Suppose

$$\int_E f \, d\mu = 0 \quad \forall E \in \mathfrak{M}$$

Then $f = 0$ a.e. on X.

Proof. Write $f = u + iv$. Let

$$E = \{x \in X : u(x) > 0\} \in \mathcal{M}$$

Then $\int_E f\,d\mu = 0 \Rightarrow \int_E u\,d\mu = 0 \Rightarrow \int_E u^+\,d\mu = 0$

$\Rightarrow u^+ = 0$ a.e. on $E \Rightarrow u^+ = 0$ a.e. on X, etc.

(3) Suppose $f \in L^1(\mu)$ and

$$\left| \int_X f\,d\mu \right| = \int_X |f|\,d\mu$$

Then $\exists\, \alpha \in \mathbb{C}$, $|\alpha| = 1$ s.t. $\alpha f = |f|$ a.e.

Proof. Set $z = \int_X f\,d\mu$. $\exists\, \alpha$, $|\alpha| = 1$, with $\alpha z = |z|$.

$$\left| \int_X f\,d\mu \right| = \alpha \int_X f\,d\mu = \int_X \alpha f\,d\mu = \int_X u\,d\mu \quad [u = \operatorname{Re} \alpha f] \quad u \le |f|$$

$$\le \int_X |f|\,d\mu$$

equality holds here by assumption

Hence $\int_X (|f| - u)\,d\mu = 0$ and $|f| - u \ge 0$ on X. By (1)

$|f| = u = \operatorname{Re} \alpha f$ a.e. on X.

$$|\alpha f| = \operatorname{Re} \alpha f \Rightarrow |f| = \alpha f \qquad \blacksquare$$

Review of Topology

X : topological space

Definition: $f : X \to [-\infty, \infty]$ is $\begin{Bmatrix} \text{upper semi-continuous} \\ \text{lower semi-continuous} \end{Bmatrix}$

if for every $\alpha \in \mathbb{R}$ $\left\{ x \in X : \begin{matrix} f(x) < \alpha \\ f(x) > \alpha \end{matrix} \right\}$ is open

Observations: (1) f is continuous iff f is both usc and lsc.

(2) The $\begin{Bmatrix} \inf \\ \sup \end{Bmatrix}$ of any family of $\begin{Bmatrix} \text{usc} \\ \text{lsc} \end{Bmatrix}$ functions is $\begin{Bmatrix} \text{usc} \\ \text{lsc} \end{Bmatrix}$

(3) χ_A is $\begin{Bmatrix} \text{usc} \\ \text{lsc} \end{Bmatrix}$ if A is $\begin{Bmatrix} \text{closed} \\ \text{open} \end{Bmatrix}$

Definition: $f : X \to \mathbb{C}$. The support of f is the closure of $\{x \in X : f(x) \neq 0\}$

Notation: If X is top. space, $C_c(X)$ denotes the collection of all complex-valued continuous functions on X with compact support.

$C_c(X)$ is a vector space.

We write $K \prec f$ to mean

(1) K compact set in $\mathbb{C}$
(2) $f : X \longrightarrow [0,1]$, $f \in C_c(X)$
(3) $f(x) = 1 \quad \forall x \in K$

We write $f \prec V$ to mean
(1) V open
(2) $f \in C_c(X)$
(3) $\text{supp} f \subset V$

<u>LEMMA</u>: X locally compact T_2-space. Suppose $K \subset U$ where K is compact and U is open. Then $\exists$ open V with compact closure s.t.

$$K \subset V \subset \overline{V} \subset U$$

Proof. Since K is compact and X is locally compact, $\exists$ G open, $G \supset K$ s.t. $\overline{G}$ is compact. Done if $X = U$. If $X \neq U$, consider closed set $C = X - U$. Consider $p \in C$. Since K is compact and X is T_2, $\exists$ open set $W_p \supset K$ s.t. $p \notin \overline{W_p}$. Consider the collection of closed sets

$$C \cap \overline{G} \cap \overline{W_p}$$

for $p \in C$. This collection has an empty intersection. Since $\overline{G}$ is compact, $\exists$ finite number of these sets with empty intersection. Suppose

$$(*) \qquad C \cap \overline{G} \cap \overline{W_{p_1}} \cap \dots \cap \overline{W_{p_M}} = \emptyset$$

Let $V := G \cap W_{p_1} \cap \dots \cap W_{p_M}$. Then V is open, $K \subset V$, $\overline{V} \subset U$ by $(*)$ since $C = X - U$, and $\overline{V}$ is compact since $\overline{V} \subset \underset{\text{compact}}{\overline{G}}$.

2/6 MEASURE THEORY

Recall that if X is a locally compact T_2 space and

$$\underset{\text{compact}}{K} \subset \underset{\text{open}}{U}$$

then $\exists$ open V with $\bar{V}$ compact and

$$K \subset V \subset \bar{V} \subset U$$

URYSOHN'S LEMMA: X loc. compact T_2 space. If compact $K \subset$ open U, then $\exists f : X \longrightarrow [0,1]$ s.t.

$$K \prec f \prec U$$

Proof. Let $r_0 = 0$, $r_1 = 1$, $(r_n)_{n=2}^{\infty}$ an enumeration of the rationals in $(0,1)$. $\exists$ open V_0 with $\overline{V_0}$ compact s.t.

$$K \subset V_0 \subset \overline{V_0} \subset U$$

Also, $\exists$ open V_1 with $\overline{V_1}$ compact s.t.

$$K \subset V_1 \subset \overline{V_1} \subset V_0 \subset \overline{V_0} \subset U$$

Suppose we have already defined V_{r_i} open with $\overline{V_{r_i}}$ compact for $0 \leq i \leq n$ and furthermore

$$r_i < r_j \Rightarrow \overline{V_{r_j}} \subset V_{r_i}$$

We specify $V_{r_{n+1}}$ as follows: Let r_i be the largest member of $\{r_0, \ldots, r_n\}$ s.t. $r_i < r_{n+1}$. Let r_j be the smallest member of $\{r_0, \ldots, r_n\}$ s.t. $r_j > r_{n+1}$. So

$$r_i < r_{n+1} < r_j$$

Hence $\overline{V_{r_j}} \subset V_{r_i}$. Let $V_{r_{n+1}}$ be open with $\overline{V_{r_{n+1}}}$ compact such that $\overline{V_{r_j}} \subset V_{r_{n+1}} \subset \overline{V_{r_{n+1}}} \subset V_{r_i}$

By induction we obtain a sequence of open sets V_r, $r \in \mathbb{Q} \cap [0,1]$ s.t. $\overline{V_r}$ is compact and

$$(*) \qquad r < s \Rightarrow \overline{V_s} \subset V_r$$

Define

$$f_r(x) := \begin{cases} r & \text{if } r \in V_r \\ 0 & \text{if } x \in X - V_r \end{cases}$$

Note f_r is lower semi-continuous. Let $f := \sup_r f_r$, so f is lsc.

Define

$$g_s(x) = \begin{cases} 1 & x \in \overline{V_s} \\ s & x \in X - \overline{V_s} \end{cases}$$

g_s is upper semi-continuous. Let

$$g := \inf_s g_s$$

Then g is usc.

$$K \subset V_r \quad \forall r \Rightarrow f(x) = 1 \quad \forall x \in K$$
$$\overline{V_r} \subset \overline{V_0} \subset U \quad \forall r \Rightarrow f(x) = 0 \quad \forall x \in X - \overline{V_0}$$

Hence $\operatorname{sup} f \subset \overline{V_0}$, a compact subset of U.

We must show f is continuous. It is sufficient to show $f = g$.

Suppose $f_r(x) > g_s(x)$ for some $x \in X$. Then $r > s$ and $x \in V_r$, $x \notin \overline{V_s}$. This contradicts construction of V_r (see (*)). Hence $\forall x \in X$ $f_r(x) \leq g_s(x) \Rightarrow f(x) \leq g(x)$. Suppose $f(x) < g(x)$. $\exists\, r,s \in \mathbb{Q} \cap [0,1]$ s.t.

$$f(x) < r < s < g(x)$$

$f_r(x) < r \Rightarrow x \notin V_r$ and $g_s(x) > s \Rightarrow x \in \overline{V_s}$. Again this contradicts (*). Hence $f = g$.

$\blacksquare$

COROLLARY: X loc. compact T_2 space, K compact $K \subset \bigcup_{i=1}^{N} V_i$, V_i open. Then $\exists\, h_i \prec V_i$ s.t. $\sum_{i=1}^{n} h_i = 1$ on K

Proof. For each $x \in K$, $\exists$ open W_x, $\overline{W_x}$ compact, $x \in W_x$ and $\overline{W_x} \subset$ some V_i. This gives open covering of K, so

$$K \subset \bigcup_{j=1}^{M} W_{x_j}$$

For $1 \leq i \leq N$, let $H_i = \bigcup \{ \overline{W_{x_j}} : \overline{W_{x_j}} \subset V_i \}$ (finite union)

H_i is compact and $K \subset \bigcup_{i=1}^{N} H_i$. Clearly $H_i \subset V_i$

By Urysohn's lemma, $\exists\, g_i$ s.t. $H_i \prec g_i \prec V_i$

Let

$$h_1 = g_1$$
$$h_2 = (1-g_1)g_2$$
$$\vdots$$
$$h_n = (1-g_1)(1-g_2)\cdots(1-g_{n-1})g_n$$

Trivially $\operatorname{supp} h_i \subset \operatorname{supp} g_i \subset V_i$.

CLAIM: $\sum_1^n h_i = 1 - \prod_1^n (1-g_i)$ $\left(\Rightarrow \sum h_i = 1 \text{ on } K \right)$

Proof. (By induction). Suppose $h_1 + \cdots + h_k = 1 - (1-g_1)\cdots(1-g_k)$

Add h_{k+1}:

$$\sum^{k+1} h_i = 1 - \prod^{k}(1-g_i) + g_{k+1}\prod_{i=1}^{k}(1-g_i)$$

$$= 1 - \left(\prod_{i=1}^{k}(1-g_i)\right)(1-g_{k+1})$$

$\boxtimes$

RIESZ REPRESENTATION THEOREM (weak version)

X lc. compact T_2 space. Suppose $\Lambda : C_c(X) \to \mathbb{C}$ is a positive linear functional. Then there is a σ-algebra $\mathfrak{M}$ of subsets of X and a unique positive measure μ on $\mathfrak{M}$ s.t.

(a) $\int_X f \, d\mu = \Lambda(f) \quad \forall f \in C_c(X)$

(b) $\mu(K) < \infty \quad K$ compact

(c) $\mu(E) = \inf \{ \mu(V) : V \text{ open}, V \supset E \} \quad \forall E \in \mathfrak{M}$

(d) $\mu(E) = \sup \{ \mu(K) : K \text{ compact}, K \subset E \} \quad \forall$ open E and $\forall E \in \mathfrak{M}$ with $\mu(E) < \infty$

(e) μ is complete

Proof of uniqueness: Suppose μ_1 and μ_2 are positive measures on $\mathfrak{M}$ which satisfies (a)-(e). By (c) and (d) it is sufficient to show $\mu_1(K) = \mu_2(K)$ $\forall$ compact $K \subset X$.

Given $\varepsilon > 0$. By (b) $\mu_2(K) < \infty$ and by (c) $\exists$ open $V \supset K$ s.t. $\mu_2(V) < \mu_2(K) + \varepsilon$. By Urysohn's lemma $\exists f$ s.t. $K \prec f \prec V$.

$$\mu_1(K) = \int_X \chi_K \, d\mu_1 \leq \int_X \chi_K f \, d\mu_1$$

$$\leq \int_X f \, d\mu_1 = \Lambda(f) = \int_X f \, d\mu_2$$

$$\leq \int_X \chi_V \, d\mu_2 = \mu_2(V) \leq \mu_2(K) + \varepsilon$$

Hence $\mu_1(K) \leq \mu_2(K)$. By symmetry $\mu_2(K) \leq \mu_1(K)$, so $\mu_1(K) = \mu_2(K)$.

2/8 MEASURE THEORY

Proof of Riesz Representation theorem (continued)

(K will always be compact, V always open)

Definition of $\mathfrak{M}$ and μ.

For V open, let $\mu(V) := \sup\{\Lambda f : f \prec V\}$. Note that μ is monotone, i.e. $V_1 \subset V_2 \Rightarrow \mu(V_1) \leq \mu(V_2)$. So for any $E \subset X$ let

$$\mu(E) := \inf\{\mu(V) : E \subset V\}$$

This is well-defined by monotonicity.

Let $\mathfrak{M}_F$ be the collection of all $E \subset X$ such that

1) $\mu(E) < \infty$
2) $\mu(E) = \sup\{\mu(K) : K \subset E\}$

Let $\mathfrak{M}$ be the collection of all $E \subset X$ s.t. $E \cap K \in \mathfrak{M}_F$ $\forall K$ (compact)

Observations: μ is monotone $(A \subset B \Rightarrow \mu(A) \leq \mu(B))$

This implies that μ is complete on $\mathfrak{M}$. Suppose $\mu(A) = 0$. Then clearly $A \in \mathfrak{M}_F$, and so $A \in \mathfrak{M}$.

If $f \leq g$, f, g real-valued in $C_c(X)$, then $\Lambda f \leq \Lambda g$.

STEP I : $E_i \subset X$, $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$

First show $\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$. Consider $f \prec V_1 \cup V_2$. Let $\operatorname{supp} f = K \subset V_1 \cup V_2$. By corollary to Urysohn's lemma, $\exists\, \underset{i=1,2}{g_i} \prec V$ and $g_1 + g_2 = 1$ on K.

CLAIM: $f = fg_1 + fg_2$. Trivial for $x \in K$. But off K $f = 0$ so all terms are 0.

$\operatorname{supp} fg_1 \subset \operatorname{supp} g_1 \subset V_1 \Rightarrow fg_1 \prec V_1$. Similarly $fg_2 \prec V_2$

Now $\Lambda f = \Lambda fg_1 + \Lambda fg_2 \leq \mu(V_1) + \mu(V_2)$, and so $\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$

In the general case there is nothing to prove if some E_i has $\mu(E_i) = \infty$.

If all the $\mu(E_i)$'s are finite, given $\varepsilon > 0$ $\forall n \in \mathbb{N}$ $\exists\, V_n$ s.t.

$$\mu(V_n) < \mu(E_n) + \frac{\varepsilon}{2^n}$$

Let $V := \bigcup_{n=1}^{\infty} V_n \supset E := \bigcup_{n=1}^{\infty} E_n$. Suppose $f \prec V$. Then $\operatorname{supp} f \subset \bigcup_{i=1}^{N} V_i$ for some $N \Rightarrow f \prec \bigcup_{i=1}^{N} V_i$

$$\Lambda f \leq \mu\left(\bigcup_{i=1}^{N} V_i\right) \leq \sum_{i=1}^{N} \mu(V_i)$$

$$\leq \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon$$

Hence $\mu(V) \leq \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon$. But $E \subset V$, so $\mu(E) \leq \mu(V)$

Hence

$$\mu(E) \le \sum_{n=1}^{\infty} \mu(E_n)$$

STEP II: If K is compact, $K \in \mathfrak{M}_F$ and

$$(*) \qquad \mu(K) = \inf\{\Lambda f : K \prec f\}$$

First note that $(*)$ implies that $K \in \mathfrak{M}_F$

Suppose $K \prec f$. Select $\alpha \in (0,1)$ and let

$$V_\alpha = \{x \in X : f(x) > \alpha\}$$

V_α is open and $K \subset V_\alpha$. Furthermore, if $g \prec V_\alpha$, then $\alpha g \le f$. For if $x \in V_\alpha$, then $f(x) > \alpha \ge \alpha g(x)$. If $x \notin V_\alpha$, then $g(x) = 0 \le f(x)$. Now $K \subset V_\alpha$, so $\mu(K) \le \mu(V_\alpha) = \sup\{\Lambda g : g \prec V_\alpha\} \le \frac{1}{\alpha}\Lambda f$

Let $\alpha \to 1$; then

$$\mu(K) \le \Lambda f$$

Hence $\mu(K) \le \inf\{\Lambda f : K \prec f\}$. In particular this shows that $\mu(K) < \infty$. Then if $\varepsilon > 0$, $\exists\, V \supset K$ s.t.

$$\mu(V) < \mu(K) + \varepsilon$$

By Urysohn's lemma $\exists f$ s.t. $K \prec f \prec V$. Then

$$\Lambda f \le \mu(V) < \mu(K) + \varepsilon$$

and so $\inf\{\Lambda f : K \prec f\} \leq \mu(K)$

STEP III: V open, $\mu(V) < \infty \Rightarrow V \in \mathfrak{M}_F$

Must show $\mu(V) = \sup\{\mu(K) : K \subset V\}$. Suppose $\beta < \mu(V)$. Then $\exists\, f \prec V$ s.t. $\Lambda f > \beta$. Let $K = \operatorname{supp} f$. Consider open $W \supset K$. Certainly $f \prec W$, so

$$\mu(W) \geq \Lambda f > \beta$$

Hence $\mu(K) \geq \beta$. Since $K \subset V$, and $\beta < \mu(V)$ is arbitrary

$$\sup\{\mu(K) : K \subset V\} \geq \mu(V)$$

But trivially $\mu(K) \leq \mu(V)$ if $K \subset V$, so $\sup\{\mu(K) : K \subset V\} \leq \mu(V)$.

2/13 MEASURE THEORY

(PROOF OF RIESZ REPRESENTATION - CONTINUED)

STEP IV. Suppose $(E_i) \subset \mathfrak{M}_F$, disjoint. Let $E = \bigcup E_i$. Then

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$$

and if $\mu(E) < \infty$, then $E \in \mathfrak{M}_F$

Proof. First suppose $K_1 \cap K_2 = \phi$. Show $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(E_2)$
Urysohn's lemma with $K = K_1$ and $V = X - K_2$ says that $\exists f$ s.t. $K_1 \prec f \prec X - K_2$. $K_1 \cup K_2$ is compact, and so if $\varepsilon > 0$, step II $\Rightarrow \exists g$ with $K_1 \cup K_2 \prec g$ and

$$\Lambda g < \mu(K_1 \cup K_2) + \varepsilon$$

Certainly $g = \underset{\underset{K_2}{\curlyvee}}{(1-f)g} + \underset{\underset{K_1}{\curlyvee}}{fg}$. Hence

$$\mu(K_1 \cup K_2) > \Lambda g - \varepsilon = \Lambda(1-f)g + \Lambda fg - \varepsilon$$

$$\geq \mu(K_2) + \mu(K_1) - \varepsilon$$

$$\Longrightarrow \mu(K_1) + \mu(K_2) \leq \mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2)$$

Therefore $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$

General case: By step I

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

$E_i \in \mathfrak{M}_F \Rightarrow \exists$ compact $H_i \subset E_i$ s.t. $\mu(H_i) > \mu(E_i) - \varepsilon/2^i$

$$\forall N \in \mathbb{N},\ \mu(E) \geq \mu\left(\bigcup_{i=1}^{N} H_i\right) = \sum_{i=1}^{N} \mu(H_i) \geq \sum_{i=1}^{N} \mu(E_i) - \varepsilon$$

$$\Rightarrow \mu(E) \geq \sum_{i=1}^{\infty} \mu(E_i)$$

Suppose $\mu(E) < \infty$. Take $\varepsilon > 0$. $\exists N$ s.t.

$$\sum_{i=1}^{N} \mu(E_i) > \mu(E) - \varepsilon$$

Let $K = \bigcup_{i=1}^{N} H_i$. K is compact, $K \subset E$, and

(as above)

$$\mu(K) = \sum_{i=1}^{N} \mu(H_i) > \mu(E) - 2\varepsilon$$

Hence $E \in \mathfrak{M}_F$

STEP V: Suppose $E \in \mathfrak{M}_F$, $\varepsilon > 0$. $\exists K \subset E \subset V$ s.t. $\mu(V - K) < \varepsilon$.

Proof. $\mu(E) < \infty \Rightarrow \exists V \supset E$ s.t. $\mu(V) < \mu(E) + \varepsilon/2$

$E \in \mathfrak{M}_F \Rightarrow \exists K \subset E$ s.t. $\mu(E) < \mu(K) + \varepsilon/2$. Then

$$\mu(V) - \varepsilon/2 < \mu(E) < \mu(K) + \varepsilon/2$$

Since $V-K \subset V$, $\mu(V) < \infty$ and $V-K$ is open, by III
$V-K \in \mathfrak{M}_F$, so

$$V = \underset{\in \mathfrak{M}_F}{K} \cup \underset{\in \mathfrak{M}_F}{(V-K)} \quad \text{(disjoint union)}$$

$$\Rightarrow \mu(V) = \mu(K) + \mu(V-K)$$

$$\Rightarrow \mu(V-K) = \mu(V) - \mu(K) < \varepsilon$$

STEP VI : $A, B \in \mathfrak{M}_F \Rightarrow A \cap B \in \mathfrak{M}_F,\ A \cup B \in \mathfrak{M}_F,\ A-B \in \mathfrak{M}_F$

Proof. $\text{V} \Rightarrow \exists\, K_1 \subset A_1 \subset V_1,\ \mu(V_1 - K_1) < \varepsilon$
$\exists\, K_2 \subset B_2 \subset V_2,\ \mu(V_2 - K_2) < \varepsilon$

$$A - B \subset V_1 - K_2 \subset (V_1 - K_1) \cup (K_1 - V_2) \cup (V_2 - K_2)$$

$$\Rightarrow \mu(A-B) \leq \mu(V_1 - K_1) + \mu(K_1 - V_2) + \mu(V_2 - K_2)$$

$$< \mu(K_1 - V_2) + 2\varepsilon$$

Note $K_1 - V_2$ is compact subset of $A-B$, and so $A - B \in \mathfrak{M}_F$

$$A \cup B = A \cup (B-A) \quad \text{disjoint}$$

Since $\mu(A\cup B) < \infty$ by I, we see by IV that $A \cup B \in \mathcal{M}_F$

$A\cap B = A - (A-B)$ difference of sets in $\mathcal{M}_F$, so by the 1st part of proof $A\cap B \in \mathcal{M}_F$

STEP VII: $\mathcal{M}$ is a σ-algebra containing the Borel sets

Proof. 1) Suppose $E \in \mathcal{M}$. Show $X - E \in \mathcal{M}$.

$$(X-E)\cap K = \underbrace{K}_{\in \mathcal{M}_F} - \underbrace{(K\cap E)}_{\in \mathcal{M}_F \text{ since } E\in\mathcal{M}} \in \mathcal{M}_F \quad (\text{by VI})$$

2) Suppose $(E_i) \in \mathcal{M}$. Show $(\bigcup E_i)\cap K \in \mathcal{M}_F$. Let

$$B_1 = E_1 \cap K \in \mathcal{M}_F$$

(induction) $$B_n = (E_n \cap K) - \underbrace{\bigcup_{i=1}^{n-1} B_i}_{\in \mathcal{M}_F \text{ by induction assumption}} \in \mathcal{M}_F$$

B_n's disjoint, in $\mathcal{M}_F$, and

$$\bigcup_{n=1}^{\infty} B_n = \left(\bigcup_{i=1}^{\infty} E_i\right) \cap K$$

$$\bigcup_{n=1}^{\infty} B_n \subset K \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} B_n\right) < \infty \underset{(\text{IV})}{\Longrightarrow} \bigcup B_n \in \mathcal{M}_F$$

3) Suppose C closed. Show $C \in \mathcal{M}$ (Then $X \in \mathcal{M}$ and $\mathcal{M}$ contains all Borel sets)

But $C \cap K \in \mathcal{M}_F$ since $C \cap K$ is compact and all compact sets are in $\mathcal{M}_F$. Hence $C \in \mathcal{M}$

STEP VIII: $\mathcal{M}_F$ is precisely the collection of members of $\mathcal{M}$ of finite measure.

Proof. Suppose $E \in \mathcal{M}_F$. Then certainly $\mu(E) < \infty$. Then if K is compact, $E \cap K \in \mathcal{M}_F$ by steps II and VI. Hence $E \in \mathcal{M}$.

Suppose $E \in \mathcal{M}$, $\mu(E) < \infty$. Show $E \in \mathcal{M}_F$.
$\exists$ open $V \supset E$ s.t. $\mu(V) < \infty$. By steps III and V, $\exists$ compact K with $K \subset V$ and $\mu(V-K) < \varepsilon$. $E \cap K \in \mathcal{M}_F \Rightarrow$ $\exists$ H compact and $H \subset E \cap K$

$$\mu(H) > \mu(E \cap K) - \varepsilon$$

Now

$$E \subset (E \cap K) \cup (V-K)$$

$$\Rightarrow \mu(E) \le \mu(E \cap K) + \mu(V-K) \le \mu(E \cap K) + \varepsilon$$

$$\le \mu(H) + 2\varepsilon$$

Hence compact $K \subset E$ and $\mu(K) \geq \mu(E) - 2\varepsilon$. Therefore $E \in \mathfrak{M}_F$

2/15 MEASURE THEORY   Chap. 2 : 4, 11, 14, 17, 20 (2/27)

(RIESZ Rep. THEOREM CONT.)

STEP IX : μ is a measure on $\mathfrak{M}$

Suppose $E_i \in \mathfrak{M}$, E_i disjoint. Step I $\Rightarrow$ it is only necessary to show

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} \mu(E_i)$$

Trivial if $\exists\, E_i$ s.t. $\mu(E_i) = +\infty$. So suppose $\mu(E_i) < \infty$ $\forall i$. By step VIII, $E_i \in \mathfrak{M}_F$ $\forall i$. But step IV is countable additivity on $\mathfrak{M}_F$, so we're done.

STEP X : $\Lambda f = \int f\, d\mu \quad \forall f \in C_c(X)$

Sufficient to do this for f real-valued, since both sides are linear functionals. Sufficient to show for all $f \in C_c(X)$ (real-valued) that

$$\Lambda f \leq \int f\, d\mu$$

for then

$$-\Lambda f = \Lambda(-f) \leq \int -f\, d\mu = -\int f\, d\mu$$

Given f real-valued, $f \in C_c(X)$, let $K = \operatorname{supp} f$. We have $f(X) \subset [a,b]$ for some $a<b$. Let $\varepsilon > 0$. Consider a partition $y_0 < a < y_1 < y_2 < \dots < y_n = b$ where

$$y_i - y_{i-1} < \varepsilon$$

Let

$$E_i = \{x \in X : y_{i-1} < f(x) \le y_i\} \cap K$$

E_i is Borel, hence $E_i \in \mathfrak{M}$. Also $\bigcup_{i=1}^n E_i = K$ and the E_i's are disjoint.

$\exists$ open $W_i \supset E_i$ s.t. $\mu(W_i) < \mu(E_i) + \varepsilon/n$. Let

$$R_i = \{x \in X : f(x) < y_i + \varepsilon\}$$

R_i is open and $R_i \supset E_i$. Let $V_i = R_i \cap W_i$. V_i open and $V_i \supset E_i$. Certainly

$$\mu(V_i) < \mu(E_i) + \varepsilon/n$$

Now
$$\bigcup_{i=1}^n V_i \supset \bigcup_{i=1}^n E_i = K$$

Corollary of Urysohn's lemma $\Rightarrow \exists\, h_i \prec V_i$ s.t. $\sum_{i=1}^n h_i = 1$ on K. Then

$$f = \sum_{i=1}^n f h_i$$

Since $K \prec \sum_{i=1}^{n} h_i$, by step II

$$\mu(K) \leq \Lambda\left(\sum_{i=1}^{n} h_i\right)$$

We also have $f h_i \leq (y_i + \varepsilon) h_i$ on X since on V_i $f < y_i + \varepsilon$ and off V_i $h_i = 0$. Also note that $f(x) > y_i - \varepsilon$ for $x \in E_i$

$$\Lambda(f) = \sum_{i=1}^{n} \Lambda(f h_i) \underset{(\Lambda \text{ positive})}{\leq} \sum_{i=1}^{n} (y_i + \varepsilon) \Lambda h_i$$

$$= \sum_{i=1}^{n} \underbrace{(|a| + y_i + \varepsilon)}_{\text{positive}} \Lambda h_i - |a| \sum_{i=1}^{n} \Lambda h_i$$

$$\leq \sum_{i=1}^{n} (|a| + y_i + \varepsilon) \Lambda h_i - |a| \mu(K)$$

$$\leq \sum_{i=1}^{n} (|a| + y_i + \varepsilon) [\mu(E_i) + \varepsilon/n] \quad [\Lambda h_i < \mu(V_i) < \mu(E_i) + \varepsilon/n]$$

$$- |a| \mu(K)$$

$$= \sum_{i=1}^{n} \left[(y_i + \varepsilon)(\mu(E_i)) + \frac{\varepsilon}{n}(|a| + y_i + \varepsilon) \right]$$

$$[\mu(K) = \sum \mu(E_i)]$$

$$= \sum_{i=1}^{n} (y_i - \varepsilon) \mu(E_i) + \frac{\varepsilon}{n} \sum_{i=1}^{n} (|a| + y_i + \varepsilon) + 2\varepsilon \mu(K)$$

$$\leq \int_X f\,d\mu + \varepsilon(|a|+|b|+\varepsilon) + 2\varepsilon\mu(K) \qquad \left[\left[f \geq \sum_{i=1}^{n}(y_i-\varepsilon)\chi_{E_i} \right]\right]$$

Let $\varepsilon \to 0$ to obtain $\Lambda f \leq \int_X f\,d\mu$

▨ ▨ ▨ ▨ ▨ !!

<u>DEFINITIONS</u>: Borel measure is a measure on the Borel sets. A Borel measure is outer regular if $\forall$ Borel E,

$$\mu(E) = \inf\{\mu(V) : V \supset E,\ V \text{ open}\}$$

and is inner regular if $\forall$ Borel E

$$\mu(E) = \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}$$

A Borel measure is regular if it is both inner and outer regular.

<u>DEFINITION</u>: If X is a topological space, we say X is σ-compact if $X = \bigcup_{n=1}^{\infty} K_n$ for K_n compact.

<u>DEFINITION</u>: If X is a measure space $(X, \mathcal{M}, \mu)$ we say X is σ-finite if $X = \bigcup_{i=1}^{\infty} E_i$ for $\mu(E_i) < \infty$

<u>THEOREM</u>: Same hypothesis as Riesz Rep. Th^m, but add X is σ-compact. Then the μ of the conclusion satisfies

a) $\forall \varepsilon > 0$ $\forall E \in \mathfrak{M}$, $\exists$ closed F, open V s.t. $F \subset E \subset V$ and $\mu(V-F) < \varepsilon$

b) μ is regular

c) $E \in \mathfrak{M} \Rightarrow \exists F_\sigma \subset E \subset G_\delta$ s.t. $\mu(G_\delta - F_\sigma) = 0$

Proof. Given $X = \bigcup_{n=1}^{\infty} K_n$, K_n compact, let $E \in \mathfrak{M}$. Then $\mu(E \cap K_n) < \infty$ so $\exists$ open $V_n \supset E \cap K_n$ s.t.

$$\mu(V_n - (E \cap K_n)) < \varepsilon/2^n$$

Let $V = \bigcup V_n$. Then $V - E \subset \bigcup_{n=1}^{\infty} (V_n - (E_n \cap K))$, so

$$\mu(V-E) \leq \sum_{n=1}^{\infty} \mu(V_n - (E_n \cap K)) < \varepsilon$$

Apply to $X-E$ as well to get open $W \supset X-E$ with

$$\mu(W-(X-E)) < \varepsilon$$

Set $F = X - W$. Then F is closed and $F \subset E \subset V$. Also $\mu(E-F) = \mu(W-(X-E)) < \varepsilon$, so

$$\mu(V-F) < 2\varepsilon$$

2/17 MEASURE THEORY

THEOREM: Same hypothesis as RRT, but add that X is σ-compact. Then $\mathfrak{M}$ and μ of conclusion of RRT

a) $E \in \mathfrak{M} \Rightarrow \forall \varepsilon > 0 \; \exists$ closed F, open V s.t. $F \subset E \subset V$ and $\mu(V - F) < \varepsilon$

b) μ is regular

c) $E \in \mathfrak{M} \Rightarrow \exists$ an F_σ-set A, G_δ-set B s.t. $A \subset E \subset B$ and $\mu(B - A) = 0$

Proof. a) done

b) We have $X = \bigcup_{n=1}^{\infty} K_n$, K_n compact. Suppose F closed. Certainly

$$F = \bigcup_{n=1}^{\infty} (K_n \cap F)$$

and $\bigcup_{n=1}^{N} (K_n \cap F)$ is compact. Then

$$(*) \qquad \mu(F) = \lim_{N \to \infty} \mu\left(\bigcup_{n=1}^{N} (K_n \cap F)\right)$$

Given $E \in \mathfrak{M}$, (a) $\Rightarrow \exists$ closed $F \subset E$ s.t. $\mu(E - F) < \varepsilon$. Combined with $(*)$, we see

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$$

Thus μ is inner regular. But μ is outer regular from RRT.

(c) $A_n \subset E \subset B_n$, A_n closed, B_n open and $\mu(B_n - A_n) < 1/n$

Set

$$A = \bigcup_{n=1}^{\infty} A_n$$

$$B = \bigcap_{n=1}^{\infty} B_n$$

Then $A \subset E \subset B$ and $\mu(B-A) \leq \mu(B_n - A_n) < 1/n \;\; \forall n \in \mathbb{N}^*$, and so $\mu(B-A) = 0$

$\boxtimes$

THEOREM: Suppose X is a locally compact T_2-space. Suppose λ is a positive Borel measure on X such that $\lambda(K) < \infty$ for every compact K. Suppose every open subset of X is σ-compact. Then λ is regular.

Proof. Define

$$\Lambda f := \int_X f \, d\lambda \quad \forall f \in C_c(X)$$

Since $\lambda(K) < \infty$ for $K = \operatorname{supp} f$, $|\Lambda f| \leq M \lambda(K) < \infty$, $M = \max|f(x)|$

Hence Λ is a positive linear functional. By the RRT, there is a positive measure μ s.t. $\forall f \in C_c(X)$

$$\int_X f \, d\lambda = \int_X f \, d\mu$$

We know μ is regular since X is σ-compact.

Suppose V is open. We want to show $\lambda(V) = \mu(V)$. By our hypothesis, $V = \bigcup_{n=1}^{\infty} H_n$, H_n compact. By Urysohn's lemma, $\exists f_1$ s.t. $H_1 \prec f_1 \prec V$. Set $K_1 = \operatorname{supp} f_1$. Certainly $H_1 \subset K_1$. Suppose $f_1, \ldots, f_n$ have been defined where $\operatorname{supp} f_j =: K_j$, and K_j is compact and $K_j \subset V$. Choose f_{n+1} s.t.

$$\underbrace{(H_1 \cup \ldots \cup H_n \cup K_1 \cup \ldots \cup K_n)}_{\text{compact}} \prec f_{n+1} \prec V$$

Claim $f_n \uparrow \chi_V$. Note $f_{n+1} = 1$ on $K_n = \operatorname{supp} f_n$ and so $f_{n+1} \geq f_n$ everywhere since $0 \leq f_n \leq 1$ everywhere. Since $\bigcup H_n = V$, we see $f_{n+1} \uparrow \chi_V$.

We apply Monotone convergence theorem twice

$$\lambda(V) = \int_X \chi_V \, d\lambda = \lim \int_X f_n \, d\lambda = \lim \int_X f_n \, d\mu$$

$$= \int_X \chi_V \, d\mu = \mu(V)$$

Suppose E is a Borel set. Suppose $V \supset E$, V open. Then $\lambda(E) \leq \lambda(V) = \mu(V)$. μ is regular, so taking inf over all $V \supset E$, V open

$$\lambda(E) \leq \mu(E)$$

Given $\varepsilon>0$, $\exists$ closed F, open V s.t. $F\subset E\subset V$ and $\mu(V-F)<\varepsilon$. Then

$$\lambda(E) \geq \lambda(F) = \lambda(V) - \lambda(V-F) = \mu(V) - \mu(V-F)$$

(not $\neq\infty$) since λ,μ agree on open sets

$$\geq \mu(E) - \varepsilon$$

and so $\lambda(E) \geq \mu(E)$. Therefore $\lambda(E)=\mu(E)$ for every Borel set E, so λ is regular since μ is regular.

LEBESGUE MEASURE ON R^1

THEOREM: There exists a positive complete regular measure m on a σ-algebra $\mathfrak{M}$ containing the Borel sets in $\mathbb{R}$ s.t.

a) $m(I)$ = length of I (I interval)

b) $E\in\mathfrak{M}$ if and only if $\exists$ F_σ-set A, G_δ-set B s.t. $A\subset E\subset B$ and $m(B-A)=0$

c) $m(x+E)=m(E)$ $\forall x\in\mathbb{R}$

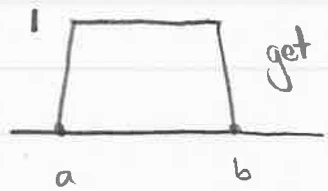
d) If μ is a positive Borel measure on $\mathbb{R}$ which is translation invariant and $\mu(K)<\infty$ $\forall K$ compact, then $\exists$ $c>0$ s.t. $\mu(E)=c\,m(E)$ $\forall$ Borel set E.

Proof. Define

$$\Lambda f := R\int_{\mathbb{R}} f(x)\,dx \quad \forall f\in C_c(\mathbb{R})$$

(Riemann integral). Λ is a positive linear functional. By RRT there is an m which is regular, complete measure on $\mathfrak{M} \supset$ Borel sets.

a) $m(I) = \sup\{\Lambda f : f \prec (a,b)\} = b-a$
$I=(a,b)$



get as close as you want

$m(\{x_0\}) = 0 \implies m(I) = \ell(I)$ for any interval I

b) shown in previous theorem

c) I open interval $\implies m(x+I) = m(I)$. If V is open,

$$V = \bigcup_{n=1}^{\infty} I_n$$

I_n disjoint, open, so $m(x+V) = m\left(\bigcup_{n=1}^{\infty}(x+I_n)\right) = m(V)$.
$m(E) = \inf\{m(V) : V \supset E, V \text{ open}\} \implies m(x+E) = m(E)$.

d) Let $\mu(0,1) = c > 0$ ($\overset{\text{WLOG}}{\text{since}}$ $c = 0 \Rightarrow \mu(\mathbb{R}) = 0 \Rightarrow \mu(E) = 0 = c\,m(E)\ \forall E$)

Suppose I = some interval, length $1/n$. Translation invariance $\implies \mu(I) = c/n$. μ is regular by the previous theorem. Also $\mu\{x_0\} = 0$ by translation invariance since otherwise we could show $\mu(0,1) = \infty$. Therefore $\mu(I) = c$ for any interval of length 1.

Given V open, $V = \bigcup I_n$, I_n disjoint, $\ell(I_n)$

is the reciprocal of some integer. Then

$$\mu(V) = c\, m(V) \quad \text{for every open } V$$

Since μ is regular, we get $\mu(E) = c\, m(E) \quad \forall$ Borel E

2/20 MEASURE THEORY

Remark: Consider the counting measure μ

$$\mu(E) = \# E$$

Certainly μ is not a scaler multiple of Lebesgue measure. Note $\mu(K) = \infty$ for lots of compact K.

Remark: Note $\displaystyle\int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} f\,dm \quad \forall f \in C_c(\mathbb{R})$

Riemann integral

In fact, we know everything necessary about Lebesgue measure now to show that every Riemann integrable function on $[a,b]$ is Lebesgue integrable (with the same value)

Recall

THEOREM: If E is a Lebesgue measurable set in $\mathbb{R}$, $\varepsilon > 0$, $m(E) < \infty$, then $\exists$ open disjoint intervals $I_1, \ldots, I_n$ s.t.

$$m\left(E \,\Delta\, \bigcup_{i=1}^{N} I_i\right) < \varepsilon$$

Sketch of proof. $\exists$ open $V \supset E$ s.t. $m(V) < m(E) + \varepsilon/2$
$\Rightarrow m(V - E) < \varepsilon/2$. V open $\Rightarrow V = \bigcup_{n=1}^{\infty} I_n$, I_n

disjoint intervals. $\exists N$ s.t. $\sum_{N+1}^{\infty} m(I_n) < \varepsilon/2$

$$\bigcup_1^N I_n - E \subset V - E \quad \text{measure} < \varepsilon/2$$

$$E - \bigcup_1^N I_n \subset \bigcup_{N+1}^{\infty} I_n \quad \text{measure} < \varepsilon/2$$

<u>LUSIN'S THEOREM</u>: X locally compact Hausdorff space. $(X, \mathfrak{M}, \mu)$ of sort produced by R.R.T. Suppose $f : X \to \mathbb{C}$ and f is $\mathfrak{M}$-measurable. Suppose $\exists A \subset X$, $\mu(A) < \infty$ s.t. $f(x) = 0$ $\forall x \in X - A$. Then $\exists g \in C_c(X)$ s.t.

$$\mu\{x \in X : f(x) \neq g(x)\} < \varepsilon$$

Moreover, if $\sup_X |f(x)| < \infty$, then g can be chosen so that $\sup |g(x)| \leq \sup |f(x)|$

Proof. First suppose $0 \leq f \leq 1$ and A is compact. $\exists$ simple functions $s_n \uparrow f$. Recall

$$t_n := s_n - s_{n-1} = 2^{-n} \chi_{T_n} \quad n > 1$$

$$t_1 := s_1 = 2^{-1} \chi_{T_1}$$

Then $f = \sum_{n=1}^{\infty} t_n$. Note $T_n \subset A$. A compact, X locally compact $\Rightarrow A \subset V \subset \overline{V}$ (compact). Since $\mu(T_n) < \infty$, $\exists$ compact K_n, open V_n s.t

$$K_n \subset T_n \subset V_n \subset V \quad \text{and} \quad \mu(V_n - K_n) < \varepsilon 2^{-n}$$

Urysohn's lemma $\Rightarrow$ $\exists\, h_n$ s.t. $K_n \prec h_n \prec V_n$. Set

$$g := \sum_{n=1}^{\infty} 2^{-n} h_n$$

Certainly g is continuous (uniform limit of continuous functions). Each $h_n = 0$ outside $V_n \subset V \Rightarrow \operatorname{supp} g \subset \overline{V}$ and hence compact. On K_n, $2^{-n} h_n = t_n$. Off V_n, $2^{-n} h_n = 0 = t_n$. Therefore $2^{-n} h_n = t_n$ off $V_n - K_n$

$$\Rightarrow \quad g = f \quad \text{off} \quad \bigcup_{n=1}^{\infty} (V_n - K_n)$$

But $\mu\left(\bigcup_{n=1}^{\infty} (V_n - K_n)\right) < \varepsilon$.

Remove simplifying assumptions. First suppose f is bounded. Work with the real and imaginary parts separately. For appropriate M and a,

$$\frac{\operatorname{Re} f}{M} + a \quad \text{takes values in } [0,1)$$

Then $M(g - a)$ should work for $\operatorname{Re} f$.

Now remove condition that A is compact. There exists a compact $K \subset A$ s.t. $\mu(A - K) < \varepsilon$. Then $f\chi_A$ agrees with off the set $A - K$ of measure $< \varepsilon$. We can obtain a suitable g for $f\chi_A$

For the general case, suppose f is unbounded and let

$$B_n := \{x \in X : |f(x)| > n\}$$

Then $\bigcap_{n=1}^{\infty} B_n = \emptyset$ (since f is complex-valued) and $B_n \subset A$, $\mu(A) < \infty$, so that $\lim_{n \to \infty} \mu(B_n) = 0$. $\exists\, n$ s.t. $\mu(B_n) < \varepsilon$.

Then $f(1 - \chi_{B_n})$ agrees with f off the set B_n, and off B_n

$$|f(1 - \chi_{B_n})| < n$$

On B_n, $f(1-\chi_{B_n}) = 0$. Now find g for $f(1-\chi_{B_n})$. This agrees with f off a set of measure $< 2\varepsilon$

Suppose $\sup |f(x)| = R < \infty$. Let

$$\varphi(z) = \begin{cases} z & |z| \le R \\ \dfrac{z}{|z|} R & |z| > R \end{cases}$$

This is continuous. We have $g \in C_c(X)$ s.t. $\mu\{x \in X : f(x) \neq g(x)\} < \varepsilon$.

Let $g_1 = \varphi \circ g$. The set $\{x \in X : f(x) \neq g_1(x)\} \subset \{x \in X : f(x) \neq g(x)\}$ and so $\mu\{x \in X : f(x) \neq g_1(x)\} < \varepsilon$. Clearly $\sup |g_1(x)| \le R$

$\blacksquare$

2/22 MEASURE THEORY

QUESTION: Suppose $f : X \to \mathbb{R}$ measurable (X locally compact Hausdorff) Does there exist a continuous g on X s.t. g approximates f well (in some sense) and, for instance, $g \leq f$?

ANSWER <u>NO</u> Consider $X = \mathbb{R}$ with Lebesgue measure

$$f = \chi_{C_\alpha} \qquad C_\alpha \subset [0,1] \text{ Cantor set of positive measure}$$

Claim: Suppose $g : [0,1] \to R$ is lower semicontinuous and $g \leq f$ on $[0,1]$. Then $g \leq 0$

If $g(x_0) > 0$, then $g > 0$ on an open interval I containing x_0

But $I \not\subset C_\alpha$ ↯ ($\exists$ points where $f(x) = 0$ but $g(x) > 0$)

Thus the best approximation to f from below is $\tilde{g} = 0$

But note $f - \tilde{g} = 1$ on a set of measure $1 - \alpha > 0$

$$\int f - \int \tilde{g} = 1 - \alpha > 0$$

THEOREM: (VITALI - CARATHÉODORY) Suppose X is a locally compact Hausdorff space and $(X, \mathcal{M}, \mu)$ of sort produced by RRT. Suppose $f : X \to \mathbb{R}$ is in $L^1(\mu)$. Then if $\varepsilon > 0$ there is an upper-semicontinuous u which is bounded above, a lower-semicontinuous v which is bounded below, such that

$$u \leq f \leq v$$

and
$$\int_X (u-v)\, d\mu < \varepsilon$$

Proof. First suppose $f \geq 0$. $\exists$ simple $s_n \uparrow f$. Let $s_0 = 0$ and

$$t_n := s_n - s_{n-1} \quad \forall n \in \mathbb{N}$$

Then $f = \sum_{n=1}^{\infty} t_n$ (converges everywhere on X). In fact

$$f = \sum_{i=1}^{\infty} c_i \chi_{E_i}$$

$c_i > 0$, E_i measurable (not in general disjoint). $f \in L^1(\mu)$ implies

$$\infty > \int_X f\, d\mu \underset{\underset{\text{MCT}}{\uparrow}}{=} \sum_{i=1}^{\infty} c_i \mu(E_i)$$

Hence $\mu(E_i) < \infty$, so because μ is "regular" on sets of finite measure, $\exists K_i \subset E_i \subset V_i$ s.t. K_i is compact, V_i open, and

$$c_i \mu(V_i - K_i) < \frac{\varepsilon}{2^{i+1}}$$

Also, $\exists N \in \mathbb{N}$ s.t. $\sum_{i=N+1}^{\infty} c_i \mu(E_i) < \varepsilon/2$.

Let

$$v = \sum_{i=1}^{\infty} c_i \chi_{V_i}$$

$$u = \sum_{i=1}^{N} c_i \chi_{K_i}$$

Certainly $u \leq f \leq v$.

$$v - u = \sum_{i=1}^{N} c_i (\chi_{V_i} - \chi_{K_i}) + \sum_{i=N+1}^{\infty} c_i \chi_{V_i}$$

$$\leq \sum_{i=1}^{\infty} c_i (\chi_{V_i} - \chi_{K_i}) + \sum_{i=N+1}^{\infty} c_i \chi_{E_i}$$

$$\Longrightarrow \int_X (v-u)\, d\mu \leq \sum_{i=1}^{\infty} c_i \mu(V_i - K_i) + \sum_{i=N+1}^{\infty} c_i \mu(E_i)$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Claim: v is lower semi-continuous. Suppose $v(x_0) > \alpha$

$\exists$ M s.t.

$$\sum_{i=1}^{M} c_i \chi_{V_i}(x_0) > \alpha$$

V_i open $\Rightarrow \sum_{i=1}^{M} c_i \chi_{V_i}(x) > \alpha$ on a nbhd of x_0

Claim: u is upper semi-continuous Suppose $u(x_0) < \alpha$

$$\sum_{i=1}^{N} c_i \chi_{K_i}(x_0) < \alpha$$

Let $I = \{ i : 1 \le i \le n ,\ x_0 \in K_i \}$. $\bigcup_{\substack{i=1 \\ i \notin I}}^{n} K_i$ closed not containing x_0, so on $X - \bigcup_{\substack{i=1 \\ i \notin I}}^{n} K_i$, we have $u(x) \le u(x_0) < \alpha$

General Case: $f = f^+ - f^-$. Obtain

$$u_1 \le f^+ \le v_1$$
$$u_2 \le f^- \le v_2$$

Then $\underset{\substack{\uparrow \\ u}}{u_1 - v_2} \le \underset{\substack{\uparrow \\ f}}{f^+ - f^-} \le \underset{\substack{\uparrow \\ v}}{v_1 - u_2}$. Certainly

$$\int_X [(v_1 - u_2) - (u_1 - v_2)]\, d\mu < 2\varepsilon$$

$u \le u_1$ bdd above, $v \ge -u_2$ bdd below. u is USC if we show the sum of two USC is USC

[Suppose h_1, h_2 are u.s.c. Show $h_1 + h_2$ is u.s.c.
Given $\alpha \in \mathbb{R}$. For real r, let

$$E_r = \{ x \in X : h_1(x) < r \} \cap \{ x \in X : h_2(x) < \alpha - r \}$$
(open)

Claim : $A := \{x \in X : h_1(x) + h_2(x) < \alpha\} = \bigcup_{r \in \mathbb{R}} E_r$

Clearly $\bigcup_{\mathbb{R}} E_r \subset A$. Given $x \in A$, $\exists r \in \mathbb{R}$ s.t.

$$0 < r - h_1(x) < \alpha - h_1(x) - h_2(x)$$

$$\Longrightarrow x \in E_r$$

]

L^p - SPACES

<u>**DEFINITION:**</u> $f : (a,b) \to \mathbb{R}$ is convex if

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y)$$

$\forall \lambda \in [0,1]$, $\forall x, y \in (a,b)$

Reminder: (1) f is continuous, differentiable off a countable set
(2) if $a < s < t < u < b$, then

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(t)}{u - t}$$

(3) <u>Jensen's</u> <u>inequality</u> for $\mathbb{R}$

<u>THEOREM (JENSEN'S INEQUALITY)</u> Suppose $(X, \mathcal{M}, \mu)$ is a measure space, $\mu(X) = 1$. Suppose $f : X \to (a,b)$ is in $L^1(\mu)$ and $\varphi : (a,b) \to \mathbb{R}$ is convex. Then

$$\varphi\left(\int_X f \, d\mu\right) \leq \int_X (\varphi \circ f) \, d\mu$$

Remarks: (1) First notice $a < \int_X f \, d\mu < b$ since $\mu(X) = 1$ and $f(x) - a > 0 \ \forall x \in X$

(2) Also, φ convex $\Rightarrow$ φ continuous $\Rightarrow$ $\varphi \circ f$ measurable.

(3) $\int \varphi \circ f \, d\mu$ could be $+\infty$

2/24 MEASURE THEORY

Proof of Jensen's inequality: Set $t = \int_X f\,d\mu \in (a,b)$. If $a < s < t < u < b$, then

$$(*) \qquad \frac{\varphi(t) - \varphi(s)}{t-s} \leq \frac{\varphi(u) - \varphi(t)}{u-t}$$

Let

$$\beta := \sup_{s<t} \frac{\varphi(t) - \varphi(s)}{t-s}$$

(left derivative of φ at t). For all $y \in (a,b)$

$$\varphi(y) \geq \varphi(t) + \beta(y-t)$$

from definition of β and $(*)$. Therefore $x \in X$ implies

$$\varphi(f(x)) \geq \underbrace{\varphi(t) + \beta(f(x)-t)}_{\in L^1(\mu)}$$

Hence $(\varphi \circ f)^- \in L^1(\mu)$. Now we consider

$$\int_X (\varphi \circ f)\,d\mu := \int_X (\varphi\circ f)^+ d\mu - \int_X (\varphi \circ f)^- d\mu$$

$$\geq \int_X \left(\varphi(t) + \beta(f(x)-t)\right) d\mu$$

$$= \varphi(t) + \beta\left(\int f\,d\mu - t\right) \quad \left(\text{used here } \mu(X)=1\right)$$

$$= \varphi(t) = \varphi\left(\int_X f\,d\mu\right)$$

$\blacksquare$

EXAMPLE: $X=(0,1)$ μ Lebesgue measure

$$f(x) := \frac{1}{\sqrt{x}} \in L^1(\mu) \qquad \varphi(t) = e^t$$

Then $$\int (\varphi\circ f)\,d\mu = \int_0^1 e^{1/\sqrt{x}}\,dx \geq \int_0^1 \frac{1}{x^2}\,dx = \infty$$

L^p - SPACES

THEOREM: Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Suppose $f: X \to [0,\infty]$ and $g: [0,\infty)$ are measurable. Suppose $1<p<\infty$ and $1/p + 1/q = 1$. Then

(i) (HÖLDER) $$\int_X fg\,d\mu \leq \left(\int f^p\,d\mu\right)^{1/p}\left(\int g^q\,d\mu\right)^{1/q}$$

(ii) (MINKOWSKI) $$\left(\int_X (f+g)^p\,d\mu\right)^{1/p} \leq \left(\int f^p\,d\mu\right)^{1/p} + \left(\int g^p\,d\mu\right)^{1/p}$$

Proof. (i) Let $A = \left(\int f^p \, d\mu\right)^{1/p}$

$$B = \left(\int g^q \, d\mu\right)^{1/q}$$

If $A = 0$, result trivial since this implies $f = 0$ a.e. If $B = \infty$, RHS is ∞, so inequality okay. So only case needing serious discussion is $0 < A, B < \infty$.

Set $F := f/A$

$$G := g/B$$

Note

$$\int F^p \, d\mu = \frac{1}{A^p} \int_X f^p \, d\mu = 1$$

$$\int G^q \, d\mu = \frac{1}{B^q} \int_X g^q \, d\mu = 1$$

Suppose $x \in X$ s.t. $F(x)G(x) > 0$. $\exists\, s, t \in \mathbb{R}$ s.t.

$$F(x) = e^{s/p}, \quad G(x) = e^{t/q}$$

Then $F(x)\,G(x) = e^{s/p + t/q} \underset{\substack{e^x \text{ convex} \\ 1/p + 1/q = 1}}{\leq} \frac{1}{p} e^s + \frac{1}{q} e^t = \frac{1}{p} F(x)^p + \frac{1}{q} G(x)^q$

In fact this holds for all x, and so

$$\int_X FG\,d\mu \le \frac{1}{p}\int F^p\,d\mu + \frac{1}{q}\int G^q\,d\mu = \frac{1}{p}+\frac{1}{q} = 1$$

$$\Rightarrow \int_X fg\,d\mu \le AB$$

(ii) Apply Holder to f and $(f+g)^{p-1}$

$$\int_X f(f+g)^{p-1}\,d\mu \le \left(\int_X f^p d\mu\right)^{1/p}\left(\int (f+g)^{(p-1)q}\,d\mu\right)^{1/q}$$

Also

$$\int_X g(f+g)^{p-1}\,d\mu \le \left(\int g^p d\mu\right)^{1/p}\left(\int (f+g)^{(p-1)q}\,d\mu\right)^{1/q}$$

Note $(p-1)q = p$. Adding

$$(*) \quad \int_X (f+g)^p d\mu \le \left(\int (f+g)^p\,d\mu\right)^{1/q}\left[\left(\int f^p d\mu\right)^{1/p} + \left(\int g^p d\mu\right)^{1/p}\right]$$

If $\int (f+g)^p\,d\mu = 0$, result is trivial. If RHS of (ii) is $+\infty$, result is trivial. Now t^p is a convex function for $0<t<\infty$ and so

$$\left(\frac{f+g}{2}\right)^p \le \frac{1}{2}\left(f^p+g^p\right)$$

Hence we may assume $\int (f+g)^p\,d\mu < \infty$.

Now divide $(*)$ by $\left(\int (f+g)^p d\mu\right)^{1/q}$ and use $1-1/q=1/p$ to obtain result

$\blacksquare$

DEFINITION: $(X,\mathcal{M},\mu)$ measure space. $(0<p<\infty)$ $L^p(\mu)$ is the set of all complex-valued measurable f s.t.

$$\|f\|_p := \int |f|^p d\mu < \infty$$

(If $X=\mathbb{N}$, μ counting measure, $\mathcal{M}=\mathcal{P}(\mathbb{N})$, then we denote $L^p(\mu)$ by ℓ_p)

DEFINITION: $(X,\mathcal{M},\mu)$ measure space, $f: X \longrightarrow [0,\infty]$

Let

$$S := \{\alpha \in \mathbb{R} : \mu\left(f^{-1}(\alpha,+\infty]\right) = 0\}$$

If $S=\phi$, set $\beta=+\infty$. If $S\neq\phi$, set $\beta := \inf S$. β is called the essential supremum of f. Note $\beta \in S$ since

$$\mu\left(f^{-1}(\beta,\infty]\right) = \bigcup_{n=1}^{\infty} \mu\left(f^{-1}(\beta+1/n,\infty]\right) = 0$$

Now let $L^\infty(\mu)$ be the set of all complex-valued measurable f such that

$$\|f\|_\infty := \operatorname{ess\,sup} |f| < \infty$$

Remark: Suppose $f \in L^\infty(\mu)$. For $0 \le \lambda < \infty$, then

$$|f(x)| \le \lambda \text{ a.e.} \iff \lambda \geq \|f\|_\infty$$

Proof. Suppose $|f(x)| \le \lambda$ a.e. Then $\lambda \in S$ (for $|f|$) and so $\operatorname{ess\,sup} |f| \le \lambda \implies \|f\|_\infty \le \lambda$.

Suppose $\|f\|_\infty \le \lambda$. Then $\lambda \in S \implies |f(x)| \le \lambda$ a.e.

THEOREM: Suppose $1 \le p$, $1/p + 1/q = 1$. Suppose $f : X \to \mathbb{C}$ is in L^p and $g : X \to \mathbb{C}$ is in L^q. Then $fg \in L^1$ and

$$\|fg\|_1 \le \|f\|_p \|g\|_q$$

Proof. For $1 < p < \infty$ this is Hölder's inequality. Suppose $p = 1$, so $g \in L^\infty(\mu)$. Then

$$|f(x) g(x)| \le |f(x)| \, \|g\|_\infty \text{ a.e.}$$

$$\implies \int |fg| \le \left(\int |f| \right) \|g\|_\infty < \infty$$

$$\implies \|fg\| \le \|f\|_1 \|g\|_\infty$$

$\boxtimes$

THEOREM: $f, g \in L^p(\mu)$, $1 \le p \le \infty$. Then

$$\|f+g\|_p \le \|f\|_p + \|g\|_q$$

Proof. If $1<p<\infty$, Minkowski.

If $p=1$: $|f+g| \le |f|+|g|$ - integrate

If $p=\infty$: $|f(x)| \le \|f\|_\infty$ a.e.

$|g(x)| \le \|g\|_\infty$ a.e.

$\Longrightarrow |f|+|g| \le \|f\|_\infty + \|g\|_\infty$ a.e.

$\Longrightarrow \|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$

2/27 MEASURE THEORY Ch. 3 8, 14, 15, 20 3/13

If $f, g \in C_c(\mathbb{R})$, let

$$d(f,g) := \int_{-\infty}^{\infty} |f(x) - g(x)|\, dx$$

What is the completion of $C_c(\mathbb{R})$ with this metric?

"DEFINITION" Given $(X, \mathcal{M}, \mu)$. We define $f \sim g$ if $f = g$ a.e. for f, g measurable. The "new" $L^p(\mu)$ is the space of equivalence classes under the above equivalence relation, of the "old" $L^p(\mu)$.

$$\|\tilde{f}\|_p = \|f\|_p \quad \text{for any } f \in \tilde{f}$$

The "new" $L^p(\mu)$ is a normed vector space. This gives a metric defined by

$$d(f,g) := \|f - g\|_p$$

<u>THEOREM</u>: For $1 \le p \le \infty$, $L^p(\mu)$ is complete.

Proof. $1 \le p < \infty$. Let (f_n) be Cauchy in L^p. We want to find $f \in L^p$ such that $\|f_n - f\|_p \to 0$.

(f_n) Cauchy $\Longrightarrow \exists N_i > 0$ such that $N_{i+1} > N_i$ and

$$n, m > N_i \implies \|f_n - f_m\|_p < 1/2^i$$

Suppose $n_i > N_i$. Then

$$\| f_{n_{i+1}} - f_{n_i} \| < 1/2^i$$

Consider

$$g_k := |f_{n_1}| + \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|$$

$$g = |f_{n_1}| + \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|$$

Then $g_k \longrightarrow g$. Moreover

$$\|g_k\|_p \leq \|f_{n_1}\|_p + \sum_{i=1}^{k} 2^{-i} < 1 + \|f_{n_1}\|_p$$

By Fatou's lemma

$$\int_X g^p \, d\mu \leq \varliminf_{k \to \infty} \int_X g_k^p \, d\mu < (1 + \|f_{n_1}\|_p)^p < \infty$$

Then g is measurable and $\int_X g^p d\mu < \infty \Rightarrow g(x) < \infty$ a.e. $[\mu]$

Define $f : X \to \mathbb{C}$ as follows

$$f(x) := \begin{cases} f_{n_1}(x) + \sum_{i=1}^{\infty} f_{n_{i+1}}(x) - f_{n_i}(x) & \text{if } g(x) < \infty \\ 0 & \text{if } g(x) = \infty \end{cases}$$

Then f is measurable, and $f(x) = \lim_{i \to \infty} f_{n_i}(x)$ a.e.

Claim: $f \in L^p(\mu)$ and $\|f_n - f\|_p \to 0$.

Let $\varepsilon > 0$. $\exists N > 0$ s.t. $n, m > N \Rightarrow \|f_n - f_m\|_p < \varepsilon$.
Let $m > N$. Then by Fatou

$$(*) \int_X |f - f_m|^p \, d\mu \le \varliminf_{i\to\infty} \int |f_{n_i}(x) - f_m(x)|^p \, d\mu \le \varepsilon^p$$

Hence $f - f_m \in L^p \Rightarrow f \in L^p$. Moreover $(*)$ shows that $\|f - f_m\|_p \to 0$.

$p = \infty$: Let (f_n) be Cauchy in L^∞. For $n, m \in \mathbb{N}$, let

$$B_{nm} := \{x \in X : |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}$$

Definition of $\|\cdot\|_\infty \Rightarrow \mu(B_{nm}) = 0$. Let

$$B := \bigcup_{n=1}^\infty \bigcup_{m=1}^\infty B_{nm}$$

Then $\mu(B) = 0$. Off B, $|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty$. Hence $(f_n(x))$ is uniformly Cauchy on $X - B$, so $\exists f$ on $X - B$ such that $f_n \to f$ uniformly on $X - B$. Let $f := 0$ on B. Then f is measurable. For large enough n, $\|f_n - f\|_\infty \le 1$, so $\|f\|_\infty \le 1 + \|f_n\|_\infty < \infty$. Hence $f \in L_\infty(\mu)$. But also $\|f_n - f\|_\infty \to 0$ since $f_n \to f$ uniformly on $X - B$.

$\square$

THEOREM: $(X, \mathfrak{M}, \mu)$ measure space, $1 \leq p < \infty$. Let

$$S := \{ s : s \text{ simple, complex-valued} \\ s(A) = 0 \text{ for some } \mu(X-A) < \infty \}$$

Then S is dense in $L^p(\mu)$.

Proof. First suppose $f \in L^p(\mu)$ and $f \geq 0$. Then there exist simple s_n, $0 \leq s_n \leq f_n$ with $s_n \uparrow f$ on X. But $\int f^p \, d\mu < \infty$ implies s_n vanishes off a set of finite measure. Hence $s_n \in S$.

Now $0 \leq (f - s_n)^p \leq f^p$ on X and $f - s_n \to 0$ a.e. on X. Also $\int f^p \, d\mu < \infty \Rightarrow f^p \in L^1(\mu)$. Hence the dominated convergence theorem says that

$$\lim_{n \to \infty} \int_X (f - s_n)^p \, d\mu = 0$$

$$\Rightarrow \| f - s_n \|_p \to 0$$

For a general f, $f = (\operatorname{Re} f)^+ - (\operatorname{Re} f)^- + i[(\operatorname{Im} f)^+ - (\operatorname{Im} f)^-]$. Approximate each term on right separately. $\square$

REMARK: This is false if $p = \infty$.

Take $f = 1$ on $\mathbb{R}$, μ = Lebesgue measure. If $s \in S$ then $\| f - s \|_\infty \geq 1$.

PROPOSITION: Suppose $(X, \mathcal{M}, \mu)$ is a measure space where μ has the properties of the conclusion of the RRT. Then $C_c(X)$ is dense in $L^p(\mu)$ for $1 \leq p < \infty$

Proof. Take $s \in S$. Lusin's theorem implies if $\varepsilon > 0$ $\exists\, g \in C_c(X)$ s.t.

$$(1) \quad \mu(\overbrace{\{x \in X : g(x) \neq s(x)\}}^{E}) < \varepsilon$$

$$(2) \quad \|g\|_\infty \leq \sup_{s \in X} |s(x)| \leq \|s\|_\infty$$

Then

$$\int_X |g-s|^p \, d\mu = \int_E |g-s|^p \, d\mu \leq (2\|s\|_\infty)^p \varepsilon$$

and so $\|g-s\|_p \leq 2\|s\|_\infty \varepsilon^{1/p}$

Given $f \in L^p(\mu)$, $\exists\, s \in S$ such that $\|f-s\|_p < \eta/2$ by the previous theorem. The above calculation shows that $\exists\, g \in C_c(X)$ s.t. $\|s-g\|_p < \eta/2$, and so $\|f-g\| \leq \eta$. ▨

This shows that $C_c(\mathbb{R})$ is a dense subset of $L^1(\mu)$ and so $L^1(\mu)$ is the completion of $C_c(\mathbb{R})$.

3/1 MEASURE THEORY

Remarks: ① Consider the proof that $L^p(\mu)$ is complete, $1 \le p \le \infty$. In the proof we also showed that if $f_n \to f$ in $L^p(\mu)$, then there is a subsequence f_{n_k} s.t. $f_{n_k}(x) \to f(x)$ a.e. $[\mu]$

② We showed that $C_c(X)$ is a dense subset of $L^p(\mu)$ where $(X, \mathcal{M}, \mu)$ is a measure space satisfying conclusions of RRT. This statement could not possibly be true if there was no relationship between the topology on X and the measurable sets $\mathcal{M}$. For example, consider $\mathbb{R}$ with the usual topology, and let μ be the counting measure on the subsets of $\mathbb{R}$. Then $C_c(\mathbb{R}) \not\subset L^1(\mu)$.

Recall: If X is a metric space, define, for Cauchy sequences (x_n) and (y_n) in X,

$$(x_n) \sim (y_n) \quad \text{if} \quad \lim_{n\to\infty} d(x_n, y_n) = 0$$

Let S = set of equivalence classes. If $s \in S$, $t \in S$, let

$$\tilde{d}(s,t) = \lim_{n\to\infty} d(x_n, y_n)$$

where $(x_n) \in s$, $(y_n) \in t$. Check

① $\lim d(x_n, y_n)$ exists

② $\tilde{d}(s,t)$ well-defined

③ $\tilde{d}$ is a metric on S

④ $(S, \tilde{d})$ is complete

Regard $X \subset S$ in following sense. Suppose $a \in X$. The constant Cauchy sequence $a, a, a, \ldots$ belongs to an equivalence class $\tilde{a} \in S$. Identify a with $\tilde{a}$. Check

⑤ X is dense in S

⑥ Any complete metric space Z of which X is a dense subset is isometric to $(S, \tilde{d})$

Recall: For $1 \leq p < \infty$, $f, g \in C_c(\mathbb{R}^k)$. Define

$$d_p(f,g) = \|f-g\|_p$$

We know $(C_c(\mathbb{R}^k), d_p)$ is a metric space which is a dense subset of L^p (Lebesgue measure on $\mathbb{R}^k$) which is itself a complete metric space. Thus $L^p(\mathbb{R}^k)$ is the completion of $(C_c(\mathbb{R}^k), d_p)$

<u>QUESTION</u>: What is the completion of $(C_c(\mathbb{R}^k), d_\infty)$ where

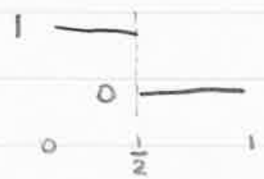
$$d_\infty(f,g) := \sup_{x \in \mathbb{R}^k} |f(x) - g(x)|$$

Take $f = 1$ on $\mathbb{R}^k$. Then $\|f-g\|_\infty \geq 1 \quad \forall g \in C_c(\mathbb{R}^k)$ and so $C_c(\mathbb{R}^k)$ is not dense in $L^\infty(\mathbb{R}^k)$.

Suppose $g_n \in C_c(\mathbb{R}^k)$, $g_n \to g$ in $L^\infty(\mathbb{R}^k)$

$$\|g_n - g\|_\infty \to 0 \implies g_n \text{ is uniformly Cauchy on } \mathbb{R}^k$$

Hence $g_n \to h$ uniformly on $\mathbb{R}^k$, h continuous, and so $h = g$ a.e, i.e. g is equal a.e. to a function continuous everywhere.

( is not an acceptable g)

<u>DEFINITION</u>: X locally compact T_2-space. $f: X \to \mathbb{C}$ vanishes at ∞ if $\forall \varepsilon > 0$ $\exists$ compact K s.t. $|f(x)| < \varepsilon$ if $x \in X - K$.

$C_0(X)$ denotes the space of continuous functions vanishing at ∞

<u>THEOREM</u>: X loc. compact T_2-space. For $f, g \in C_c(X)$ let

$$d(f,g) := \sup_{x \in X} |f(x) - g(x)|$$

The completion of $(C_c(X), d)$ is $(C_0(X), d)$.

Proof. Clearly $C_c(X) \subset C_0(X)$. Must show $C_c(X)$ is dense in $C_0(X)$ and $C_0(X)$ is complete.

Choose $f \in C_0(X)$ and let $\varepsilon > 0$. $\exists K$ compact in X s.t. $|f(x)| < \varepsilon$ if $x \in X - K$. By Urysohn's lemma there is a $g \in C_c(X)$ s.t. $K \prec g \prec X$. Set $h := fg$. Certainly h is continuous, and $\operatorname{supp} h \subset \operatorname{supp} g$, and so is compact.

$$h - f = f(1-g) = 0 \quad \text{on } K$$
$$|h - f| < \varepsilon \quad \text{on } X-K$$

Hence $d(f,h) < \varepsilon$, so $C_c(X)$ is dense in $C_0(X)$.

Suppose (f_n) Cauchy in $C_0(X)$. Let $\varepsilon > 0$. Since (f_n) is uniformly Cauchy on X, there is a continuous $f : X \to \mathbb{C}$ s.t. $f_n \to f$ uniformly. Then $\exists N$ s.t.

$$\sup_{x \in X} |f_N(x) - f(x)| < \varepsilon/2$$

There is a compact K s.t. $|f_N(x)| < \varepsilon/2$ for $x \in X-K$. Hence $|f(x)| < \varepsilon$ for $x \in X-K$, so $f \in C_0(X)$. Therefore $C_0(X)$ is complete.

Things to look out for in Hilbert space chapter

Riesz-Fisher thm
Parseval's thm
Bessel Inequality
Fejer thm
Characterization of the continuous linear functionals on Hilbert space

3/3 Measure Theory

DEFINITION: Suppose H is a vector space over $\mathbb{C}$. If there is a function $(\cdot|\cdot) : H \times H \to \mathbb{C}$ satisfying the following conditions, we say H is an inner product space

(i) $(x|y) = \overline{(y|x)}$
(ii) $(x_1 + x_2 | y) = (x_1|y) + (x_2|y)$
(iii) $(\alpha x | y) = \alpha (x|y)$
(iv) $(x|x) \geq 0$
(v) $(x|x) = 0 \iff x = 0$

Properties: (a) $(0|y) = (y|0) = 0$ 〚(i) and (iii)〛
(b) For a fixed y, the map $(\cdot|y)$ is a linear functional on H.
(c) $(x|\alpha y) = \bar{\alpha}(x|y)$
$(x|y_1 + y_2) = (x|y_1) + (x|y_2)$

DEFINITION: For $x \in H$ let $\|x\| := (x|x)^{1/2}$

SCHWARZ INEQUALITY: $|(x|y)| \leq \|x\| \, \|y\|$

Proof: Let $A = \|x\|$, $B = |(x|y)|$, $C = \|y\|$. There is an $\alpha \in \mathbb{C}$ s.t. $\alpha(y|x) = B$. For every $r \in \mathbb{R}$,

$$0 \leq (x - \alpha r y \,|\, x - \alpha r y) = (x|x) - \alpha r(y|x) - \bar{\alpha} r (x|y) + r^2 (y|y)$$

$$\Rightarrow 0 \leq A^2 - rB - rB + r^2 C^2 \quad \forall r \in \mathbb{R}$$

If $C=0$, then $B=0$, so result holds. If $C \neq 0$, then

$$(2B)^2 - 4A^2C^2 \leq 0$$

(otherwise quadratic is <0 for some r)

$$\Rightarrow \quad B \leq AC$$

<u>Triangle Inequality</u> — $\|x+y\| \leq \|x\| + \|y\|$.

$$\|x+y\|^2 = (x+y|x+y) = (x|x) + (y|x) + (x|y) + (y|y)$$

$$= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\,(x|y).$$

$$\leq \|x\|^2 + \|y\|^2 + 2\|x\|\,\|y\| \qquad [\![\text{Schwartz}]\!]$$

$$= (\|x\| + \|y\|)^2$$

<u>DEFINITION</u>: If $x, y \in H$, let

$$d(x,y) := \|x-y\|$$

This defines a metric on H. H is called a Hilbert space if H is complete in this metric.

EXAMPLE: (a) Consider a measure space $(X, \mathcal{M}, \mu)$. For $f \in L^2(\mu)$, $g \in L^2(\mu)$, define

$$(f|g) := \int_X f\,\bar{g}\,d\mu$$

(Note that Holder $\Rightarrow f\bar{g} \in L^1(\mu)$) The Hilbert space norm obtained from this inner product is just the L^2 norm, and so $L^2(\mu)$ is a Hilbert space.

(b) Let $\mathbb{C}^n := \{(x_1, \ldots, x_n) : x_k \in \mathbb{C}\}$. Define

$$(x|y) := \sum_{k=1}^{n} x_k \overline{y_k}$$

This is $L^2(\mu)$ where $X = \mathbb{N}_n^*$ and μ is the counting measure. This is also a Hilbert space

(c) $H = C[0,1]$. $f, g \in H$

$$(f|g) := \int_0^1 f(t)\,\overline{g(t)}\,dt$$

This gives an inner product space. Let $h = \chi_{(1/2, 1]} \in L^2[0,1]$. $\exists$ continuous g_n s.t. $\|g_n - h\|_2 \to 0$. Then $(g_n) \subset H$ and is Cauchy in (H, d). If $g_n \to g$ in (H, d), then $g = h$ a.e. But no continuous function on $[0,1]$ can equal h a.e.

Remark: The map $(\cdot|y)$ is a continuous functional on H

$$|(x_1|y)-(x_2|y)| = |(x_1-x_2|y)| \leq \|x_1-x_2\|\,\|y\|$$

The maps $(x|\cdot)$ and $\|\cdot\|$ are also continuous.

Definition: $M\subset H$ is a closed subspace if it is a vector space which is closed in the topology of H.

Theorem: Suppose $E\subset H$ is a closed convex set. Then E contains a unique element of smallest norm.

Proof. Note

$$(x+y|x+y) = (x|x)+(y|y)+(y|x)+(x|y)$$
$$(x-y|x-y) = (x|x)+(y|y)-(y|x)-(x|y)$$

and so $\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+\|y\|^2$.

Suppose $x,y\in E$. Let

$$\delta := \inf_{z\in E}\|z\|$$

Since E is convex, $\frac{1}{2}(x+y)\in E$. Therefore

$$\frac{1}{4}\|x-y\| = \frac{1}{2}\|x\|^2+\frac{1}{2}\|y\|^2-\frac{1}{4}\|x+y\|^2$$

$$\leq \frac{1}{2}\|x\|^2+\frac{1}{2}\|y\|^2-\delta^2$$

$$\|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 - 4\delta^2$$

If $\|x\| = \|y\| = \delta$, then $\|x-y\| = 0$ from the above, so $x = y$.
This shows uniqueness.

$$\exists\, (y_n) \subset E \text{ s.t. } \|y_n\| \to \delta$$

$$\|y_n - y_m\|^2 \leq 2\|y_n\|^2 + 2\|y_m\|^2 - 4\delta^2 \xrightarrow[n,m \to \infty]{} 0$$

Hence (y_n) is Cauchy, so $\exists\, x_0 \underset{\substack{\uparrow \\ E \text{ closed}}}{\in} E$ s.t. $y_n \to x_0$

$\|\cdot\|$ continuous $\Rightarrow$ $\|x_0\| = \lim \|y_n\| = \delta$ ▨

DEFINITION: $x \perp y$ means $(x|y) = 0$ (x is "orthogonal" to y). If $x \in H$

$$x^\perp := \{ y \in H : (x|y) = 0 \}$$

[$x^\perp$ is a closed subspace (= inverse image of $\{0\}$ under $(x|\cdot)$)]
If $M \subset H$ is a subspace,

$$M^\perp := \bigcap_{x \in M} x^\perp$$

[$M^\perp$ is also a closed subspace]

3/6 MEASURE THEORY

THEOREM: Suppose H is a Hilbert space, M closed subspace of H. Then $\exists\ P: H \to M$ and $Q: H \to M^\perp$ such that $\forall x \in H$, $x = Px + Qx$. P and Q are unique. Moreover

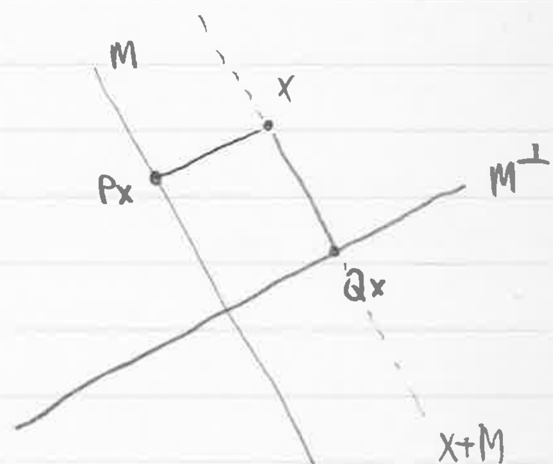
i) $x \in M \Rightarrow Px = x$, $Qx = 0$

ii) $x \in M^\perp \Rightarrow Qx = x$, $Px = 0$

iii) $\|Px - x\| = \inf\{\|y - x\| : y \in M\}$

iv) $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$

v) P, Q are linear



Proof. Note $x + M$ is closed and convex. Let Qx be the unique element of $x+M$ with smallest norm. Let $Px := x - Qx$. Since $Qx \in x + M$, $Px \in M$. Want to show $Qx \in M^\perp$.

Let $z := Qx$. Select $y \in M$, $\|y\| = 1$, and show $(z|y) = 0$. For any scalar α, $z - \alpha y \in x + M$. Hence for all α

$$(z|z) \leq (z - \alpha y \mid z - \alpha y) = (z|z) - \alpha(y|z) - \bar{\alpha}(z,y) + |\alpha|^2$$

Set $\alpha := (z|y)$. Then

$$0 \leq -|\alpha|^2 - |\alpha|^2 + |\alpha|^2 = -|\alpha|^2$$

and so $\alpha = 0$. Therefore $Qx \in M^\perp$.

Uniqueness: Suppose $x = x_1 + x_2$, where $x_1 \in M$, $x_2 \in M^\perp$

Then $Px - x_1 = x_2 - Qx$. But $Px - x_1 \in M$ and $x_2 - Qx \in M^\perp$ and $M \cap M^\perp = \{0\}$. Therefore $x_1 = Px$ and $x_2 = Qx$.

(i), (ii) follows immediately from uniqueness since $x = x + 0$

(iii) follows from definition of Qx

(iv) $(x|x) = (Px + Qx \mid Px + Qx) = (Px|Px) + (Qx|Qx)$

(v)
$$\alpha x = P(\alpha x) + Q(\alpha x)$$
$$\beta y = P(\beta y) + Q(\beta y)$$
$$\alpha x + \beta y = P(\alpha x + \beta y) + Q(\alpha x + \beta y)$$

Subtract
$$0 = \underbrace{P(\alpha x + \beta y) - P(\alpha x) - P(\beta y)}_{\in M} + \underbrace{Q(\alpha x + \beta y) - Q(\alpha x) - Q(\beta y)}_{\in M^\perp}$$

Hence P and Q are linear ▨

Example: $H = L^2[-\pi, \pi]$ $M = C[-\pi, \pi]$.
M is a dense subspace of H (not closed). Hence $M^\perp = (0)$ so we can't write $x = Px + Qx$ for $x \notin M$ with $Px \in M$ and $Qx \in M^\perp$.

<u>COROLLARY</u>: If M is a closed subspace of H, $M \neq H$, then $M^\perp \neq \{0\}$.

Proof. Let $x \notin M$. Then $Px \neq x$, so $Qx \neq 0$.

(*) **THEOREM**: Suppose $L : H \to \mathbb{C}$ is linear and continuous. Then there is a unique $y \in H$ such that

$$Lx = (x|y) \quad \forall x \in H$$

Proof. If $L = 0$ then $y = 0$ works. Note $y = 0$ is the only choice since $Ly = \|y\|^2 \neq 0$ if $y \neq 0$.

Suppose $L \neq 0$. Set

$$M := \{ x \in H : Lx = 0 \}$$

Then M is a closed proper subspace of H. Let $z \in M^{\perp}$, $\|z\| = 1$. Define for $x \in H$,

$$u_x := (Lx)z - (Lz)x$$

Note that $L(u_x) = 0$, so $u_x \in M$. Therefore

$$0 = (u_x|z) = Lx\,(z|z) - Lz\,(x|z)$$

$$= Lx - (x|(\overline{Lz})z)$$

Set $y := (\overline{Lz})z$. Then the above shows that $\forall x \in H$, $Lx = (x|y)$.

$\blacksquare$

<u>DEFINITION</u>: H Hilbert space. $\{u_\alpha : \alpha \in A\} \subset A$ is an orthonormal family if

$$(u_\alpha \mid u_\beta) = \begin{cases} 0 & \alpha \neq \beta \\ 1 & \alpha = \beta \end{cases}$$

If $(u_\alpha : \alpha \in A)$ is an orthonormal family, then for any $x \in H$, $(x \mid u_\alpha)$ is called the α^{th} Fourier coefficient of x (relative to $(u_\alpha : \alpha \in A)$)

Classical case: $H = L^2\left([-\pi,\pi], \frac{d\theta}{2\pi}\right)$ ← Lebesgue measure divided by 2π

Let $u_n(t) := e^{int}$ for $n \in \mathbb{Z}$.
Gives an orthonormal family
If $f \in H$, its n^{th} Fourier coefficient is

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} f(t)\, dt$$

The Fourier series of f is $\sum_n \hat{f}(n) e^{int}$.

<u>PROPOSITION</u>: Suppose $\{u_i : i \in \mathbb{N}_n\}$ is an orthonormal set in H. Let

$$x = \sum_{i=0}^{n} c_i u_i$$

Then $c_i = (x \mid u_i)$ and $\|x\|^2 = \sum_{i=0}^{n} |c_i|^2$. In particular $(u_i : i \in \mathbb{N}_n)$ is linearly independent

3/8 MEASURE THEORY

Recall: If $(u : u \in F)$ is a finite orthonormal family in H and

$$x = \sum_F c_u u$$

then $c_u = (x|u)$ and $\|x\|^2 = \sum_F |c_u|^2$.

Rephrase as follows: Given F a finite orthonormal family, let $M = \text{span}\{u : u \in F\}$. The map from M into ℓ^2 (counting measure on $\mathbb{N}_k$) (where $k = |F|$), given by

$$\forall x \in M \qquad x \longmapsto \left((x|u_1), (x|u_2), \dots (x|u_k) \right)$$

is norm preserving.

__THEOREM__: Suppose F is a finite orthonormal family in H. For every $x \in H$,

$$(*) \qquad \left\| x - \sum_F (x|u)\, u \right\| \leq \left\| x - \sum_F \lambda_u u \right\|$$

for any family $(\lambda_u : u \in F)$ of scalars. Equality holds if and only if $\lambda_u = (x|u)$ $\forall u \in F$

The projection of x into the (necessarily closed) subspace M of H spanned by F is $\sum_F (x|u)\, u$. If $\delta = d(x, M)$, then

$$(**) \qquad \sum_F |(x|u)|^2 = \|x\|^2 - \delta^2$$

Proof. $(*)$ is equivalent to

$$(x|x) - \sum_F \overline{(x|u)}(x|u) - \sum_F (x|u)(u|x) + \sum_F (x|u)\overline{(x|u)}$$
$$\leqslant (x|x) - \sum_F \lambda_u (u|x) - \sum_F \overline{\lambda_u}(x|u) + \sum_F \lambda_u \overline{\lambda_u}$$

which is equivalent to

$$2\,\mathrm{Re} \sum_F \lambda_u (u|x) \leqslant \sum_F |(x|u)|^2 + \sum_F |\lambda_u|^2$$

Now

$$\mathrm{Re} \sum_F \lambda_u (u|x) \underset{(\dagger)}{\leqslant} \sum_F |\lambda_u|\,|(u|x)| \leqslant \Big(\sum_F |\lambda_u|^2\Big)^{1/2} \Big(\sum |(u|x)|^2\Big)^{1/2}$$

Schwartz inequality in $\ell^2(\mathbb{N}_{|F|})$

$$\leqslant \frac{1}{2}\Big(\sum_F |\lambda_u|^2 + \sum_F |(u|x)|^2 \Big)$$

geo. mean $\leqslant$ arith. mean

Equality holds iff $\underset{(\dagger)}{\lambda_u (u|x) \geqslant 0}$ and $\underset{(\text{Schwartz})}{|\lambda_u| = c\,|(u|x)|}$

and $\underset{(\text{geo = arith})}{c = 1}$. Hence $\lambda_u (u|x) \geqslant 0$ and $|\lambda_u| = |(u|x)|$. Therefore

$$\lambda_u = \overline{(u|x)} = (x|u) \quad \forall u \in F$$

If $x \notin M$, $(*) \Rightarrow \mathrm{dist}(x, M) \geqslant \|x - \sum (x|u)u\| > 0$

Hence M is closed.

Recall that if $P: H \to M$ is the projection onto M (so $x = Px + Qx$, $Qx \in M^\perp$), then $\|x - Px\| \le \|x - y\|$ $\forall y \in M$ so by (*), $Px = \sum_i' (x|u)u$. Also

$$\delta^2 = \|x - \sum_F' (x|u)u\|^2 = \|x\|^2 - \sum_F |(x|u)|^2 \qquad \square$$

<u>COROLLARY</u>: If $(u_\alpha : \alpha \in A)$ is an orthonormal family in H, then

$$\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 \le \|x\|^2 \qquad \left(\begin{array}{l}\text{Bessel's}\\ \text{Inequality}\end{array}\right)$$

(where $\hat{x}(\alpha)$ is the α^{th} Fourier coefficient of x)

$[\![\ \sum_{\alpha \in A}$ is sup of all sums over finite subsets of A $]\!]$

Proof: Follows from (**) $\square$

<u>COROLLARY</u>: Only countably many $\hat{x}(\alpha) \neq 0$ for any particular $x \in H$.

Let $\ell^2(A) = L^2(A$, counting measure$)$. Notice that Bessel's inequality tells us that the mapping from H into $\ell^2(A)$ given by $x \longrightarrow \hat{x}$ is a linear norm-decreasing mapping

RIESZ - FISCHER THEOREM: Let H be a Hilbert space and $(u_\alpha : \alpha \in A)$ an orthonormal family. Given $\varphi \in \ell^2(A)$, then $\exists\, x \in H$ such that $\hat{x} = \varphi$ (In other words, $x \to \hat{x}$ maps H onto $\ell^2(A)$)

Proof. For $n \in \mathbb{N}$, let

$$A_n := \{\alpha \in A : |\varphi(\alpha)| > 1/n\}$$

Since $\varphi \in \ell^2(A)$, A_n is finite. Define

$$x_n := \sum_{\alpha \in A_n} \varphi(\alpha)\, u_\alpha \qquad \text{(finite sum)}$$

(1) CLAIM: $\hat{x}_n = \varphi \chi_{A_n}$

If $\beta \in A_n$, $\hat{x}_n(\beta) = (x_n | u_\beta) = \varphi(\beta)$ ✓
If $\beta \notin A_n$, $\hat{x}_n(\beta) = (x_n | u_\beta) = 0$

(2) CLAIM: $\hat{x}_n \to \varphi$ pointwise on A

If $\varphi(\beta) = 0$, then $\hat{x}_n(\beta) = 0$ since $\beta \notin A_n$ for any n. If $\varphi(\beta) \neq 0$, then $\hat{x}_n(\beta) = \varphi(\beta)$ eventually.

Claim (1) also shows that $|\hat{x}_n - \varphi|^2 \leq |\varphi|^2$ on A. Since $\hat{x}_n - \varphi \to 0$ pointwise on A and is dominated by an integrable function ($|\varphi|^2$), the DCT says that $\|\hat{x}_n - \varphi\|_2 \to 0$ in $\ell^2(A)$. Hence $\hat{x}_n$ is Cauchy in $\ell^2(A)$. But

$$\|\hat{x}_n - \hat{x}_m\|_2 = \|x_n - x_m\|_H \qquad [\![x_k \text{ finite sum }]\!]$$

Hence (x_n) is Cauchy in H, and so converges to some $x \in H$.
For any $\alpha \in A$

$$\hat{x}(\alpha) = (x \mid u_\alpha) = \lim_{n\to\infty} (x_n \mid u_\alpha) = \lim_{n\to\infty} \hat{x}_n(\alpha) = \varphi(\alpha)$$

Hence $\hat{x} = \varphi$. $\boxtimes$

3/10 MEASURE THEORY

<u>**THEOREM**</u>: Suppose H is a Hilbert space and $\{u_\alpha : \alpha \in A\}$ is an orthonormal family in H. TFAE

i) $\{u_\alpha : \alpha \in A\}$ is a maximal orthonormal family

ii) The set S of finite linear combinations of members of this family is dense in H

iii) $x \in H \Rightarrow \|x\|^2 = \sum_{\alpha \in A} |\hat{x}(\alpha)|^2$ (Parseval's Theorem)

iv) $\forall x, y \in H$, $(x|y) = \sum_{\alpha \in A} \hat{x}(\alpha)\overline{\hat{y}(\alpha)}$

Proof. i) $\Rightarrow$ (ii) Suppose ii) does not hold, i.e. $M := cl(S) \neq H$. Note that M is a subspace and closed. Since $M \neq H$, $M^\perp \neq \{0\}$. Let $u \in M^\perp$, $u \neq 0$. Then $u/\|u\| \in M^\perp$ and

$$\left(\frac{u}{\|u\|} \,\Big|\, u_\alpha\right) = 0 \quad \forall \alpha$$

Adjoin $u/\|u\|$ to $\{u_\alpha : \alpha \in A\}$ to obtain a larger orthonormal family.

(ii) $\Rightarrow$ (iii) Given $\varepsilon > 0$, $x \in H$, (ii) says $\exists F \subset A$ finite and $(c_\alpha : \alpha \in F) \subset \mathbb{C}$ s.t.

$$\left\|x - \sum_{\alpha \in F} c_\alpha u_\alpha\right\| < \varepsilon$$

Recall that the best approximation is with $(x|u_\alpha)$, so that

$$\left\|x - \sum_{\alpha \in F} (x|u_\alpha) u_\alpha\right\| \leq \left\|x - \sum_{\alpha \in F} c_\alpha u_\alpha\right\| < \varepsilon$$

Then

$$\|x\| \leq \Big\| \sum_F (x|u_\alpha) u_\alpha \Big\| + \varepsilon = \Big(\sum_{\alpha\in F} |(x|u_\alpha)|^2 \Big)^{1/2} + \varepsilon$$

and so

$$(\|x\| - \varepsilon)^2 \leq \sum_{\alpha\in F} |(x|u_\alpha)|^2 \underset{\substack{\uparrow \\ \text{Bessel}}}{\leq} \|x\|^2$$

Therefore $\|x\|^2 = \sum_{\alpha\in A} |(x|u_\alpha)|^2$.

(iii) $\Rightarrow$ (iv) What (iii) says is that $\|x\| = \|\hat{x}\|_2 \quad \forall x\in H$
Let $\lambda \in \mathbb{C}$. Then

$$(x+\lambda y \,|\, x+\lambda y) = (\widehat{x+\lambda y} \,|\, \widehat{x+\lambda y}) = (\hat{x}+\lambda\hat{y} \,|\, \hat{x}+\lambda\hat{y})$$

$$\Rightarrow \qquad \lambda(y|x) + \bar{\lambda}(x|y) = \lambda(\hat{y}|\hat{x}) + \bar{\lambda}(\hat{x}|\hat{y})$$

$$\Rightarrow \qquad \mathrm{Re}\, \lambda(y|x) = \mathrm{Re}\, \lambda(\hat{y}|\hat{x}) \quad \forall \lambda\in\mathbb{C}$$

Setting $\lambda = 1$ and then $\lambda = i$, we see that $(y|x) = (\hat{y}|\hat{x})$, i.e.

$$(y|x) = (\hat{y}|\hat{x}) = \sum_{\alpha\in A} \hat{y}(\alpha)\overline{\hat{x}(\alpha)}$$

(iv) $\Rightarrow$ (i) Suppose $(u_\alpha : \alpha\in A)$ is not maximal. Then
$\exists\, u \notin (u_\alpha : \alpha \in A)$ such that $\hat{u}(\alpha) = (u|u_\alpha) = 0 \quad \forall \alpha\in A$.
$u \neq 0$

Hence

$$(u|u) \neq 0 = \sum_{\alpha \in A} \hat{u}(\alpha)\, \overline{\hat{u}(\alpha)}$$

so (4) does not hold.

$\boxed{\mathbin{/\!/}}$

Summary: If $(u_\alpha : \alpha \in A)$ is a maximal orthonormal family, then the mapping from H onto $\ell^2(A)$ given by $x \to \hat{x}$ is a Hilbert space isomorphism.

Remark: Every orthonormal family in H is contained in some maximal orthonormal family. Hence any Hilbert space is isomorphic to $\ell^2(A)$ for some A.

Classical Case

$$H = L^2\left([-\pi, \pi],\ \frac{d\theta}{2\pi} \right)$$

normalized Lebesgue measure

$$T = \{ z \in \mathbb{C} : |z| = 1 \}$$

$C(T)$ = continuous complex-valued functions on T
($\Leftrightarrow$ cont. complex-valued functions on $\mathbb{R}$ with period 2π)

Claim: $\{ e^{int} : n \in \mathbb{Z} \}$ is an orthonormal family in H

$$\left(e^{int} \mid e^{imt}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int} e^{-imt}\,dt = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

Let $S_N(x,f) = S_N(x) := \sum_{k=-N}^{N} \hat{f}(k) e^{ikx}$ for $f \in L^2[-\pi,\pi]$

S_N is the N^{th} partial sum of the Fourier series of f.

<u>FEJÉR'S THEOREM</u>: Suppose $f \in C(T)$. Let

$$\sigma_N(x,f) = \sigma_N(x) := \frac{1}{N+1}\sum_{k=0}^{N} S_k(x)$$

Then $\sigma_N \to f$ uniformly on $[-\pi,\pi]$

Proof later

Clearly $\sigma_N(x) = \sum_{j=-N}^{N} c_j e^{ijx}$ for some choice of c_j's

trigonometric polynomial of degree N

We know $C(T)$ are dense in H. Then by Fejér's theorem
S = set of finite linear combinations of $\{e^{inx} : n \in \mathbb{Z}\}$ (trig poly)
is dense in H, and so $\{e^{inx} : n \in \mathbb{Z}\}$ is maximal.

Suppose $f \in L^2[-\pi,\pi]$

$$\hat{f}(n) = (f \mid e^{inx}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int}\,dt$$

Then $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\,dt = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$. Also, if $g \in L^2[-\pi,\pi]$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$$

For $f \in L^2[-\pi,\pi]$

$$\widehat{f - S_N}(k) = \begin{cases} \hat{f}(k) & |k| > N \\ 0 & |k| \leq N \end{cases}$$

By Parseval's theorem

$$\| f - S_N \|_2^2 = \sum_{|k|>N} |\hat{f}(k)|^2 \to 0 \quad \text{as } N \to \infty$$

Hence $S_N \to f$ in $L^2[-\pi,\pi]$, so $\exists\, S_{n_j}$ s.t. $S_{N_j}(x) \to f(x)$ a.e.

What trig polynomial of degree N best approximates f in L^2 sense?

Answer - S_N

$$\| f - S_N \| \leq \| f - \sum_{k=-N}^{N} c_k e^{ikx} \| \quad \forall c_k \in \mathbb{C}$$

3/13 MEASURE THEORY

PROPOSITION: Define the Dirichlet kernel by

$$D_m(x) := \sum_{k=-m}^{m} e^{ikx}$$

for $m \in \mathbb{N}$. Define the Fejér kernel by

$$K_n(x) := \frac{1}{n+1} \sum_{m=0}^{n} D_m(x)$$

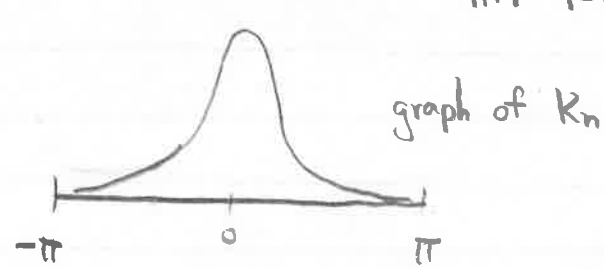
for $n \in \mathbb{N}$. Then

(1) $$D_m(x) = \frac{\sin\left(m+\frac{1}{2}\right)x}{\sin\frac{x}{2}}$$

(2) $$K_n(x) = \frac{1}{n+1} \frac{1-\cos(n+1)x}{1-\cos x}$$

(3) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_m(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1$$

(4) $0 \le K_n(x)\ \forall x$ and $K_n(x) \le \frac{2}{n+1}\frac{1}{1-\cos\delta}$ for $\delta \le |x| \le \pi$



Proof.

$$1*) \quad (e^{ix}-1)\, D_m(x) = e^{i(m+1)x} - e^{-imx}$$

Multiply this by $e^{-ix/2}$:

$$2i \sin\frac{x}{2}\, D_m(x) = e^{i(m+1/2)x} - e^{-i(m+1/2)x} = 2i \sin(m+1/2)x$$

$$\Rightarrow D_m(x) = \frac{\sin(m+1/2)x}{\sin x/2}$$

(*) also implies that

$$(n+1)\, K_n(x)(e^{ix}-1) = \sum_{m=0}^{n} \left(e^{i(m+1)x} - e^{-imx}\right)$$

$$= \sum_{j=-n}^{n+1} c_j\, e^{ijx} \qquad c_j = \begin{cases} 1 & 1 \le j \le n+1 \\ -1 & -n \le j \le 0 \end{cases}$$

Hence

$$(n+1)\, K_n(x)(e^{ix}-1)(e^{-ix}-1) = -e^{i(n+1)x} - e^{-i(n+1)x} + 2$$

$$= 2 - 2\cos(n+1)x$$

$$\Rightarrow (n+1)\, K_n(x) = \frac{2-2\cos(n+1)x}{2-2\cos x} = \frac{1-\cos(n+1)x}{1-\cos x}$$

(4) now follows immediately from (2). (3) also easy.

$\boxtimes$

FEJÉR'S THEOREM: Suppose $f \in C(T)$ (i.e. f is continuous, complex-valued, period 2π). Let

$$S_N(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{ikx}$$

and

$$\sigma_N(x) = \frac{1}{N+1} \sum_{k=0}^{N} S_k(x)$$

Then $\sigma_N \to f$ uniformly on $[-\pi, \pi]$

Proof.

$$S_N(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{ikx} = \sum_{k=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt \right) e^{ikx}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{k=-N}^{N} e^{ik(x-t)} dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_N(x-t)\, dt \qquad [\![u = x-t]\!]$$

$$= \frac{1}{2\pi} \int_{x+\pi}^{x-\pi} f(x-u) D_N(u)(-du)$$

$\left(f(x-u) D_N(u) \right.$ has period 2π, so may replace x by 0 in limits$\left.\right)$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t)\, dt$$

Then

$$\sigma_n(x) = \frac{1}{n+1} \sum_{N=0}^{n} S_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) \left[\frac{1}{n+1} \sum_{N=0}^{n} D_N(t) \right] dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)\, K_n(t)\, dt$$

Hence

$$\sigma_n(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[f(x-t) - f(x) \right] K_n(t)\, dt$$

(since $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(t)\, dt = 1$), and so

$$\left| \sigma_n(x) - f(x) \right| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(x-t) - f(x) \right| K_n(t)\, dt$$

(since $K_n(t) \geq 0$!) Since f is continuous, $\exists\, M$ s.t. $|f(y)| \leq M \;\; \forall y \in [-\pi, \pi]$. Also, given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/2$$

Since $K_n(t) \leq \frac{2}{n+1} \frac{1}{1-\cos\delta}$ $\forall$ $\delta \leq |t| \leq \pi$, $\exists$ $L \in \mathbb{N}$ such that $\forall n \geq L$,

$$\delta \leq |t| \leq \pi \Rightarrow K_n(t) < \frac{\varepsilon}{4M}$$

Thus, for all $n \geq L$

$$\frac{1}{2\pi}\int_{-\delta}^{\delta} |f(x-t) - f(x)|\, K_n(t)\,dt \leq \frac{\varepsilon}{2}$$

$$\frac{1}{2\pi}\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) \leq 2M \cdot \frac{1}{2\pi}\int_{\delta \leq |t| \leq \pi} K_n(t)\,dt$$

$$\leq 2M \cdot \frac{1}{2\pi} \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{4\pi} < \frac{\varepsilon}{2}$$

Hence $\forall x$

$$|\sigma_n(x) - f(x)| \leq \varepsilon \quad \forall n \geq L$$

$\boxtimes$

Note

$$|S_N(x) - f(x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)|\, |D_N(t)|\,dt$$

and $\int_{-\pi}^{\pi} |D_N(t)|\,dt > c \cdot \log N$

COROLLARY: $\{e^{inx} : n \in \mathbb{Z}\}$ is a maximal orthonormal system in $L^2[-\pi,\pi]$.

[Add to our list of observations on $L^2[-\pi,\pi]$:

Given $(c_n : n \in \mathbb{Z}) \in \ell^2(\mathbb{Z})$, i.e. $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$, $\exists\, f \in L^2[-\pi,\pi]$ s.t.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt$$

(Riesz-Fischer)]

Question - Does $S_N(x) \to f(x)$ say for $f \in C(T)$?

Not true $\forall f$ and $\forall x$. However, it is true if $f \in BV[-\pi,\pi]$ (in fact uniform convergence)

THEOREM: If $f \in L^2[-\pi,\pi]$, then $S_N(f) \to f(x)$ a.e.

(Proof mid 1960's)

3/15 MEASURE THEORY

BANACH SPACES

Examples of Banach spaces

1) $L^p(\mu)$ $1 \le p \le \infty$
2) Hilbert spaces
3) $\mathbb{C}$
4) $C(T)$ with supremum norm

RECALL: BAIRE CATEGORY THEOREM If X is a complete metric space and $\mathcal{O}_n$ is a seq. of dense open sets. Then $\bigcap \mathcal{O}_n$ is dense (and hence non-empty)

COROLLARY: X complete metric space, G_n seq. of dense G_δ-sets. Then $\bigcap G_n$ is dense

Proof: Each $G_n = \bigcap_{i=1}^{\infty} \mathcal{O}_{n,i}$, each $\mathcal{O}_{n,i}$ open, dense

$$\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \bigcap_{i=1}^{\infty} \mathcal{O}_{n,i} \quad \text{dense in } X$$

$\blacksquare$

UNIFORM BOUNDEDNESS THEOREM (BANACH-STEINHAUS)
Suppose X is a Banach space and Y is a normed linear space. $\{\Lambda_\alpha : \alpha \in A\} \subset \mathcal{B}(X,Y)$. Then one of the following (dramatically different) alternatives must occur

(1) $\exists M > 0$ s.t. $\|\Lambda_\alpha\| \leq M \quad \forall \alpha \in A$

(2) $\sup\limits_{\alpha \in A} \|\Lambda_\alpha x\| = \infty$ for a dense G_δ subset of X

(Hence pointwise boundedness $\Rightarrow$ uniform boundedness)

Proof. Define $\varphi : X \longrightarrow [0, \infty]$ by

$$\varphi(x) = \sup_{\alpha \in A} \|\Lambda_\alpha x\|$$

Define for $n \in \mathbb{N}$

$$V_n := \{x \in X : \varphi(x) > n\}$$

Note for a fixed $\alpha \in A$, $\Lambda_\alpha x$ is a continuous function of x. Hence $\|\Lambda_\alpha x\|$ is continuous. Therefore $\sup \|\Lambda_\alpha x\|$ is lower semi-continuous. Hence V_n is open

Suppose $\exists N \in \mathbb{N}$ s.t. V_N is not dense. Then $\exists x_0 \in X$ and $r > 0$ such that

$$\|x\| \leq r \implies x + x_0 \notin V_N$$

Therefore $\|x\| \leq r \implies \varphi(x + x_0) \leq N \Rightarrow \|\Lambda_\alpha(x + x_0)\| \leq N \quad \forall \alpha \in A$

In particular $\|\Lambda_\alpha(x_0)\| \leq N \quad \forall \alpha \in A$. Hence if $\|x\| \leq r$

$$\|\Lambda_\alpha x\| = \|\Lambda_\alpha(x + x_0) - \Lambda_\alpha(x)\| \leq 2N \quad \forall \alpha \in A$$

Therefore if $\|u\| = 1$

$$\|\Lambda_\alpha u\| \leq \frac{2N}{r} \quad \forall \alpha \in \Lambda$$

and so (1) holds $(M = 2N/r)$

If each V_N is dense, then $E = \bigcap_{n=1}^{\infty} V_n$ is dense
If $x \in E$, then $\varphi(x) > n \;\; \forall n \in \mathbb{N}$, i.e. $\varphi(x) = \infty$. Hence (2) holds.

$\blacksquare$

Question: Suppose $f \in C(T)$. Does $S_n(x,f) \to f(x)$ $\forall x \in [-\pi,\pi]$?

Answer: No

Recall $S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\, D_n(t)\, dt$, where

$$D_n(t) = \frac{\sin(n+1/2)t}{\sin t/2}$$

Define $\Lambda_n : C(T) \to \mathbb{C}$ by

$$\Lambda_n(f) := S_n(0,f) \quad \forall f \in C(T)$$

$$|\Lambda_n(f)| = \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-t)\, D_n(t)\, dt \right|$$

$$\leq \|f\|_\infty \, \|D_n\|_1$$

Hence $\|\Lambda_n\| \leq \|D_n\|_1$

We will show that $\|\Lambda_n\| = \|D_n\|_1 \longrightarrow \infty$ as $n \to \infty$. Hence by the uniform boundedness principal, there is a dense set of f in $C(T)$ s.t.

$$\sup_n |\Lambda_n f| = \sup_n |S_n(0,f)| = \infty$$

and so for this large collection we have $S_n(0,f) \not\to f(0)$.

3/17 MEASURE THEORY

Consider $\Lambda_n : C(T) \longrightarrow \mathbb{C}$ given by

$$\Lambda_n(f) = S_n(f,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) D_n(0-t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) D_n(t)\,dt$$

By Hölder's inequality

$$|\Lambda_n f| \leq \|f\|_\infty \|D_n\|_1$$

and so $\|\Lambda_n\| \leq \|D_n\|_1$

CLAIM: $\|\Lambda_n\| = \|D_n\|_1 \longrightarrow \infty$

$$\|D_n\|_1 = \frac{1}{\pi}\int_0^{\pi} \frac{|\sin(n+1/2)t|}{\sin t/2}\,dt$$

$$\geq \frac{2}{\pi}\int_0^{\pi} \frac{|\sin(n+1/2)t|}{t}\,dt \qquad [[\sin x \leq x \text{ for } x \geq 0]]$$

$$= \frac{2}{\pi}\int_0^{(n+1/2)\pi} \frac{|\sin u|}{u}\,du$$

$$\geq \frac{2}{\pi}\sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi} \frac{|\sin u|}{u}\,du$$

$$\geq \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin u| \, du$$

$$= \frac{4}{\pi^2} \sum_{k=1}^{n} \frac{1}{k} \longrightarrow \infty \quad \text{as } n \to \infty$$

Define

$$g_n(t) = \begin{cases} 1 & \text{if } D_n(t) \geq 0 \\ -1 & \text{if } D_n(t) < 0 \end{cases}$$

g_n is a step function. It's elementary that $\exists f_j \in C(T)$ such that $\|f_j\|_\infty = 1$ and $f_j(t) \to g_n(t)$ as $j \to \infty$ for each $t \in [-\pi, \pi]$. Now

$$\|\Lambda_n\| = \sup_{\|f\|_\infty = 1} |\Lambda_n f|$$

and $\Lambda_n f_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_j(t) D_n(t) \, dt$. By D.C.T, as $j \to \infty$

$$\Lambda_n f_j \longrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(t) D_n(t) \, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| \, dt$$

$$= \|D_n\|$$

Hence $\|\Lambda_n\| \geq \|D_n\|$. But we saw earlier that $\|D_n\| \geq \|\Lambda_n\|$.

This establishes the claim. Now the Uniform Boundedness principle says $\exists$ dense G_δ-set $E \subset C(\mathbb{T})$ s.t.

$$\sup_{n \in \mathbb{N}} |S_n(f,0)| = +\infty \quad \forall f \in E$$

and so $S_n(f,0)$ does not converge

There is nothing special about 0. $\forall x \in [-\pi,\pi]$ $\exists$ a dense G_δ-set $E_x \subset C(\mathbb{T})$ s.t.

$$\sup |S_n(f,x)| = \infty \quad \forall f \in E_x$$

Let $S^*(f,x) = \sup_n |S_n(f,x)|$. Fix f. Then $S^*(f,x)$ is a supremum of continuous functions and so is lower semicontinuous. It is the case that for each $f \in C(\mathbb{T})$ $\{x : S^*(f,x) = \infty\}$ is a G_δ-set.

Let (x_n) be a dense sequence in $[-\pi,\pi]$. Associate with each x_i a set $E_{x_i} \subset C(\mathbb{T})$ s.t. E_{x_i} is a dense G_δ-set and

$$S^*(f,x_i) = \infty \quad \forall f \in E_{x_i}$$

Let $E = \bigcap_{i=1}^{\infty} E_{x_i}$. E is a dense G_δ set. Suppose $f \in E$

$$S^*(f,x_i) = \infty \quad \forall i \in \mathbb{N}$$

Then for each $f \in E$, $\{x \in [-\pi,\pi] : S^*(f,x) = \infty\}$ is a dense

G_δ - set

SUMMARY: There is a dense G_δ set $E \subset C(T)$ s.t. for every $f \in E$, $S_n(f,x)$ diverges for all $x \in F$, where F is a dense G_δ set in $[-\pi,\pi]$.

Remark: If X is a complete metric space with no isolated points, a dense G_δ is uncountable.

OPEN MAPPING THEOREM

Suppose X and Y are both Banach spaces. Suppose $\Lambda : X \to Y$ is a bounded linear transformation onto Y. Let $U = \{x \in X : \|x\| \le 1\}$ and $V = \{y \in Y : \|y\| \le 1\}$. Then $\exists\, \delta > 0$ s.t.

$$\delta V \subset \Lambda(U)$$

Remark: It follows from the linearity of Λ that the image of every open set in X is an open set in Y.

Observation: Suppose X is a complete metric space. If $X = \cup E_n$, then $\exists\, n$ s.t. $\mathrm{int}(\overline{E_n}) \neq \emptyset$ (Baire Cat. Thm)

Proof: Since Λ is onto

$$Y = \bigcup_{k=1}^{\infty} \Lambda(kU)$$

Y complete $\Rightarrow \exists\, k$ s.t. $\overline{\Lambda(kU)}$ contains W open, $W \neq \emptyset$.
$\exists\, y_0 \in W$, $\eta > 0$ s.t. $\|y\| \le \eta \Rightarrow y_0 + y \in W$. $\exists\, x_i' \in kU$ s.t. $\Lambda x_i' \to y_0$. For $\|y\| \le \eta$, $\exists\, x_i'' \in kU$ s.t. $\Lambda x_i'' \to y_0 + y$. Let $x_i = x_i'' - x_i'$. Then $\Lambda x_i \to y$
$(x_i) \subset (2k)U$

If $\|y\| = \eta$, $\exists (x_i) \subset (2k)U$ s.t. $\Lambda x_i \to y$
Let $\delta = \eta/2k$. If $\|y\| = \eta$, $\exists (x_i)$ s.t. $\|x_i\| \le \delta^{-1}\|y\|$ s.t. $\Lambda x_i \to y$. But now Λ linear $\Rightarrow \forall y$, $\exists (x_i)$ s.t. $\|x_i\| \le \delta^{-1}\|y\|$ and $\Lambda x_i \to y$

(*) For any $\varepsilon > 0$, $y \in Y$, $\exists\, x \in X$ s.t. $\|x\| < \delta^{-1}\|y\|$ s.t. $\|\Lambda x - y\| < \varepsilon$.

Suppose $\|y\| < \delta$. By (*) $\exists\, x_1$, $\|x_1\| < 1$ s.t. $\|\Lambda x_1 - y\| < \frac{1}{2}\delta\varepsilon$. Suppose $x_1, \dots, x_n$ have been chosen s.t.

$$\|y - \Lambda x_1 - \Lambda x_2 - \dots - \Lambda x_n\| < 2^{-n}\delta\varepsilon$$

Choose, by (*), $x_{n+1} \in X$, $\|x_{n+1}\| < 2^{-n}\varepsilon$ s.t.

$$\|(y - \Lambda x_1 - \dots - \Lambda x_n) - \Lambda x_{n+1}\| < 2^{-(n+1)}\delta\varepsilon$$

Let $s_n = x_1 + \dots + x_n$. Then s_n is Cauchy in X since $\|x_{n+1}\| < 2^{-n}\varepsilon$. Therefore $s_n \to x$. $\Lambda s_n \to \Lambda x$. But $\Lambda s_n \to y$ also, so $y = \Lambda x$. Now $\|x\| < 1 + \varepsilon$, so

$$\delta V \subset \Lambda((1+\varepsilon)U)$$

and so

$$(1+\varepsilon)^{-1}\,\delta V \subset \Lambda(U) \quad \forall \varepsilon>0$$

$$\Rightarrow \quad \delta V \subset \Lambda(U)$$

$\blacksquare$

3/27 ANALYSIS

COROLLARY : $\Lambda : X \longrightarrow Y$ 1-1, onto, linear, and bounded. X, Y Banach spaces. Then $\exists\, \delta > 0$ s.t.

$$\|\Lambda x\| \geq \delta \|x\|$$

$\forall x \in X$ (and so Λ^{-1} is bounded, with $\|\Lambda^{-1}\| \leq 1/\delta$).

Proof. Let δ be as in Open Mapping Theorem. If $\|\Lambda x\| < \delta$, then $\|x\| < 1$, and so if $\|x\| \geq 1$, we have $\|\Lambda x\| \geq \delta$. In particular

$$\left\|\Lambda\left(\frac{x}{\|x\|}\right)\right\| \geq \delta \quad \forall x \neq 0$$

$$\Longrightarrow \|\Lambda x\| \geq \delta \|x\| \quad \forall x \in X$$

$\boxtimes$

RIEMANN - LEBESQUE LEMMA : If $f \in L^1[-\pi, \pi]$, then

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt \longrightarrow 0$$

as $|n| \to \infty$.

Proof. There is a continuous g on $[-\pi, \pi]$ such that

$$\|g - f\|_1 < \varepsilon$$

WLOG, assume $g(-\pi) = g(\pi)$ (can modify g on a small set)
Thus $g \in C(\mathbb{T})$, so by Fejér's theorem, there is a trig. polynomial P s.t.

$$\|P - g\|_1 \leq \|P - g\|_\infty < \varepsilon$$

Hence $\|f - P\|_1 < 2\varepsilon$.

Suppose $|n| > \deg P$. Then

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(t) - \underset{\substack{\uparrow \\ \text{contributes 0 to integral}}}{P(t)})\, e^{-int}\, dt$$

and so

$$|\hat{f}(n)| \leq \|f - P\|_1 \, \|e^{-int}\|_\infty = \|f - P\|_1 < 2\varepsilon$$

whenever $|n| > \deg P$.

$\blacksquare$

QUESTION: If $(a_n : n \in \mathbb{Z}) \to 0$ as $|n| \to \infty$, does there exist $f \in L^1[-\pi,\pi]$ s.t. $\hat{f}(n) = a_n$?

ANSWER: NO

Recall, Riesz-Fischer theorem tells us that every (a_n) s.t. $\sum a_n^2 < \infty$ is of the form $a_n = \hat{f}(n)$ for some $f \in L^2[-\pi,\pi]$

<u>THEOREM</u>: Define $\Lambda : L^1[-\pi,\pi] \to c_0(\mathbb{Z})$ given by

$$(\Lambda f)_n := \hat{f}(n)$$

($(\hat{f}(n)) \in c_0$ by Riemann-Lebesgue Lemma) Then Λ is a bounded 1-1 linear transformation, but Λ is not onto.

Proof.

$$\|\Lambda f\| = \sup_n |\hat{f}(n)| \underset{\uparrow \text{ Hölder}}{\leq} \|f\|_1 \quad \forall f \in L^1$$

Hence $\|\Lambda f\| \leq 1$.

Suppose $f \in L^1[-\pi,\pi]$ and $\hat{f}(n) = 0$ $\forall n \in \mathbb{Z}$. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt = 0$$

$\forall n$, and so

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, p(t)\, dt = 0$$

for any trig. polynomial p. Suppose $g \in C(\mathbb{T})$. There exist trig. polynomials P_n s.t. $\|P_n - g\|_\infty \to 0$. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, g(t)\, dt = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(g-P_n)}_{\to 0} + \underbrace{\frac{1}{2\pi}\int fP_n}_{=0}$$

Therefore

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt = 0 \qquad \forall g \in C(T)$$

In fact

$$(*) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,g(t)\,dt = 0 \quad \forall g \text{ cont. on } [-\pi,\pi]$$

(can modify g on a small set so $g(\pi)=g(-\pi)$).

By Lusin's theorem, for any measurable $E \subset [-\pi,\pi]$ $\exists$ cont. g_n such that $\|g_n\|_\infty \leq 1$ and $g_n \to \chi_E$ a.e. Therefore

$$\frac{1}{2\pi}\int_E f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\chi_E \underset{\text{D.C.T.}}{=} \lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} f g_n \underset{(*)}{=} 0$$

and so $f=0$ a.e. Hence Λ is 1-1.

If Λ was onto, then the corollary to the Open Mapping theorem would imply that Λ^{-1} is bounded. But consider

$$D_n(t) = \sum_{k=-n}^{n} e^{ikt}$$

$\|D_n\|_1 \to \infty$, while $\|\hat{D}_n\|_{c_0} = 1$. Thus there is no $\delta>0$ s.t.

$$\forall n \quad \|\Lambda D_n\|_{c_0} \geqslant \delta\,\|D_n\|$$

Therefore Λ can not be onto.

$\blacksquare$

3/29 ANALYSIS

DEFINITION: X vector space over $\mathbb{C}$. We say $f: X \to \begin{Bmatrix} \mathbb{R} \\ \mathbb{C} \end{Bmatrix}$ is $\begin{Bmatrix} \text{real} \\ \text{complex} \end{Bmatrix}$ - linear if $f(x+y) = f(x) + f(y)$ and $f(\alpha x) = \alpha f(x)$ for every $\begin{Bmatrix} \text{real} \\ \text{complex} \end{Bmatrix}$ scalar α.

Remark: If f is complex-linear, then $\operatorname{Re} f$ is real-linear

LEMMA: X vector space over $\mathbb{C}$

(1) Suppose $f: X \to \mathbb{C}$ is complex-linear. Let $u = \operatorname{Re} f$. Then $\forall x \in X$, $f(x) = u(x) - i\,u(ix)$

(2) If $u: X \to \mathbb{R}$ is real-linear, then $f(x) := u(x) - i\,u(ix)$ is complex-linear.

(3) X normed linear space over $\mathbb{C}$. If u is a real-linear bounded functional, the complex-linear functional $f(x) = u(x) - i\,u(ix)$ satisfies $\|f\| = \|u\|$

Proof. $u = \operatorname{Re} f \Rightarrow \|u\| \leq \|f\|$. Suppose $x \in X$. $\exists\, \alpha \in \mathbb{C}$ s.t. $|\alpha| = 1$ and $\alpha f(x) = |f(x)|$

since $f(\alpha x)$ is Real

$$|f(x)| = \alpha f(x) = f(\alpha x) = u(\alpha x) \leq \|u\|\,\|\alpha x\| = \|u\|\,\|x\|$$

Hence $\|f\| \leq \|u\|$.

<u>HAHN-BANACH THEOREM</u>: X normed linear space (over $\mathbb{R}$ or $\mathbb{C}$). Let M be a proper subspace. Suppose f is a bounded functional on M. Then f extends to a bounded functional on X, say F, with $\|F\| = \|f\|$.

Specifically, we want to treat these cases:
(1) Field of scalers $= \mathbb{R}$, f real-linear
(2) Field of scalers $= \mathbb{C}$, f real-linear
(3) Field of scalers $= \mathbb{C}$, f complex-linear

Proof. Assume f is real-linear (case (1)).
Nothing to prove if $\|f\| = 0$, so WLOG $\|f\| = 1$. Consider $x_0 \in X - M$, and set

$$M_1 := sp(M \cup \{x_0\})_{\mathbb{R}} = \{ x + \lambda x_0 : x \in M, \lambda \in \mathbb{R} \}$$

Note each member of M_1 is uniquely expressible in the form $x + \lambda x_0$ for $x \in M$ and $\lambda \in \mathbb{R}$. Thus it makes sense to define $f : M_1 \to \mathbb{R}$ by

$$f(x + \lambda x_0) = f(x) + \lambda \alpha$$

where α is a fixed real number at our disposal. Then f is a real-linear functional on M_1 which agrees with its old self on M.

Question: do there a choice of α so that $\|f\|=1$, regarding f as defined on M_1? That is, is there a real α s.t.

$$(*) \quad |f(x)+\lambda\alpha| = |f(x+\lambda x_0)| \leq \|x+\lambda x_0\|$$

Note: if $x \in M$, $y \in M$, then

$$f(x)-f(y) = f(x-y) \leq |f(x-y)| \leq \|x-y\| \leq \|x-x_0\|+\|y-x_0\|$$

$$\Longrightarrow \quad f(x)-\|x-x_0\| \leq f(y)+\|y-x_0\|$$
$$\forall x,y \in M$$

Choose $\alpha \in R$ s.t.

$$\sup_{x\in M}\left(f(x)-\|x-x_0\|\right) \leq \alpha \leq \inf_{y\in M}\left(f(y)+\|y-x_0\|\right)$$

Given $x \in M$ and $\lambda \in \mathbb{R}$, we want to show $(*)$. WLOG $\lambda \neq 0$.
Let $y = -x/\lambda \in M$

$$f(x)+\lambda\alpha = f(-\lambda y)+\lambda\alpha = -\lambda\left(f(y)-\alpha\right)$$

$$\begin{aligned} |f(x)+\lambda\alpha| &= |\lambda|\,|f(y)-\alpha| \\ &\leq |\lambda|\,\|y-x_0\| \\ &= \|-\lambda y+\lambda x_0\| \\ &= \|x+\lambda x_0\| \end{aligned}$$

Thus f has a norm-preserving extension to M_1

Let P be the collection of order pairs (M', f') where M' is a subset closed under addition and multiplication by real scalers, $M' \supset M$, $f' : M' \to R$ is real linear and $\|f'\| = 1$. Partially order P as follows

$$(M', f') \leq (M'', f'') \text{ iff } M' \subset M'' \text{ and } f''|_{M'} = f'$$

Hausdorff Maximality Theorem says $\exists$ a maximal totally ordered subset Ω of P

Let

$$\widetilde{M} = \bigcup \{M' : (M', f') \in \Omega\}$$

① $\widetilde{M}$ is a subspace of X

② Define $F : \widetilde{M} \to \mathbb{R}$ by $F(x) := f'(x)$ if $x \in M'$

F is well-defined and linear. If $x \in \widetilde{M}$,

$$|F(x)| = \underset{\text{some } f'}{|f'(x)|} \leq \|x\|$$

Hence F is bounded. Finally $F|_M = f$ since each f' has this property, so in fact $\|F\| = 1$.

The fact that Ω is a maximal chain implies that $\widetilde{M} = X$, for otherwise we could repeat first part of proof with $\widetilde{M}$ to produce a larger chain. Hence F is the desired extension.

3/31 ANALYSIS

Case ② : X vector space over $\mathbb{C}$; $f : M \to \mathbb{R}$ real linear.
Simply regard X and M as a vector space over $\mathbb{R}$. Then follows from case ①

Case ③ : X vector space over $\mathbb{C}$; $f : M \to \mathbb{C}$ linear.
Let $u := \operatorname{Re} f$ on M. Then u is a real-linear functional.
$\forall x \in M$

$$f(x) = u(x) - i\,u(ix)$$

and $\|f\| = \|u\|$. By case 2, there is an extension $U : X \to \mathbb{R}$ of u with $\|U\| = \|u\|$. Set

$$F(x) := U(x) - i\,U(ix)$$

$\forall x \in X$. Then F is complex linear and $\|F\| = \|U\| = \|u\| = \|f\|$.
Moreover, if $x \in M$,

$$F(x) = U(x) - i\,U(ix) = u(x) - i\,u(ix) = f(x)$$

$\boxtimes$

COROLLARIES:

① X normed linear space. M subspace. Then

$$x \in \overline{M} \text{ if and only if } \left(f(M) = 0 \Rightarrow f(x) = 0 \quad \forall f \in X^* \right)$$

Proof. Suppose $f \in X^*$, $f(M) = 0$, and $x \in \overline{M}$. Then by continuity, $f(x) = 0$.

Suppose $x_0 \notin \overline{M}$. Then $\exists\, \delta > 0$ s.t. $\|x - x_0\| \geq \delta \quad \forall x \in M$.
Let $M_1 = sp(M \cup \{x_0\}) = \{x + \lambda x_0 : x \in M, \lambda \in \mathbb{F}\}$. Define $f \in M_1^*$ by

$$f(x + \lambda x_0) := \lambda$$

$f(M) = 0$
Note $f(x_0) = 1$. Must check f is actually bounded. If $\lambda \neq 0$ then $\|x_0 + x/\lambda\| \geq \delta$. Hence

$$|f(x + \lambda x_0)| = |\lambda| \leq \frac{\|\lambda x_0 + x\|}{\delta} \Longrightarrow \|f\| \leq \frac{1}{\delta}$$

By Hahn-Banach, we can extend f to $F \in X^*$. Then $F(M) = 0$ and $F(x_0) = 1$.

$\blacksquare$

2) X normed linear space, $x_0 \neq 0$. $\exists\, f \in X^*$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$

Proof: Let $M = sp\{x_0\}$. This is a subspace of X. Define $f : M \to \mathbb{F}$ by $f(\lambda x_0) = \lambda \|x_0\|$. Then $f \in M^*$ and $\|f\| = 1$. Extend to all of X.

$\blacksquare$

Complex Measures

Definition: Suppose $\mathfrak{M}$ is a σ-algebra of subsets of X. If $E \in \mathfrak{M}$ and

$$E = \bigcup_{i=1}^{\infty} E_i$$

where $(E_i) \subset \mathfrak{M}$ and $E_i \cap E_j = \phi$, $i \neq j$, then we call (E_i) a partition of E

Definition: Suppose $\mathfrak{M}$ is a σ-algebra of subsets of X. A complex measure is a function $\mu : \mathfrak{M} \to \mathbb{C}$ which is countably additive, i.e. if (E_i) is a partition of $E \in \mathfrak{M}$, then

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$$

Remark: Since $\sum_{i=1}^{\infty} \mu(E_i)$ is required to be independent of permutations of the sets E_i, we are in fact requiring $\sum_i \mu(E_i)$ to be absolutely convergent.

Definition: Define the total variation $|\mu|$ of μ to be

$$|\mu|(E) := \sup \left\{ \sum_{i=1}^{\infty} |\mu(E_i)| : (E_i) \text{ partition of } E \right\}$$

$\forall E \in \mathfrak{M}$.

So $|\mu| : \mathcal{M} \to [0, \infty]$.

PROPOSITION: $|\mu|$ is a positive measure on $\mathcal{M}$.

4/3 MEASURE THEORY — CH 5 #6 (without H-B), #13, #16 (4/12)

Remark: Suppose λ is a positive measure on $\mathfrak{M}$ s.t.

$$\lambda(E) \geq |\mu(E)| \quad \forall E \in \mathfrak{M}$$

Then $\lambda(E) \geq |\mu|(E) \quad \forall E \in \mathfrak{M}$. [[Suppose $E = \bigcup_{i=1}^{\infty} E_i$, E_i disjoint

$$\Longrightarrow \quad \lambda(E) = \sum \lambda(E_i) \geq \sum |\mu(E_i)|$$

$$\Longrightarrow \quad \lambda(E) \geq |\mu|(E)$$

(sup over all partitions)]]

THEOREM: If μ is a complex measure on $\mathfrak{M}$, then $|\mu|$ is a positive measure.

Proof. Suppose $E \in \mathfrak{M}$, (E_i) partition of E. Must show $|\mu|(E) = \sum |\mu|(E_i)$

Suppose $t_i < |\mu|(E_i)$. By definition of $|\mu|$, there is a partition $(A_{ij} : j \in \mathbb{N})$ of E_i s.t.

$$\sum_{j=1}^{\infty} |\mu(A_{ij})| > t_i$$

Then $(A_{ij} : i,j \in \mathbb{N})$ is a partition of E, and so

$$|\mu|(E) \geq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\mu(A_{ij})| \geq \sum_{i=1}^{\infty} t_i$$

Since $t_i < |\mu|(E_i)$ is arbitrary, we get

$$|\mu|(E) \geq \sum_{i=1}^{\infty} |\mu|(E_i)$$

Let (A_j) be any partition of E.

$$\sum_{j=1}^{\infty} |\mu(A_j)| = \sum_{j=1}^{\infty} \Big| \sum_{i=1}^{\infty} \mu(A_j \cap E_i) \Big| \leq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |\mu(A_j \cap E_i)|$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\mu(A_j \cap E_i)|$$

$$\leq \sum_{i=1}^{\infty} |\mu|(E_i)$$

$((A_j \cap E_i : j \in \mathbb{N})$ partition of $E_i)$

Now sup over all partitions (A_j) of E, we get

$$|\mu|(E) \leq \sum_{i=1}^{\infty} |\mu|(E_i)$$

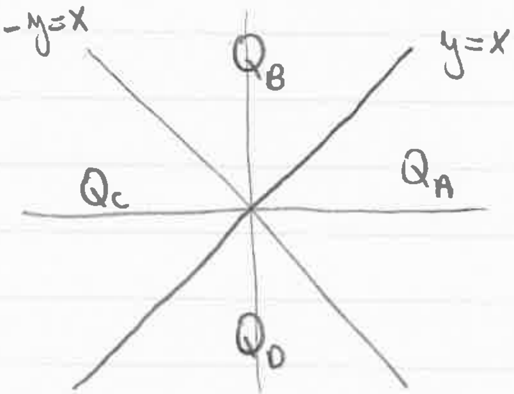
Since $|\mu|(\emptyset) = 0$, $|\mu|$ is not identically zero.

$\blacksquare$

LEMMA: Suppose $z_1, \dots, z_n$ are in $\mathbb{C}$. $\exists S \subset \{1, \dots, n\}$ s.t.

$$\Big| \sum_{j \in S} z_j \Big| \geq \frac{1}{6} \sum_{j=1}^{n} |z_j|$$

Proof. Let $W = \sum_{j=1}^{n} |z_j|$



WLOG: If we let $S = \{ j : 1 \le j \le n \text{ s.t. } z_j \in Q_A \}$, then

$$\sum_{j \in S} |z_j| \ge \frac{W}{4}$$

Then

$$\left| \sum_{j \in S} z_j \right| \ge \operatorname{Re} \sum_{j \in S} z_j \ge \frac{1}{\sqrt{2}} \sum_{j \in S} |z_j| > \frac{W}{\sqrt{2} \cdot 4} > \frac{W}{6}$$

$\blacksquare$

Proposition: Suppose μ is a complex measure on a σ-algebra $\mathcal{M}$ of subsets of X. Then $|\mu|(X) < \infty$. (In particular, $\{\mu(E) : E \in \mathcal{M}\}$ is a bounded subset of $\mathbb{C}$)

Proof. Suppose $|\mu|(E) = \infty$ for some $E \in \mathcal{M}$. Then we claim $E = A \cup B$, where $A \cap B = \phi$, $|\mu|(A) = +\infty$ and $|\mu(B)| \ge 1$.

For every $t > 0$, there is a partition (E_i) of E s.t.

$$\sum_{j=1}^{\infty} |\mu(E_j)| > t$$

Apply with $t = 6(1+|\mu(E)|)$. By the lemma, there is a finite set S of integers s.t.

$$\left| \sum_{j\in S} \mu(E_j) \right| > t/6 \geq 1$$

Let $A = \bigcup_{j\in S} E_j$. Then $|\mu(A)| \geq 1$. Let $B = E - A$

Then $\mu(B) = \mu(E) - \mu(A)$, and so

$$|\mu(B)| = |\mu(A) - \mu(E)| \geq |\mu(A)| - |\mu(E)|$$

$$> \frac{t}{6} - |\mu(E)| \underset{\substack{\uparrow \\ \text{choice of } t}}{\geq} 1$$

Thus $|\mu(A)| \geq 1$, $|\mu(B)| \geq 1$.
Now

$$|\mu|(A) + |\mu|(B) = |\mu|(E) = \infty$$

so suppose WLOG that $|\mu|(A) = \infty$. This establishes the claim

Thus if $|\mu|(X) = \infty$, then $X = A_0 \cup B_0$, disjoint union, with $|\mu|(A_0) = \infty$ and $|\mu(B_0)| \geq 1$. Then $A_0 = A_1 \cup B_1$ disjoint with $|\mu|(A_1) = \infty$ and $|\mu(B_1)| \geq 1$. Continuing by induction, $\exists$ disjoint $B_j \in \mathcal{M}$ s.t. $|\mu(B_j)| \geq 1$ $\forall j$

But

$$\mu\left(\bigcup_{j=1}^{\infty} B_j\right) \underset{\substack{\uparrow \\ (B_j) \text{ disjoint}}}{=} \sum_{j=1}^{\infty} \mu(B_j)$$

and the above shows that $\sum \mu(B_j)$ does not converge.

Thus $|\mu|(X) < \infty$.

$\boxtimes$

<u>DEFINITION</u>: Fix a σ-algebra $\mathcal{M}$ of subsets of X. Suppose λ, μ are complex measures on $\mathcal{M}$, $c \in \mathbb{C}$.

$$\forall E \in \mathcal{M} \qquad (\mu+\lambda)(E) := \mu(E) + \lambda(E)$$
$$(c\mu)(E) := c\,\mu(E)$$

(Then $\mu+\lambda$ and $c\mu$ are complex measures) Define

$$\|\mu\| := |\mu|(X)$$

Then the set of complex measures on $\mathcal{M}$ with this norm is a normed linear space

$$\|\mu_1+\mu_2\| = |\mu_1+\mu_2|(X) = \sup \sum |(\mu_1+\mu_2)(E_i)|$$

$$\leq \sup\left(\sum |\mu_1(E_i)| + |\mu_2(E_i)|\right)$$

$$\leq \sup \sum |\mu_1(E_i)| + \sup \sum |\mu_2(E_i)|$$

$$= |\mu_1|(X) + |\mu_2|(X) = \|\mu_1\| + \|\mu_2\|$$

$$\|\mu\| = 0 \iff |\mu|(X) = 0 \iff |\mu| = 0 \iff \mu = 0$$

$$|\mu(E)| \leq |\mu|(E)$$

DEFINITION: Let μ a complex measure taking only real values

$$\mu^+ := \frac{1}{2}(|\mu| + \mu)$$

$$\mu^- := \frac{1}{2}(|\mu| - \mu)$$

Then μ^+ and μ^- are positive measures (since $|\mu(E)| \leq |\mu|(E)$)

$$|\mu| = \mu^+ + \mu^- \ ; \quad \mu = \mu^+ - \mu^-$$

4/5 MEASURE THEORY

For this part μ will denote a positive measure on $\mathfrak{M}$ and λ a (complex or positive) measure on $\mathfrak{M}$

DEFINITION: λ is absolutely continuous w.r.t. μ $(\lambda \ll \mu)$ if $\mu(E) = 0 \Rightarrow \lambda(E) = 0$

DEFINITION: If $A \in \mathfrak{M}$, we say λ is concentrated on A if

$$\lambda(E) = \lambda(E \cap A) \quad \forall E \in \mathfrak{M}$$

Remark - λ is concentrated on A iff $\lambda(E) = 0$ whenever $E \in \mathfrak{M}$ and $E \cap A = \phi$

Proof. Suppose $\lambda(B) = 0$ $\forall B$ s.t. $B \cap A = \phi$. Given any $E \in \mathfrak{M}$,

$$\lambda(E) = \lambda(E \cap A) + \lambda(E - A) = \lambda(E \cap A)$$

Conversely, if λ is concentrated on A and $E \subset X - A$, then

$$\lambda(E) = \lambda(E \cap A) = \lambda(\phi) = 0$$

DEFINITION: λ_1 and λ_2 are mutually singular $(\lambda_1 \perp \lambda_2)$ if λ_1 and λ_2 are concentrated on disjoint sets.

PROPOSITION: $(X, \mathcal{M})$ σ-algebra. μ positive measure; $\lambda, \lambda_1, \lambda_2$ complex measures

★ (a) If λ concentrated on A, then $|\lambda|$ concentrated on A

Proof. $\lambda(B) = 0$ $\forall B \subset A^c$. If $B \subset A^c$ and (B_i) is a partition of B, then $B_i \subset B \subset A^c$. Hence

$$|\lambda|(B) = \sup_{\substack{\text{all} \\ \text{partitions}}} \sum |\lambda(B_i)| = 0 \qquad \forall B \subset A^c$$

★ (b) If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$

Proof. λ_1 concentrated on A_1, λ_2 concentrated on A_2, with $A_1 \cap A_2 = \phi$. Then (a) $\Rightarrow$ $|\lambda_1|$ concentrated on A_1 and $|\lambda_2|$ concentrated on A_2

★ (c) $\lambda_1 << \mu$ and $\lambda_2 << \mu \Rightarrow \lambda_1 + \lambda_2 << \mu$

Proof. Suppose $\mu(E) = 0$. Then $\lambda_1(E) = \lambda_2(E) = 0$, and so $(\lambda_1 + \lambda_2)(E) = \lambda_1(E) + \lambda_2(E) = 0$.

★ (d) $\lambda_1 \perp \lambda$, $\lambda_2 \perp \lambda \Rightarrow \lambda_1 + \lambda_2 \perp \lambda$

Proof. λ_1 concentrated on A_1, λ concentrated on B_1 with $A_1 \cap B_1 = \phi$. λ_2 concentrated on A_2, λ concentrated on B_2 with $A_2 \cap B_2 = \phi$. Then $\lambda_1 + \lambda_2$ is concentrated on $A_1 \cup A_2$ and λ is concentrated on $B_1 \cap B_2$

$$\left(Y \subset X - (B_1 \cap B_2) = (X - B_1) \cup (X - B_2) \Rightarrow \right.$$

$$\lambda(Y) = \lambda(Y \cap (X - B_1)) + \lambda(Y \cap (X - B_2))$$

$$= 0 + 0 = 0$$

Note $(A_1 \cup A_2) \cap (B_1 \cap B_2) = \phi$.

★ (e) If $\lambda \ll \mu$, then $|\lambda| \ll \mu$

Proof. If $\mu(E) = 0$ and (E_i) is a partition of E, $\lambda(E_i) = 0 \;\; \forall i$. Hence $|\lambda|(E) = 0$

★ (f) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.

Proof. λ_2 concentrated on A, μ concentrated on B, with $A \cap B = \phi$. Then $\mu(E) = 0 \;\; \forall E \subset X - B \Rightarrow \lambda_1(E) = 0 \;\; \forall E \subset X - B$
Hence λ_1 is concentrated on B.

★ (g) If $\lambda \ll \mu$, $\lambda \perp \mu$, then $\lambda = 0$

Proof. By (f), $\lambda \perp \lambda$. So $\exists$ A, B with $A \cap B = \phi$ and λ concentrated on A and concentrated on B. Hence $\lambda(E) = 0$ for any $E \in \mathfrak{M}$

$$\lambda(E) = \lambda(\underbrace{E \cap A}_{\subset X - B}) + \lambda(\underbrace{E \cap B}_{\subset X - A}) = 0 + 0 = 0$$

$\boxtimes$

LEMMA: Suppose μ is a positive measure on $(X, \mathfrak{M})$ and $\mu(X) < \infty$. If $f \in L^1(\mu)$ and S a closed set in $\mathbb{C}$ s.t. for every $E \in \mathfrak{M}$ with $\mu(E) > 0$ and

$$\frac{1}{\mu(E)} \int_E f \, d\mu \in S$$

then $f(x) \in S$ for almost all x.

Proof. Let $\Delta := \{ z : |z - \alpha| \leq r \} \subset \mathbb{C} - S$. Sufficient to show $\mu(E) = 0$ where $E = f^{-1}(\bar{\Delta})$ since $\mathbb{C} - S$ is a countable union of such $\bar{\Delta}$'s

Suppose $\mu(E) > 0$

$$\left| \frac{1}{\mu(E)} \int_E f \, d\mu - \alpha \right| = \left| \frac{1}{\mu(E)} \int_E (f - \alpha) \, d\mu \right|$$

$$\leq \frac{1}{\mu(E)} r \cdot \mu(E) = r$$

$$\Rightarrow \frac{1}{\mu(E)} \int_E f \, d\mu \in \bar{\Delta} \subset \mathbb{C} - S \quad \lightning$$

Hence $\mu(E) = 0$.

$\blacksquare$

<u>RADON-NIKODYM THEOREM</u>: Suppose λ and μ are both positive bounded measures on a σ-algebra $\mathfrak{M}$ in X. Then there exists a unique pair of measures λ_a and λ_s such that λ_a is absolutely continuous w.r.t. μ, λ_s is singular w.r.t. μ and $\lambda = \lambda_a + \lambda_s$. λ_a and λ_s are positive measures and $\lambda_a \perp \lambda_s$.

Moreover, there is a unique $h \in L^1(\mu)$ s.t.

$$(*) \qquad \lambda_a(E) = \int_E h\, d\mu \quad \forall E \in \mathfrak{M}$$

Proof. Suppose $\lambda = \lambda_a' + \lambda_s'$ where $\lambda_a' \ll \mu$ and $\lambda_s' \perp \mu$. Then $\lambda_a - \lambda_a' = \lambda_s - \lambda_s'$. Thus $(\lambda_a - \lambda_a') \ll \mu$ and $(\lambda_a - \lambda_a') \perp \mu$ $\Rightarrow \lambda_a - \lambda_a' = 0$, so $\lambda_a = \lambda_a'$ and $\lambda_s = \lambda_s'$.

Recall by (5), $\lambda_a \ll \mu$ and $\lambda_s \perp \mu \Rightarrow \lambda_a \perp \lambda_s$

Suppose there were another $h_1 \in L^1(\mu)$ satisfying $(*)$. Then

$$\int_E (h_1 - h)\, d\mu = 0 \quad \forall E \in \mathfrak{M}$$

and so $h = h_1$ a.e., i.e. $h = h_1$ in $L^1(\mu)$.

4/7 MEASURE THEORY

(writing λ as $\lambda_a + \lambda_s$ where $\lambda_a \ll \mu$ and $\lambda_s \perp \mu$ is called the Lebesgue decomposition of λ w.r.t. μ)

(Continuation of proof of R-N)

Set $\varphi = \lambda + \mu$. Note $\varphi(X) < \infty$. $E \in \mathfrak{M} \Rightarrow$
$\varphi(E) = \lambda(E) + \mu(E)$, so

$$(*) \qquad \int_X f\,d\varphi = \int_X f\,d\lambda + \int f\,d\mu$$

for $f = \chi_E$, $E \in \mathfrak{M}$. Hence $(*)$ holds for $f =$ simple function, and so for non-negative measurable functions by M.C.T. Therefore $(*)$ holds for all $f \in L^1(\varphi)$

If $f \in L^2(\varphi)$, then

$$\left|\int_X f\,d\lambda\right| \leq \int_X |f|\,d\lambda \leq \int_X |f|\,d\varphi \leq \left(\int_X |f|^2\,d\varphi\right)^{1/2} \varphi(X)^{1/2}$$

Hence $f \to \int f\,d\lambda$ is a bounded linear functional on $L^2(\varphi)$, so there exists $g \in L^2(\varphi)$ s.t.

$$(**) \qquad \int_X f\,d\lambda = (f, \bar{g}) = \int_X fg\,d\varphi \quad \forall f \in L^2(\varphi)$$

Take $f = \chi_E$, $E \in \mathfrak{M}$ for which $\varphi(E) > 0$. By $(**)$

$$\lambda(E) = \int_E g\, d\varphi$$

$$\Rightarrow \frac{1}{\varphi(E)} \int_E g\, d\varphi = \frac{\lambda(E)}{\varphi(E)} \in [0,1]$$

Lemma from previous section $\Rightarrow$ $g(x) \in [0,1]$ a.e. $[\varphi]$. WLOG $g(x) \in [0,1]$ $\forall x \in X$.

$$(*) \text{ and } (**) \Rightarrow \int_X f(1-g)\, d\lambda = \int_X fg\, d\mu \quad \forall f \in L^2(\varphi) \quad (\dagger)$$

$f \in L^2(\varphi)$, $g \in L^2(\varphi) \Rightarrow fg \in L^1(\varphi)$ by Holder

Set

$$A := \{x \in X : 0 \le g(x) < 1\}$$

$$B := \{x \in X : g(x) = 1\}$$

Define

$$\lambda_a(E) := \lambda(E \cap A) \quad \forall E \in \mathfrak{M}$$
$$\lambda_s(E) := \lambda(E \cap B) \quad \forall E \in \mathfrak{M}$$

Clearly λ_a and λ_s are positive measures on $\mathfrak{M}$ since λ is, and $\lambda = \lambda_a + \lambda_s$ (since $A \cap B = \emptyset$)

If $Y \cap B = \emptyset$, then $\lambda_s(Y) = \lambda(\emptyset) = 0$, whence λ_s is concentrated on B. Set $f = \chi_B$ in $(\dagger)$

$$0 = \int_B (1-g)\,d\lambda = \int_B g\,d\mu = \mu(B)$$

since $g=1$ on B

Therefore $\mu \perp \lambda_s$

In (†) set $f = (1+g+g^2+\dots+g^n)\chi_E$. Note $f \in L^2(\varphi)$

$$\int_E (1-g^{n+1})\,d\lambda = \int_E (g+g^2+\dots+g^{n+1})\,d\mu$$

On B, $1-g^{n+1} = 0$. On A, $(1-g^{n+1}) \uparrow 1$. Therefore $1-g^{n+1} \uparrow \chi_A$

$$\text{MCT} \Rightarrow \text{LHS} \longrightarrow \int_E \chi_A\,d\lambda = \lambda(E\cap A) = \lambda_a(E)$$

As $n\to\infty$, $g+g^2+\dots+g^{n+1} \longrightarrow g/1-g$. Set

$$h = \begin{cases} +\infty & \text{if } g(x)=1 \iff x\in B \\ g/1-g & \text{otherwise} \end{cases}$$

Then

$$\text{RHS} \longrightarrow \int_E h\,d\mu$$

Hence $\forall E \in \mathfrak{M}$

$$\lambda_a(E) = \int_E h\,d\mu$$

Set $E = X$

$$\infty > \lambda_a(X) = \int_X h\,d\mu \implies h \in L^1(\mu)$$

Also, if $\mu(E) = 0$

$$\lambda_a(E) = \int_E h\,d\mu = 0$$

and so $\lambda_a \ll \mu$.

$\blacksquare$

Extensions

Case I: $\lambda(X) < \infty$, X σ-finite w.r.t. μ OK
Case II: λ complex measure, X σ-finite w.r.t. μ OK

$\mathrm{I} \Rightarrow \mathrm{II}$: Write $\lambda = \lambda_1 + i\lambda_2$ where λ_1, λ_2 are real-valued

$$\left.\begin{aligned} \lambda_1^+ &= \tfrac{1}{2}\left(|\lambda_1| + \lambda_1\right) \\ \lambda_1^- &= \tfrac{1}{2}\left(|\lambda_1| - \lambda_1\right) \end{aligned}\right\} \text{ positive, bounded measures}$$

Then $\lambda_1^+ = (\lambda_1^+)_a + (\lambda_1^+)_s$ where $(\lambda_1^+)_a \ll \mu$, $(\lambda_1^+)_s \perp \mu$ and

$$(\lambda_1^+)_a = \int_E h_1\,d\mu$$

$h_1 \geq 0$, $h_1 \in L^1(\mu)$. Also, $\lambda_1^- = (\lambda_1^-)_a + (\lambda_1^-)_s$ etc.

$$\lambda_1 = \left[(\lambda_1^+)_a - (\lambda_1^-)_a\right] + \left[(\lambda_1^+)_s - (\lambda_1^-)_s\right]$$

absolutely cont. w.r.t. μ — singular w.r.t. μ

Similar for imaginary part

Sketch of proof for case I: WLOG, $X_n \cap X_m = \emptyset$. Define

$$\mu_n(E) := \mu(E \cap X_n)$$
$$\lambda_n(E) := \lambda(E \cap X_n)$$

Then μ_n and λ_n satisfy hypothesis of R-N, so

$$\lambda_n = (\lambda_n)_a + (\lambda_n)_s$$

where $(\lambda_n)_a \ll \mu_n$ and $(\lambda_n)_s \perp \mu$, and

$$(\lambda_n)_a(E) = \int_E h_n \, d\mu_n$$

WLOG $h_n = 0$ on $X - X_n$.

Note $\lambda = \sum \lambda_n$ (If $E \in \mathcal{M}$, $\lambda(E) = \lambda\left(\bigcup (E \cap X_n)\right)$ $= \sum \lambda(E \cap X_n) = \sum \lambda_n(E)$). Let

$$\lambda_a := \sum_{n=1}^{\infty} (\lambda_n)_a$$

$$\lambda_s := \sum_{n=1}^{\infty} (\lambda_n)_s$$

Check λ_a, λ_s measures on $\mathcal{M}$; $\lambda_a \ll \mu$, $\lambda_s \perp \mu$; for $E \in \mathcal{M}$

$$\lambda_a(E) = \int_E h \, d\mu$$

where $h = \sum_{n=1}^{\infty} h_n \in L^1(\mu)$

4/10 MEASURE THEORY

PROPOSITION: Let μ be a positive measure and λ a complex measure. TFAE

(a) $\lambda \ll \mu$

(b) $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $E \in \mathcal{M}$ with $\mu(E) < \delta$, then $|\lambda|(E) < \varepsilon$

Proof. Suppose (b) holds. Suppose $E \in \mathcal{M}$, $\mu(E) = 0$. Let $\varepsilon > 0$. Then by (b) $|\lambda|(E) < \varepsilon$. Hence $|\lambda|(E) = 0$, so that $\lambda(E) = 0$

Suppose (b) doesn't hold. $\exists \varepsilon > 0$ and $(E_n) \subset \mathcal{M}$ s.t.

$$\mu(E_n) < 1/2^n$$
$$|\lambda|(E_n) \geq \varepsilon$$

Let $A_n = \bigcup_{j=n}^{\infty} E_j$ and $A = \bigcap_{n=1}^{\infty} A_n$. Then

$$\mu(A) = \lim_{n\to\infty} \mu(A_n) \leq \lim_{n\to\infty} 2^{1-n} = 0$$

However

$$|\lambda|(A_n) \geq |\lambda|(E_n) \geq \varepsilon \quad \forall n \in \mathbb{N}$$

and so

$$|\lambda|(A) = \lim_{n\to\infty} |\lambda|(A_n) \geq \varepsilon$$

Therefore $|\lambda|$ is not absolutely continuous w.r.t. μ, so that λ is not absolutely cont. w.r.t. μ

$\blacksquare$

(Can replace statement in (b) by $|\lambda(E)| < \varepsilon$)

THEOREM: Suppose λ is a complex measure on $(X, \mathcal{M})$. Then there exist a measurable $h : X \longrightarrow \{z \in \mathbb{C} : |z| = 1\}$ s.t.

$$\forall E \in \mathcal{M} \qquad \lambda(E) = \int_E h \, d|\lambda|$$

(also written $d\lambda = h\, d|\lambda|$)

Proof. Certainly $\lambda \ll |\lambda|$. Now

$$\lambda = \operatorname{Re}\lambda + i \operatorname{Im}\lambda$$

and so
$$(\operatorname{Re}\lambda)^+ \ll |\lambda|$$
$$(\operatorname{Im}\lambda)^+ \ll |\lambda|$$
$$(\operatorname{Re}\lambda)^- \ll |\lambda|$$
$$(\operatorname{Im}\lambda)^- \ll |\lambda|$$

Recall $|\lambda|(X) < \infty$. By the Radon-Nikodym Theorem

$$(\operatorname{Re}\lambda)^+(E) = \int_E h_1 \, d|\lambda|$$

for some $h_1 \geq 0$, $h_1 \in L^1[|\lambda|]$. If we do this for each part, we see that

$$\lambda(E) = \int_E h \, d|\lambda|$$

for some $h \in L^1[|\lambda|]$.

Must show h can be chosen so that $|h(x)| = 1$ everywhere. Select $r < 1$, and let

$$A_r = \{x : |h(x)| \leq r\}$$

Let $\{E_i\}$ be any partition of A_r.

$$\sum_{j=1}^{\infty} |\lambda(E_j)| = \sum_{j=1}^{n} \left| \int_{E_j} h \, d|\lambda| \right| \leq \sum_{j=1}^{\infty} \int_{E_j} |h| \, d|\lambda|$$

$$\leq r \sum_{j=1}^{\infty} \int_{E_j} d|\lambda| = r \, |\lambda|(A_r)$$

Sup over all partitions:

$$|\lambda|(A_r) \leq r \, |\lambda|(A_r)$$

But $r < 1$, so we must have $|\lambda|(A_r) = 0$. Therefore $|h(x)| \geq 1$ a.e.

Suppose $E \in \mathfrak{M}$, $|\lambda|(E) > 0$.

$$\left| \frac{1}{|\lambda|(E)} \int_E h \, d|\lambda| \right| = \frac{1}{|\lambda|(E)} |\lambda(E)| \leq 1$$

Hence all averages of h over E s.t. $|\lambda|(E) > 0$ lie in $\{|z| \le 1\}$.
Therefore $|h(x)| \le 1$ a.e.

Hence $|h(x)| = 1$ a.e. Redefine h as follows: if $|h(x)| \neq 1$, change so that $h(x) = 1$; Don't change h anywhere else.

$\boxed{/\!/}$

(will use this ↑ to define complex integration)

<u>HAHN DECOMPOSITION THEOREM</u>: μ real measure on $(X, \mathfrak{M})$. Then there are sets $A, B \in \mathfrak{M}$ such that

$$A \cap B = \emptyset$$
$$A \cup B = X$$

such that

$$\mu^+(E) = \mu(E \cap A)$$
$$\mu^-(E) = -\mu(E \cap B)$$

(Note: If $E \subset A$, then $\mu(E) = \mu^+(E) \ge 0$ and if $E \subset B$ then $\mu(E) = -\mu^-(E) \le 0$.)

Proof $\exists\, h : X \longrightarrow T$ s.t. $\forall E \in \mathfrak{M}$

$$\mu(E) = \int_E h \, d|\mu|$$

Let $E = \{x : \operatorname{Im} h(x) > 0\}$

$$\int_E \operatorname{Im} h(x)\, d|\mu| = \operatorname{Im} \int_E h(x)\, d|\mu| = 0$$

Hence $|\mu|(E) = 0$ $\qquad \int h(x)\, d|\mu| = \mu(E) \in \mathbb{R}$

Therefore $h(x) = \pm 1$ a.e. $[|\mu|]$. Modify h s.t. $h(x) = 1$ if previously $\operatorname{Im} h(x) \neq 0$. Then $h(x) = \pm 1$ everywhere.

$$\mu^+ = \frac{1}{2}(|\mu| + \mu)$$

$$E \in \mathfrak{M} \Rightarrow \mu^+(E) = \frac{1}{2}\int_E (1+h)\, d|\mu| = \int_{E\cap A} h\, d|\mu| = \mu(E \cap A)$$

$$\left(\text{Note} \quad \frac{1}{2}(1+h) = \begin{cases} h & x \in A := \{x : h(x) = 1\} \\ 0 & x \in B := \{x : h(x) = -1\} \end{cases} \right)$$

Now

$$\mu(E) = \mu^+(E) - \mu^-(E)$$

$$\mu(E) = \mu(A \cap E) + \mu(B \cap E)$$

and so $\mu^-(E) = -\mu(B \cap E)$.

$\boxtimes$

Corollary: If μ is a real measure on X and

$\mu = \lambda_1 - \lambda_2$, where λ_1, λ_2 are positive, then $\mu^+ \le \lambda_1$ and $\mu^- \le \lambda_2$

Proof. Let A be as in Hahn Decomposition Theorem. Let $E \in \mathfrak{M}$.

$$\mu^+(E) = \mu(E \cap A) \underset{\lambda_2 \ge 0}{\le} \lambda_1(E \cap A) \underset{\lambda_1 \ge 0}{\le} \lambda_1(E)$$

Also

$$\mu^- = \mu^+ - \mu = \mu^+ - \lambda_1 + \lambda_2 \underset{\mu^+ - \lambda_1 \le 0}{\le} \lambda_2$$

$\blacksquare$

PROPOSITION: μ positive measure on $(X, \mathfrak{M})$, $g \in L^1(\mu)$. Let

$$\lambda(E) := \int_E g \, d\mu \qquad \forall E \in \mathfrak{M}$$

(Note λ is a complex measure by D.C.T.) Then

$$|\lambda|(E) = \int_E |g| \, d\mu \qquad \forall E \in \mathfrak{M}$$

Proof. $\exists\, h : X \longrightarrow T$ s.t.

$$\lambda(E) = \int_E h \, d|\lambda|$$

Also

$$\lambda(E) = \int_E g \, d\mu$$

Therefore

$$\int_E h \, d|\lambda| = \int_E g \, d\mu \qquad \forall E \in \mathfrak{M}$$

Hence

$$\int_E f h \, d|\lambda| = \int_E f g \, d\mu$$

for $f = \chi_{E'}$ $\Rightarrow$ for $f =$ simple function $\Rightarrow$ for $f =$ unif. limit of simple functions. Now $\bar{h}$ can be unif. approx. by simple functions, so

$$E \in \mathfrak{M} \Rightarrow \int_E g\bar{h} \, d\mu = \int_E h\bar{h} \, d|\lambda| = \int_E d|\lambda| = |\lambda|(E)$$

Now left to show that $g\bar{h} \geq 0$ a.e. Hence $g\bar{h} = |g\bar{h}| = |g|$ a.e. Therefore

$$|\lambda|(E) = \int_E |g| \, d\mu$$

$\blacksquare$

4/12 MEASURE THEORY

THEOREM: μ positive σ-finite measure. $\Phi : L^p(\mu) \to \mathbb{C}$ bounded linear functional ($1 \le p < \infty$). Then there is a unique $g \in L^q(\mu)$ such that

$$(*) \qquad \Phi(f) = \int_X fg \, d\mu$$

where $1/p + 1/q = 1$. Furthermore, $\|g\|_q = \|\Phi\|$.

Proof: uniqueness

If $\int fg\,d\mu = \int fg'\,d\mu$ $\forall f \in L^p(\mu)$, then $\int_E (g-g')\,d\mu = 0$ for all $E \in \mathcal{M}$ with $\mu(E) < \infty$. Hence $g = g'$ a.e. (need σ-finiteness here)

If $(*)$ holds, then $\|\Phi\| \le \|g\|_q$ by Hölder. So, we must produce $g \in L^q(\mu)$ and show $\|g\|_q \le \|\Phi\|$ and $(*)$.

First suppose $\mu(X) < \infty$. For $E \in \mathcal{M}$, define

$$\lambda(E) := \Phi(\chi_E)$$

Finitely additive since

$$E_1 \cap E_2 = \emptyset \Rightarrow \lambda(E_1 \cup E_2) = \Phi(\chi_{E_1 \cup E_2}) = \Phi(\chi_{E_1} + \chi_{E_2})$$

$$= \Phi(\chi_{E_1}) + \Phi(\chi_{E_2}) = \lambda(E_1) + \lambda(E_2)$$

Now suppose (E_i) is a partition of $E \in \mathcal{M}$. Let

$$A_k = \bigcup_{i=1}^{k} E_i$$

Then $E - A_k \supset E - A_{k-1}$ and $\bigcap (E - A_k) = \phi$, so

$$\| \chi_E - \chi_{A_k} \|_p = (\mu(E - A_k))^{1/p} \longrightarrow 0$$

Thus $\chi_{A_k} \longrightarrow \chi_E$ in L^p, so $\Phi(\chi_{A_k}) \longrightarrow \Phi(\chi_E)$, i.e.

$$\sum_{i=1}^{k} \lambda(E_i) \longrightarrow \lambda(E)$$

Therefore λ is a complex measure. Moreover, $\lambda \ll \mu$, for if $\mu(E) = 0$, then $\chi_E = 0$ in L^p, so $\lambda(E) = \Phi(0) = 0$.
By the Radon-Nikodym theorem, there is a $g \in L^1(\mu)$ such that

$$\lambda(E) = \int_E g \, d\mu$$

Hence

$$(**) \qquad \Phi(f) = \int_X f g \, d\mu$$

if $f = \chi_E \Rightarrow$ if f = simple function. If $f = \lim f_n$, f_n simple and limit uniform, then $\Phi(f_n) \to \Phi(f)$ since $\mu(X) < \infty \Rightarrow$ (uniform convergence $\Rightarrow L^p$ convergence). Therefore

$$\Phi(f) = \lim \Phi(f_n) = \lim \int f_n g \, d\mu = \int_X f g \, d\mu$$

Hence $(**)$ holds for all bounded measurable f.

Case I: $p=1$

Let $f = \chi_E$, $\mu(E) > 0$. Then

$$\left| \int_E g \, d\mu \right| = \left| \Phi(\chi_E) \right| \leq \|\Phi\| \, \mu(E)$$

and so

$$\left| \frac{1}{\mu(E)} \int_E g \, d\mu \right| \leq \|\Phi\|$$

Therefore $|g| \leq \|\Phi\|$ a.e., whence $\|g\|_\infty \leq \|\Phi\|$

Case II: $1 < p < \infty$

For $n \in \mathbb{N}$, let $E_n = \{x \in X : |g(x)| \leq n\}$. $\exists$ measurable α such that $\alpha(x) g(x) = |g(x)|$ $\forall x \in X$. Consider the bounded measurable f on X given by

$$\forall x \in X \qquad f(x) := |g(x)|^{q-1} \alpha(x) \chi_{E_n}(x)$$

Note $f(x) g(x) = \chi_{E_n}(x) \, |g(x)|^q$. Also

$$|f(x)|^p = |g(x)|^{(q-1)p} \chi_{E_n}(x) = |g(x)|^q \chi_{E_n}(x)$$

$$\int_{E_n} |g|^q \, d\mu = \int_X f(x)g(x)\, d\mu \underset{\substack{\uparrow \\ f \text{ bounded, meas}}}{=} \Phi(f) \leq \|\Phi\| \, \|f\|_p$$

$$= \|\Phi\| \left(\int_X |f|^p \, d\mu \right)^{1/p} = \|\Phi\| \left(\int_{E_n} |g|^q d\mu \right)^{1/p}$$

and so

$$\int_{E_n} |g|^q \, d\mu = \left(\int_{E_n} |g|^q \right)^{q - q/p} \leq \|\Phi\|^q$$

Let $n \to \infty$ MCT shows that

$$\int |g|^q \, d\mu \leq \|\Phi\|^q$$

so that $g \in L^q(\mu)$ and $\|g\|_q \leq \|\Phi\|$.

Recall the set of bounded measurable functions is dense in $L^p(\mu)$. Therefore, given $f \in L^p$, $\exists (f_n) \subset L^p$ s.t. $\|f_n - f\|_p \to 0$ and

$$\Phi(f) = \lim \Phi(f_n) = \lim \int g f_n \, d\mu \underset{\substack{\uparrow \\ \text{Holder since } g \in L^q}}{\to} \int g f \, d\mu$$

Done for $\mu(X) < \infty$.

Now suppose $X = \bigcup_{n=1}^{\infty} X_n$ with $0 < \mu(X_n) < \infty$ and X_n disjoint

Define $h : X \to (0,\infty)$ by

$$h(x) := \frac{1}{n^2} \frac{1}{\mu(X_n)} \qquad x \in X_n$$

Then $h \in L^1(\mu)$.

For $E \in \mathfrak{M}$, let

$$\tilde{\mu}(E) := \int_E h \, d\mu$$

$\tilde{\mu}$ is a finite, positive measure on X. Recall

$$\int r(x)\, d\tilde{\mu} = \int r(x) h(x)\, d\mu$$

if $r(x) \geq 0$ is measurable. This also holds for $r \in L^1(\tilde{\mu})$.
Consider the mapping $F \mapsto h^{1/p} F$ for $F \in L^p(\tilde{\mu})$. This maps $L^p(\tilde{\mu})$ onto $L^p(\mu)$ and is 1-1, linear, norm-preserving

$$\int |F|^p \, d\tilde{\mu} = \int |F|^p h \, d\mu = \int \left(|F| h^{1/p}\right)^p d\mu$$

If $k \in L^p(\mu)$, then $h^{-1/p} k(x) \in L^p(\tilde{\mu})$ since

$$\int_X h^{-1} |k|^p \, d\tilde{\mu} = \int h h^{-1} |k|^p \, d\mu = \int |k|^p d\mu < \infty$$

Define $\psi : L^p(\tilde{\mu}) \to \mathbb{C}$ by

$$\psi(F) = \Phi(h^{1/p} F)$$

ψ bounded linear functional on $L^p(\tilde{\mu})$ with $\|\psi\| = \|\Phi\|$

$$\|\psi\| = \sup_{F \in L^p(\tilde{\mu})} \frac{|\psi(F)|}{\left(\int |F|^p \, d\tilde{\mu}\right)^{1/p}} = \sup_{F \in L^p(\tilde{\mu})} \frac{|\Phi(h^{1/p} F)|}{\left(\int |h^{1/p} F|^p \, d\mu\right)^{1/p}} = \sup_{K \in L^p(\mu)} \frac{|\Phi(K)|}{\left(\int |K|^p \, d\mu\right)^{1/p}} = \|\Phi\|$$

4/14 MEASURE THEORY

For $F \in L^p(\tilde{\mu})$, let $\psi(F) = \Phi(h^{1/p} F)$. ψ is a bounded linear functional on $L^p(\tilde{\mu})$ with $\|\psi\| = \|\Phi\|$. By the first part of the proof, $\exists\, G \in L^q(\tilde{\mu})$ s.t.

$$\psi(F) = \int FG \, d\tilde{\mu} \qquad \forall F \in L^p(\tilde{\mu})$$

Also $\|\psi\| = \|G\|_q$

Case I: $p = 1$ Set $g = G$

$$\underset{\text{w.r.t. } \mu}{\|g\|_\infty} \underset{\mu(E)>0 \Leftrightarrow \tilde{\mu}(E)>0}{=} \|G\|_\infty \leftarrow \text{w.r.t. } \tilde{\mu}$$

Then $\|g\|_\infty = \|G\|_\infty = \|\psi\| = \|\Phi\|$. Hence $g \in L^\infty(\mu)$

Case II: $1 < p < \infty$. Set $g = h^{1/q} G$

$$\int_X |g|^q \, d\mu = \int_X h |G|^q \, d\tilde{\mu} = \int_X |G|^q \, d\tilde{\mu}$$

Hence $g \in L^q(\mu)$ and $\|g\|_q = \|G\|_q = \|\psi\| = \|\Phi\|$

Back to case I:

Suppose $f \in L^p(\mu)$

$$\Phi(f) = \psi(h^{-1}f) = \int_X h^{-1} f G \, d\tilde{\mu} = \int_X h(h^{-1} f G)\, d\mu$$

$h^{-1}fG \in L^1(\tilde{\mu})$

$$= \int_X f G \, d\mu = \int_X f g \, d\mu$$

For case II: $\forall f \in L^p(\mu)$

$$\Phi(f) = \psi(h^{-1/p} f) = \int_X h^{-1/p} f G \, d\tilde{\mu} = \int_X h(h^{-1/p} f G)\, d\mu$$

$$= \int_X h^{1/q} f G \, d\mu = \int_X f g \, d\mu$$

▨

LEMMA: Suppose $\Phi : C_0(X) \to \mathbb{C}$ is a bounded $(\|\Phi\| = 1)$ linear functional (X locally compact T_2 space) $\exists \Lambda : C_c(X) \to \mathbb{C}$ positive linear functional s.t.

$$|\Phi(f)| \le \Lambda(|f|) \le \|f\|_\infty$$

Proof. Let $C_c^+(X) = \{ f \in C_c(X) : f(x) \geq 0 \ \forall x \in X \}$

For $f \in C_c^+(X)$, define

$$\Lambda f := \sup\{|\Phi(h)| : h \in C_c(X),\ |h| \le f\} < \infty$$

First show if $f, g \in C_c^+(X)$, then $\Lambda(f+g) = \Lambda f + \Lambda g$.
Suppose $\varepsilon > 0$. There exists $h_1 \in C_c(X)$ s.t. $|h_1| \le f$ and

$$|\Phi(h_1)| + \varepsilon > \Lambda f$$

$\exists\, h_2 \in C_c(X)$ s.t. $|h_2| \le g$ and

$$|\Phi(h_2)| + \varepsilon > \Lambda g$$

$\exists\ |\alpha_1| = 1, |\alpha_2| = 1$ s.t. $\alpha_j \Phi(h_j) = |\Phi(h_j)|$ $j = 1, 2$. Then

$$\Lambda f + \Lambda g < |\Phi(h_1)| + |\Phi(h_2)| + 2\varepsilon$$

$$= \alpha_1 \Phi(h_1) + \alpha_2 \Phi(h_2) + 2\varepsilon$$

$$= \Phi(\alpha_1 h_1 + \alpha_2 h_2) + 2\varepsilon$$

Note that $|\alpha_1 h_1 + \alpha_2 h_2| \le f+g$, and so

$$\Lambda f + \Lambda g \le \Lambda(f+g) + 2\varepsilon$$

Therefore $\Lambda f + \Lambda g \le \Lambda(f+g)$
Suppose $h \in C_c(X)$ and $|h| \le f+g$. Let

$$V = \{x \in X : f(x) + g(x) > 0\}$$

Define
$$h_1 = \begin{cases} \frac{f}{f+g}\, h & \text{on } V \\ 0 & \text{off } V \end{cases} \qquad h_2 = \begin{cases} \frac{g}{f+g}\, h & \text{on } V \\ 0 & \text{off } V \end{cases}$$

Then $h_1 + h_2 = h$ on all of X. Also $|h_j| \le |h|$ on all of X, $j=1,2$
Moreover h_j is continuous on X: clear on V; off V $h_j = h = 0$; result follows from $|h_j| \le |h|$ and h continuous. Also $|h_j| \le |h|$
$\Rightarrow$ supp h_j compact.

$$\Phi(h) = \Phi(h_1 + h_2) \underset{\substack{\uparrow \\ h_1, h_2 \in C_c(X)}}{=} \Phi(h_1) + \Phi(h_2)$$

$$\Rightarrow |\Phi(h)| \le |\Phi(h_1)| + |\Phi(h_2)| \le \Lambda f + \Lambda g$$

Since h was arbitrary, by taking sups we have

$$\Lambda(f+g) \le \Lambda f + \Lambda g$$

$|h_1| = \frac{|h|}{f+g} f \le f$ on V

$|h_1| = 0 \le f$ on $X - V$

same for h_2

If $f \in C_c(X)$ and f real, define

$$\Lambda f := \Lambda f^+ - \Lambda f^-$$

If $f \in C_c(X)$, let

$$\Lambda f := \Lambda(\operatorname{Re} f) + i\,\Lambda(\operatorname{Im} f)$$

Definition of Λ of positive function $\Rightarrow$ $|\Phi(f)| \leq \Lambda(|f|)$
If $|h| \leq |f|$ on X, $\|\Phi\| = 1$ $\Rightarrow$ $|\Phi(h)| \leq 1 \cdot \|h\|_\infty \leq \|f\|_\infty$
Sup over all $h \in C_c(X)$, $|h| \leq |f|$ gives

$$\Lambda(|f|) \leq \|f\|_\infty$$

4/17 MEASURE THEORY

Chapter 3 Urysohn $\Rightarrow$ $C_c(X)$ dense in $C_0(X)$

Remark: If $\Phi : C_c(X) \to \mathbb{C}$ is a bounded linear functional then $\exists$ extension $\tilde{\Phi} : C_0(X) \to \mathbb{C}$ of the same norm

Integration with respect to a complex measure

Suppose μ is a complex measure on $(X, \mathfrak{M})$. Then there is a measurable h with $|h| = 1$ everywhere s.t. $d\mu = h\, d|\mu|$, i.e.

$$(*) \qquad \mu(E) = \int_E h\, d|\mu|$$

Note, if h_1 also satisfies $(*)$ and $|h_1| = 1$, then

$$\int_E (h - h_1)\, d|\mu| = 0 \quad \forall E \in \mathfrak{M}$$

and so $h_1 = h$ a.e. $[|\mu|]$. Thus we can define unambiguously for $f \in L^1(\mu)$

$$\int_X f\, d\mu := \int_X f h\, d|\mu|$$

Set $f = \chi_E$ for $E \in \mathfrak{M}$. Then

$$\int_X \chi_E\, d\mu = \int_X \chi_E h\, d|\mu| = \int_E h\, d|\mu| = \mu(E)$$

$$\int_X \chi_E \, d(\mu+\lambda) = (\mu+\lambda)(E) = \mu(E) + \lambda(E)$$
$$= \int_X \chi_E \, d\mu + \int_X \chi_E \, d\lambda$$

Hence $\int f \, d(\mu+\lambda) = \int f \, d\mu + \int f \, d\lambda$ for f = simple function

$\Rightarrow$ for $f \in L^1(|\mu|+|\lambda|)$ (since simple functions dense in this space - given $f \in L^1(|\mu|+|\lambda|)$, $\exists (f_n)$ simple with $f_n \to f$ in $L^1(|\mu|+|\lambda|)$

$$\left| \int (f_n - f) \, d\mu \right| = \left| \int (f_n - f) h \, d|\mu| \right|$$
$$\leq \int |f_n - f| \, d|\mu| \to 0$$

so $\int f_n \, d\mu \to \int f \, d\mu$)

Now take $f \in L^1(|\mu|)$.

$$\mu = (\mathrm{Re}\,\mu)^+ - (\mathrm{Re}\,\mu)^- + i\{(\mathrm{Im}\,\mu)^+ - \mathrm{Im}(\mu)^-\}$$

so that the above shows that

$$\int f \, d\mu = \int f \, d((\mathrm{Re}\,\mu)^+) - \int f \, d(\mathrm{Re}\,\mu^-)$$
$$+ i\left(\int f \, d(\mathrm{Im}\,\mu^+) - \int f \, d(\mathrm{Im}\,\mu^-) \right)$$

Let X be a locally compact T_2-space and μ a positive measure on $(X, \mathfrak{M})$, where $\mathfrak{M} \supset$ Borel sets. Recall that μ is regular if for every Borel set E

$$\sup\{\mu(K) : K \subset E,\ K \text{ compact}\} = \mu(E) = \inf\{\mu(V) : E \subset V,\ V \text{ open}\}$$

<u>DEFINITION</u>: If μ is a complex measure, we call μ regular if $|\mu|$ is regular.

Suppose μ is a complex measure on X. Then

$$f \overset{\Phi}{\longmapsto} \int_X f\, d\mu$$

is a linear functional on $C_0(X)$ and

$$\left| \int f\, d\mu \right| = \left| \int f h\, d|\mu| \right| \le \|f\|_\infty |\mu|(X)$$

so $\Phi : C_0(X) \to \mathbb{C}$ is a bounded linear functional with norm $\le |\mu|(X)$

<u>RIESZ REPRESENTATION THEOREM</u> (#2). Let X be a locally compact T_2-space, $\Phi : C_0(X) \to \mathbb{C}$ a bounded linear functional. Then there is a unique regular complex Borel measure μ s.t. $\|\Phi\| = |\mu|(X)$ and

$$(*) \qquad \Phi(f) = \int_X f\, d\mu$$

Proof. Uniqueness: First show if μ_1 and μ_2 are both regular Borel measures on X, then $\mu_1 - \mu_2$ is a regular Borel measure. Suppose E is a Borel set. Let $\varepsilon > 0$. μ_1 regular $\Rightarrow \exists$ open $V_1 \supset E$ s.t.

$$|\mu_1|(V_1 - E) < \varepsilon$$

μ_2 regular $\Rightarrow \exists$ open $V_2 \supset E$ s.t.

$$|\mu_2|(V_2 - E) < \varepsilon$$

Let $V = V_1 \cap V_2 \supset E$.

$$|\mu_1 - \mu_2|(V - E) \le |\mu_1|(V-E) + |\mu_2|(V-E) < 2\varepsilon$$

Hence $\mu_1 - \mu_2$ is outer regular. Inner regularity works the same.

Suppose μ_1 and μ_2 are regular complex Borel measures satisfying $(*)$. Then

$$\int_X f \, d(\mu_1 - \mu_2) = 0$$

$\forall f \in C_0(X)$. Let $\mu = \mu_1 - \mu_2$ and write $d\mu = h \, d|\mu|$. Consider $(f_n) \subset C_0(X)$ and

$$\int_X (\bar{h} - f_n) h \, d|\mu| = \int_X d|\mu| - \int_X f_n h \, d|\mu| = |\mu|(X) - \int f_n d\mu$$

$$= |\mu|(X)$$

Hence

$$|\mu|(X) = \int (\bar{h} - f_n) h \, d|\mu| \leq \int_X |\bar{h} - f_n| \, d|\mu| \to 0$$

(By chapter 3, since $|\mu|$ is regular, there exist $(f_n) \subset C_c(X)$ s.t. $f_n \to \bar{h}$ in $L^1(|\mu|)$). Hence $|\mu|(X) = 0 \Rightarrow |\mu| = 0 \Rightarrow \mu = 0$

4/19 MEASURE THEORY

By the last lemma, $\exists\, \Lambda : C_c(X) \to \mathbb{C}$ positive linear functional s.t.

$$(*) \qquad |\Phi(f)| \leq \Lambda(|f|) \leq \|f\|_\infty \qquad \forall f \in C_c(X)$$

By the first Riesz Representation theorem, there is a positive measure λ on the Borel sets of X s.t.

$$\Lambda(f) = \int f \, d\lambda \quad \forall f \in C_c(X)$$

Recall

$$\lambda(X) = \sup\{\Lambda f : f \prec X\} = \sup\{\Lambda f : f \in C_c(X), 0 \leq f \leq 1\}$$

So if $f \in C_c(X)$, $0 \leq f \leq 1$ on X, then $(*)$ gives

$$\Lambda f \leq \|f\|_\infty \leq 1$$

and thus $\lambda(X) \leq 1$. By part (d) of RRT #1, $\lambda(X)$ finite implies λ is regular. Moreover $\lambda(X) < \infty \Rightarrow C_c(X) \subset L^1(\lambda)$
For $f \in C_c(X)$

$$|\Phi(f)| \leq \Lambda(|f|) = \int_X |f| \, d\mu = \|f\|_1$$

Therefore $\Phi\big|_{C_c(X)}$ is a bounded linear functional of norm ≤ 1
(regarded as a subspace of $L^1(\lambda)$)

By the Hahn-Banach theorem, Φ extends to a bounded linear functional $\tilde{\Phi}$ on $L^1(\lambda)$ with $\|\tilde{\Phi}\| \leq 1$. Therefore $\exists\, g \in L^\infty(\lambda)$ with $\|g\|_\infty \leq 1$ (so can take $|g(x)| \leq 1$ everywhere) such that

$$\tilde{\Phi}(f) = \int_X fg\,d\lambda \quad \forall f \in L^1(\lambda)$$

Hence if $f \in C_c(X)$

$$\Phi(f) = \int_X fg\,d\lambda$$

Given $f \in C_0(X)$, take $f_n \in C_c(X)$ s.t. $\|f_n - f\|_\infty \to 0$. Then

$$\Phi(f_n) \longrightarrow \Phi(f)$$

$$\left|\int f_n g\,d\lambda - \int fg\,d\lambda\right| \leq \|f_n - f\|_\infty \lambda(X) \longrightarrow 0$$

Hence

$$\Phi(f) = \int_X fg\,d\lambda \quad \forall f \in C_0(X)$$

Define a measure μ by

$$\mu(E) := \int_E g\,d\lambda \quad (E \text{ Borel set})$$

Then

$$\int_X f\,d\mu = \int_X fg\,d\lambda$$

for $f = \chi_E$, E Borel set $\Rightarrow$ for f simple $\Rightarrow$ for f uniform limit of simple functions $\Rightarrow$ for f bounded measurable functions $\Rightarrow$ for $f \in C_0(X)$. Hence

$$\Phi(f) = \int_X fg \, d\lambda = \int_X f \, d\mu$$

for $f \in C_0(X)$

Recall if $\mu(E) := \int_E g \, d\lambda$, then $|\mu|(E) = \int |g| \, d\lambda$

We know λ is regular.

Given a Borel set A and $\varepsilon > 0$, $\exists$ open $V \supset A$ s.t. $\lambda(V - A) < \varepsilon$. By taking ε sufficiently small and setting $E = V - A$, we see that $|\mu|(V-A)$ can be made as small as we wish. Hence μ is regular

Now

$$\int |g| \, d\lambda \geq \sup\left\{ |\Phi(f)| : f \in C_c(X), \|f\|_\infty \leq 1 \right\} = \|\Phi\|$$

so that

$$1 \leq \int |g| \, d\lambda \leq \lambda(X) \leq 1$$

Hence $\lambda(X) = 1$ and $\int |g| \, d\lambda = 1$, so that

$$|\mu|(X) = \int |g| \, d\lambda = 1 = \|\Phi\|$$

$\blacksquare$

INTEGRATION ON PRODUCT SPACES

$(X, \mathcal{S})$, $(Y, \mathcal{T})$ measurable spaces

DEFINITION: $A \times B \subset X \times Y$ is a measurable rectangle if $A \in \mathcal{S}$ and $B \in \mathcal{T}$

DEFINATION: An elementary set is a finite, disjoint union of measurable rectangles

$\mathcal{E}$ = collection of all elementary sets
$\mathcal{S} \times \mathcal{T} :=$ smallest σ-algebra containing the measurable rectangles

DEFINITION: A monotone class of subsets of a set Z is a collection Ω of subsets of Z satisfying

$$E_i \subset E_{i+1} \ , \ E_i \in \Omega \implies \bigcup_{i=1}^{\infty} E_i \in \Omega$$

$$A_{i+1} \subset A_i \ , \ A_i \in \Omega \implies \bigcap_{i=1}^{\infty} A_i \in \Omega$$

If $E \subset X \times Y$, and $x \in X$, $y \in Y$, then

$$E_x := \{ y \in Y : (x,y) \in E \} \subset Y$$

$$E^y := \{ x \in X : (x,y) \in E \} \subset X$$

PROPOSITION: $(X, \mathcal{S}), (Y, \mathcal{T})$ measurable spaces. If $E \in \mathcal{S} \times \mathcal{T}$, then $E_x \in \mathcal{T}$ $\forall x \in X$ and $E^y \in \mathcal{S}$ $\forall y \in Y$

Proof. Let Ω be the collection of all members E of $\mathcal{S} \times \mathcal{T}$ such that $E_x \in \mathcal{T}$ $\forall x \in X$. Sufficient to show Ω is a σ-algebra containing all the measurable rectangles. Then $\mathcal{S} \times \mathcal{T} \subset \Omega$.

Suppose $A \times B$ is a measurable rectangle. Then

$$(A \times B)_x = \begin{cases} B & x \in A \\ \phi & x \notin A \end{cases}$$

and so $A \times B \in \Omega$.

Ω is a σ-algebra:

i) $X \times Y \in \Omega$ $\quad$ $X \times Y$ measurable rectangles

ii) $E \subset X \times Y$, then

$$(E_x)^c = (E^c)_x$$

and so $E \in \Omega \Rightarrow E_x \in \mathcal{T} \Rightarrow (E_x)^c \in \mathcal{T} \Rightarrow E^c \in \Omega$

iii) $E_i \in X \times Y$, then

$$\left(\bigcup_{i=1}^{\infty} E_i\right)_x = \bigcup_{i=1}^{\infty} (E_i)_x$$

Hence $(E_i) \subset \Omega \Rightarrow \bigcup E_i \in \Omega$

Hence $\Omega = \mathcal{S} \times \mathcal{T}$. Do same thing for E^y's.

$\boxtimes$

4/21 MEASURE THEORY

DEFINITION: $f: X\times Y \to Z$ (top. space). Then $f_x : Y \to Z$ is given by

$$f_x(y) = f(x,y)$$

and $f^y : X \to Z$ is given by

$$f^y(x) = f(x,y)$$

PROPOSITION: If $f: X\times Y \to Z$ (top. space) is $\mathcal{S}\times\mathcal{T}$ - measurable, then f_x is $\mathcal{T}$- measure and f^y is $\mathcal{S}$- measurable

Proof. Take V open in Z

$$f_x^{-1}(V) = \{y\in Y : f_x(y)\in V\} = \{y\in Y : f(x,y)\in V\}$$

$$= (f^{-1}(V))_x$$

Now $f^{-1}(V)\in \mathcal{S}\times\mathcal{T}$, so that $(f^{-1}(V))_x \in \mathcal{T}$.
Same thing for f_y.

$\blacksquare$

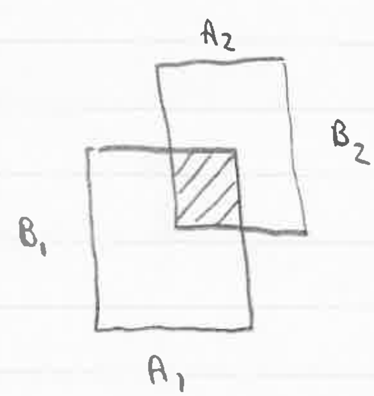
PROPOSITION: $\mathcal{S}\times\mathcal{T}$ is the smallest monotone class containing $\mathcal{E}$

Proof. Let $\mathcal{M}$ be the intersection of all monotone classes containing $\mathcal{E}$ (so $\mathcal{M}$ is the smallest monotone class containing $\mathcal{E}$) Also, $\mathcal{M} \subset \mathcal{S} \times \mathcal{T}$. To show $\mathcal{S} \times \mathcal{T} \subset \mathcal{M}$ it suffices to show that $\mathcal{M}$ is a σ-algebra.

First note if $A_1 \times B_1$ and $A_2 \times B_2$ are measurable rectangles, then

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$$

(measurable rectangles)



$$(A_1 \times B_1) - (A_2 \times B_2) = (A_1 - A_2) \times B_1 \cup (A_1 \cap A_2) \times (B_1 - B_2)$$

(elementary set)

Suppose $P \in \mathcal{E}$, $Q \in \mathcal{E}$. Claim $P \cap Q \in \mathcal{E}$.

$$P = \bigcup_{i=1}^{n} (A_i \times B_i)$$

(disjoint unions)

$$Q = \bigcup_{j=1}^{m} (C_j \times D_j)$$

Then

$$P \cap Q = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (A_i \times B_i) \cap (C_j \times D_j)$$

$$= \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (A_i \cap C_j) \times (B_i \cap D_j) \in \mathcal{E}$$

(disjoint union)

Claim: $P - Q \in \mathcal{E}$

$$P - Q = \bigcap_{j=1}^{m} \bigcup_{i=1}^{n} (A_i \times B_i) - (C_j \times D_j)$$

$A_i \times B_i$'s disjoint $\Rightarrow$ for each j

$$\bigcup_{i=1}^{n} \left((A_i \times B_i) - (C_j \times D_j) \right) \in \mathcal{E}$$

By first claim (extended by induction), $P - Q \in \mathcal{E}$

Claim: $P \cup Q \in \mathcal{E}$.

$$P \cup Q = (P - Q) \cup Q \in \mathcal{E}$$

$\uparrow$ since $Q \cap (P-Q) = \phi$

For $P \subset X \times Y$, let

$$\Omega(P) := \{ Q \subset X \times Y : P - Q \in \mathcal{M},\ Q - P \in \mathcal{M},\ P \cup Q \in \mathcal{M} \}$$

Remarks: a) $Q \in \Omega(P)$ iff $P \in \Omega(Q)$

b) $\forall P$, $\Omega(P)$ is a monotone class.

Proof. Suppose $Q_i \in \Omega(P)$, Q_i monotone $\uparrow$. $Q := \bigcup_{i=1}^{\infty} Q_i$

$$P - Q = P - \left(\bigcup Q_i\right) = \bigcap (P - Q_i) \in \mathcal{M}$$

$\uparrow$

$P - Q_i \in \mathcal{M}\ \forall i$, $(P - Q_i) \downarrow$

Similarly, $Q - P = \bigcup_i (Q_i - P) \in \mathcal{M}$

$\uparrow$

$Q_i - P \in \mathcal{M}\ \forall i$, $(Q_i - P) \uparrow$

Finally, $P \cup Q = \bigcup_i (P \cup Q_i) \in \mathcal{M}$

$\uparrow$

$P \cup Q_i \in \mathcal{M}\ \forall i$, $(P \cup Q_i) \uparrow$

Suppose $P \in \mathcal{E}$. If $Q \in \mathcal{E}$, we know $Q \in \Omega(P)$ and so $\mathcal{E} \subset \Omega(P)$. Definition of $\mathcal{M} \Rightarrow \mathcal{M} \subset \Omega(P)$. Now suppose $Q \in \mathcal{M}$. If $P \in \mathcal{E}$, then $Q \in \Omega(P) \Rightarrow P \in \Omega(Q)$. Hence $\mathcal{E} \subset \Omega(Q) \Rightarrow \mathcal{M} \subset \Omega(Q)$

If $P \in \mathcal{M}$, $Q \in \mathcal{M}$, then $P \in \Omega(Q) \Rightarrow P \cup Q \in \mathcal{M}$ and $\underset{(*)}{P - Q} \in \mathcal{M}$.

Claim: $\mathcal{M}$ is a σ-algebra.

(a) $X \times Y \in \mathcal{M}$ (since $X \times Y \in \mathcal{E}$)

(b) $\mathcal{M}$ closed under complementation by (a) and (*)

(c) $(Q_i) \subset \mathcal{M}$, $Q = \bigcup Q_i$. Let $P_N = \bigcup_{i=1}^{N} Q_i \in \mathcal{M}$

$P_N \uparrow Q \Rightarrow Q \in \mathcal{M}$ ($\mathcal{M}$ monotone class) ▨

PROPOSITION: $(X, \mathcal{S}, \mu)$, $(Y, \mathcal{T}, \lambda)$ positive σ-finite measure spaces. Given $Q \in \mathcal{S} \times \mathcal{T}$, define $\varphi : X \to [0,\infty]$ and $\psi : Y \to [0,\infty]$ by

$$\varphi(x) := \lambda(Q_x)$$

$$\psi(y) := \mu(Q^y)$$

Then φ is $\mathcal{S}$-measure, ψ is $\mathcal{T}$-measurable and

$$(*) \qquad \int_X \varphi(x)\, d\mu(x) = \int_Y \psi(y)\, d\lambda(y)$$

Remark: $\varphi(x) = \lambda(Q_x) = \int_Y \chi_{Q_x}(y)\, d\lambda(y) = \int_Y \chi_Q(x,y)\, d\lambda(y)$

LHS of $(*)$ is thus

$$\int_X \left(\int_Y \chi_Q(x,y)\, d\lambda(y) \right) d\mu(x)$$

But notice that $\psi(y) = \mu(Q^y) = \int_X \chi_{Q^y}(x)\, d\mu(x) = \int_X \chi_Q(x,y)\, d\mu(x)$

so that the RHS is then

$$\int_Y \left(\int_X \chi_Q(x,y)\, d\mu(x) \right) d\lambda(y)$$

(187)

Proof. Let Ω be the collection of all $Q \in S \times T$ for which the conclusion holds. Show

(a) Ω contains all measurable rectangles

Let $Q = A \times B$ (measurable rectangle). Then

$$Q_x = \begin{cases} B & \text{if } x \in A \\ \phi & \text{if } x \notin A \end{cases}$$

and so

$$\varphi(x) = \lambda(Q_x) = \lambda(B)\,\chi_A(x)$$

Hence φ is S-measurable. Similarly

$$\lambda(y) = \mu(Q^y) = \mu(A)\,\chi_B(y)$$

which is T-measurable. Also

$$\int \varphi(x)\,d\mu(x) = \lambda(B)\,\mu(A)$$

$$\int \psi(y)\,d\lambda(y) = \mu(A)\,\lambda(B)$$

(b) Suppose $Q_i \in \Omega$ with $Q_{i+1} \supset Q_i$. Let $Q = \cup Q_i$. Then $Q \in \Omega$

Associated with each Q_i are functions φ_i (S-measurable) and ψ_i (T measurable) with

$$\int_X \varphi_i(x)\, d\mu(x) = \int_Y \psi_i(y)\, d\lambda(y)$$

Now $Q_i \uparrow Q \Rightarrow (Q_i)_x \uparrow Q_x \Rightarrow$

$$\varphi(x) = \lambda(Q_x) = \lim \lambda((Q_i)_x) = \lim \varphi_i(x)$$

Hence $\varphi_i(x) \uparrow \varphi(x)$ $\forall x \in X$, so that φ is $\mathcal{S}$-measurable and

$$\text{MCT} \Rightarrow \lim_{i \to \infty} \int \varphi_i(x)\, d\mu(x) = \int \varphi(x)\, d\mu(x)$$

Similarly, $\psi(y)$ is $\mathcal{T}$-measurable and

$$\lim_{i \to \infty} \int \psi_i(y)\, d\lambda(y) = \int \psi(y)\, d\lambda(y)$$

4/24 MEASURE THEORY

(Proof continued)

Claim: $(Q_i) \subset \Omega$, Q_i disjoint $\Rightarrow \bigcup_{i=1}^{\infty} Q_i \in \Omega$

Let $P = \bigcup_{i=1}^{N} Q_i$. For $x \in X$,

$$P_x = \bigcup_{i=1}^{N} (Q_i)_x$$

(disjoint union) Therefore

$$\lambda(P_x) = \sum_{i=1}^{N} \lambda((Q_i)_x)$$

If $\varphi_i(x) = \lambda((Q_i)_x)$ and $\psi_i(y) = \mu((Q_i)^y)$, then φ_i is $\mathcal{S}$-measurable, ψ_i is $\mathcal{T}$-measurable and

$$\int_X \varphi_i \, d\mu = \int_Y \psi_i \, d\lambda$$

Then $\varphi(x) = \lambda(P_x) = \sum_{i=1}^{N} \varphi_i(x)$, so φ is $\mathcal{S}$-measurable and

$$\int_X \varphi \, d\mu = \sum_{i=1}^{N} \int_X \varphi_i \, d\mu$$

Similarly $\psi(y) = \mu(P^y) = \sum_{i=1}^{N} \psi_i(y)$, so ψ is $\mathcal{T}$-measurable and

$$\int \psi(y)\, d\lambda(y) = \sum_{i=1}^{N} \int \psi_i \, d\lambda$$

Hence $P \in \Omega$.

Now $\cup Q_i$ is an increasing sequence of sets of the form P, so $\cup Q_i \in \Omega$.

CLAIM: If $\mu(A) < \infty$ and $\lambda(B) < \infty$, and if

$$A \times B \supset Q_1 \supset Q_2 \supset Q_3 \supset \ldots$$

Then $\bigcap_{i=1}^{\infty} Q_i \in \Omega$.

Proof. Let $\varphi_i(x) = \lambda((Q_i)_x)$, $\psi_i(y) = \mu((Q_i)^y)$, $Q = \cap Q_i$. Note

$$Q_x = \bigcap_{i=1}^{\infty} (Q_i)_x$$

$\lambda(B) < \infty$ implies $\lambda(Q_x) = \lim_{i \to \infty} \lambda((Q_i)_x)$. Let $\varphi(x) = \lambda(Q_x)$. Then $\varphi(x) = \lim \varphi_i(x)$, so φ is $\mathcal{S}$-measurable. Similarly, $\psi(y) = \lim \psi_i(y)$ is $\mathcal{T}$-measurable.

Note also

$$\int_X \varphi_i \, d\mu = \int_Y \psi_i \, d\lambda$$

Now $\varphi_i(x) = \lambda((Q_i)_x) \le \lambda((A \times B)_x) = \lambda(B) \chi_A(x) \in L^1(\mu)$

and $\psi_i(y) = \mu((Q_i)^y) \leq \mu((A\times B)^y) = \mu(A)\chi_B(y) \in L^1(\lambda)$
By the Dominated Convergence Theorem

$$\int_X \varphi_i \, d\mu \longrightarrow \int_X \varphi \, d\mu$$

$$\int_Y \psi_i \, d\lambda \longrightarrow \int_Y \psi \, d\lambda$$

Hence $Q \in \Omega$

Write $X = \bigcup_{n=1}^{\infty} X_n$ (disjoint) and $Y = \bigcup_{m=1}^{\infty} Y_m$ (disjoint), where $\mu(X_n) < \infty$, $\lambda(Y_m) < \infty$. If $Q \in \mathcal{S}\times\mathcal{T}$, let

$$Q_{mn} := Q \cap (X_n \times Y_m)$$

Let $\mathcal{M}$ be the collection of all $\underline{Q \in \mathcal{S}\times\mathcal{T}}$ s.t. $Q_{mn} \in \Omega$ $\forall n, m$

i) Every measurable rectangle in in $\mathcal{M}$
ii) Every elementary set is in $\mathcal{M}$
iii) $\mathcal{M}$ is a monotone class

Hence $\mathcal{S}\times\mathcal{T} \subset \mathcal{M}$ ($\mathcal{S}\times\mathcal{T}$ smallest monotone class containing $\mathcal{E}$). But $\mathcal{M} \subset \mathcal{S}\times\mathcal{T}$, so $\mathcal{M} = \mathcal{S}\times\mathcal{T}$

Now the second to last claim $\Rightarrow$ every $Q \in \mathcal{S}\times\mathcal{T}$ belongs to Ω

$\boxtimes$

DEFINITION: $(X, \mathcal{S}, \mu)$, $(Y, \mathcal{T}, \lambda)$ σ-finite. For $Q \in \mathcal{S} \times \mathcal{T}$, let

$$(\mu \times \lambda)(Q) := \int_X \lambda(Q_x)\, d\mu(x)$$

$$= \int_Y \mu(Q^y)\, d\lambda(y)$$

PROPOSITION: $\mu \times \lambda$ is a σ-finite measure.

Proof: Take $Q = \bigcup_{i=1}^{\infty} Q_i$ (disjoint). $\forall x \in X$

$$Q_x = \bigcup_{i=1}^{\infty} (Q_i)_x \quad \text{(disjoint)}$$

$$\Longrightarrow \lambda(Q_x) = \sum_{i=1}^{\infty} \lambda((Q_i)_x)$$

Therefore

$$(\mu \times \lambda)(Q) = \int_X \lambda(Q_x)\, d\mu(x) = \int_X \sum_{i=1}^{\infty} \lambda((Q_i)_x)\, d\mu(x)$$

MCT

$$= \sum_{i=1}^{\infty} \int_X \lambda((Q_i)_x)\, d\mu(x)$$

$$= \sum_{i=1}^{\infty} (\mu \times \lambda)(Q_i)$$

Consider $A\times B$, where $\mu(A)<\infty$, $\mu(B)<\infty$.

$$(\mu\times\lambda)(A\times B) = \int_X \lambda((A\times B)_x)\, d\mu(x) = \int_X \lambda(B)\chi_A(x)\, d\mu(x)$$

$$= \lambda(B)\mu(A) < \infty$$

$\boxtimes$

<u>FUBINI'S THEOREM</u> $(X, \mathcal{S}, \mu)$, $(Y, \mathcal{T}, \lambda)$ σ-finite measure spaces. Let $f(x,y)$ be measurable w.r.t. $\mathcal{S}\times\mathcal{T}$.

(a) Suppose $f \geq 0$. Let

$$\varphi(x) := \int_Y f_x(y)\, d\lambda(y)$$

(1)

$$\psi(y) := \int_X f^y(x)\, d\mu(x)$$

Then

$$(*) \quad \int_X \varphi(x)\, d\mu(x) = \int_{X\times Y} f(x,y)\, d(\mu\times\lambda) = \int_Y \psi(y)\, d\lambda(y)$$

(b) Let

$$\varphi^*(x) := \int_Y |f_x(y)|\, d\lambda(y)$$

Then if $\int \varphi^*(x)\,d\mu(x) < \infty$, we have $f \in L^1(\mu \times \lambda)$

(c) If $f \in L^1(\mu\times\lambda)$, then $f_x \in L^1(\lambda)$ for almost all x $[\mu]$ and $f^y \in L^1(\mu)$ for almost all y $[\lambda]$, and the functions φ and ψ defined a.e. by equations (1) are in $L^1(\mu)$ and $L^1(\lambda)$ respectively. Furthermore, (*) holds

Remark about (a):

$$\int_X\int_Y f(x,y)\,d\lambda(y)\,d\mu(x) = \int_{X\times Y} f(x,y)\,d(\mu\times\lambda) = \int_Y\int_X f(x,y)\,d\mu(x)\,d\lambda(y)$$

Proof of (a). We know this holds if $f = \chi_Q$ where $Q \in \mathcal{S}\times\mathcal{T}$ so it holds for simple functions. Given $f \geq 0$, there exist simple $s_n \uparrow f$ on $X\times Y$ with each s_n $\mathcal{S}\times\mathcal{T}$-measurable. Let

$$\varphi_n(x) = \int_Y s_n(x,y)\,d\lambda(y)$$

$$\psi_n(y) = \int_X s_n(x,y)\,d\mu(x)$$

Let

$$\varphi(x) = \int_Y f(x,y)\,d\lambda(y)$$

$$\psi(y) = \int_X f(x,y)\,d\mu(x)$$

By the Monotone Convergence theorem, $\varphi_n(x) \uparrow \varphi(x)$ and $\psi_n(y) \uparrow \psi(y)$. Since (a) holds for each s_n,

$$\int \varphi_n(x)\,d\mu(x) = \int s_n(x,y)\,d(\mu\times\lambda) = \int \psi_n(y)\,d\lambda(y)$$

MCT $\quad\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$

$$\int \varphi(x)\,d\mu(x) \qquad \int f(x,y)\,d(\mu\times\lambda) \qquad \int \psi(y)\,d\lambda(y)$$

4/26 MEASURE THEORY

Remarks about Fubini theorem: Summary - a) iterated integrals are equal if $f \geq 0$

(b),(c): If one of the iterated integrals of $|f|$ is finite then the two iterated integrals of f are equal

Proof of (b) Apply a to $|f|$. (*) becomes

$$\int |f(x,y)| \, d(\mu\times\lambda) = \int_X \varphi^*(x) \, d\mu(x) < \infty$$

(c) First assume f is real. Write $f = f^+ - f^-$. Apply (a) to f^+, f^-. Let

$$\varphi_1(x) := \int_Y f_x^+(y) \, d\lambda(y)$$

$$\varphi_2(x) := \int_Y f_x^-(y) \, d\lambda(y)$$

φ_j is $\mathcal{S}$-measurable and

$$\int_X \varphi_1(x) \, d\mu(x) = \int_{X\times Y} f^+(x,y) \, d(\mu\times\lambda) < \infty$$

Hence $\varphi_1(x) < \infty$ a.e. $[\mu]$. Similarly $\varphi_2(x) < \infty$ a.e. $[\mu]$

$\Rightarrow (f^+)_x \in L^1[\lambda]$ $\qquad \Rightarrow (f^-)_x \in L^1[\lambda]$

But $f_x = (f^+)_x - (f^-)_x$, so $f_x \in L^1(\mu)$ for almost all x $[\mu]$.
For x s.t. $\varphi_j(x) < \infty$

$$\varphi(x) = \int_Y f_x(y)\, d\lambda(y) = \varphi_1(x) - \varphi_2(x)$$

so $\varphi(x) = \varphi_1(x) - \varphi_2(x)$ a.e. $[\mu]$. Since $\varphi_j \in L^1(\mu)$, we get $\varphi \in L^1(\mu)$.

$$\int_X \varphi(x)\, d\mu(x) = \int_X \varphi_1(x)\, d\mu(x) - \int \varphi_2(x)\, d\mu(x)$$

$$= \int_{X\times Y} f(x,y)\, d(\mu\times\lambda)$$

(Part of proof for ψ is done in the same way)

Now consider $f = u + iv$, so $u \in L^1(\mu\times\lambda)$ and $v \in L^1(\mu\times\lambda)$. Then $u_x \in L^1(\lambda)$ a.e. $[\mu]$ and $v_x \in L^1(\lambda)$ a.e. $[\mu]$ whence $f_x \in L^1(\lambda)$ a.e. $[\mu]$. We also have

$$\int_Y u_x(y)\, d\lambda(y) \in L^1(\mu)$$

$$\int_Y v_x(y)\, d\lambda(y) \in L^1(\mu)$$

$$\Longrightarrow \int_Y f_x(y)\, d\lambda(y) \in L^1(\mu), \text{ i.e. } \varphi(x) \in L^1(\mu)$$

$$\int_X \varphi(x)\, d\mu(x) = \int_X (u_x + iv_x)\, d\mu(x) = \int_{X\times Y} u(x,y) + iv(x,y)\, d(\mu\times\lambda)$$

$\boxtimes$

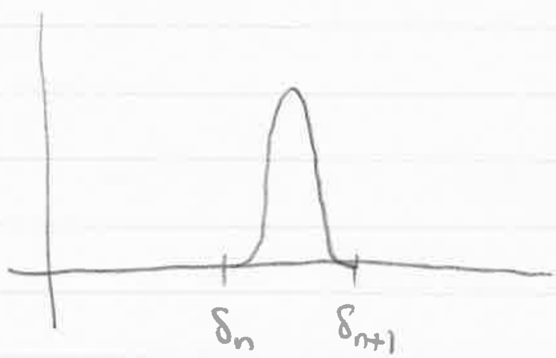
EXAMPLES

I. $X = Y = [0,1]$, Lebesgue measure. Take

$$0 = \delta_1 < \delta_2 < \cdots < \delta_n \longrightarrow 1$$

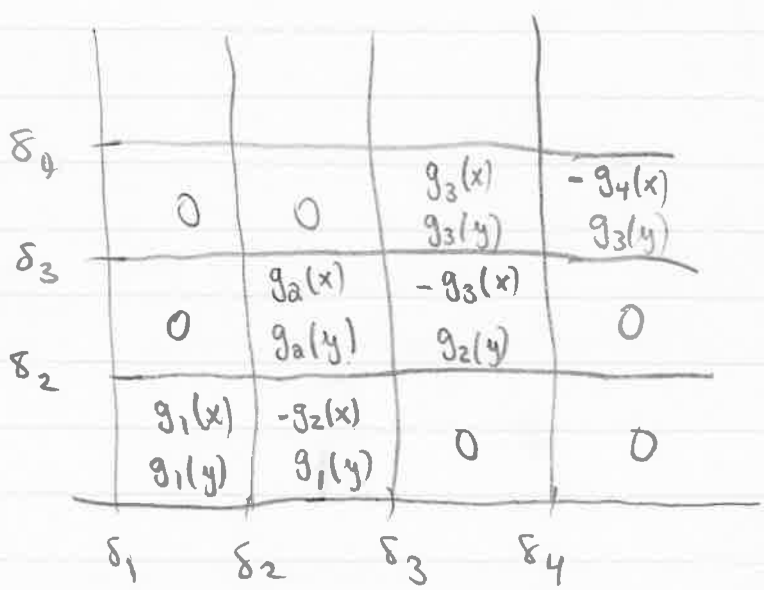
Define $g_n : [0,1] \to [0,\infty)$ s.t. $\operatorname{supp} g_n \subset (\delta_n, \delta_{n+1})$ and
(continuous)

$$\int_0^1 g_n(t)\,dt = 1$$



Define

$$f(x,y) := \sum_{n=1}^{\infty} \left[g_n(x) - g_{n+1}(x) \right] g_n(y)$$



For $\delta_N < y \leq \delta_{N+1}$

$$\int_0^1 f(x,y)\,dx = g_N(y) \int_0^1 [g_N(x) - g_{N+1}(x)]\,dx = 0$$

$$\Longrightarrow \int_0^1 \int_0^1 f(x,y)\,dx\,dy = 0$$

For $N \geq 2$, $\delta_N < x \leq \delta_{N+1}$

$$\int_0^1 f(x,y)\,dy = g_N(x) \int_0^1 [g_N(y) - g_{N-1}(y)]\,dy = 0$$

For $\delta_1 < x \leq \delta_2$

$$\int_0^1 f(x,y)\,dy = g_1(x) \int_0^1 g_1(y)\,dy = g_1(x)$$

$$\Longrightarrow \int_0^1 \int_0^1 f(x,y)\,dy\,dx = \int_0^1 g_1(x)\,dx = 1$$

<u>NOTE</u>

$$\int_0^1 |f(x,y)|\,dx = 2\,g_N(y)$$

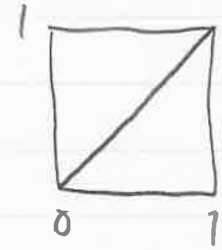
for $\delta_N < y \leq \delta_{N+1}$, and so

$$\int_0^1\int_0^1 |f(x,y)|\,dx\,dy = \infty$$

so that $f \notin L^1(\mu\times\lambda)$

II: $X = Y = [0,1]$ Lebesgue measure on X μ
Counting measure on Y λ

$$f := \chi_D$$



D = diagonal of square
$\mathcal{S}\times\mathcal{T}$ measurable

$$\int_0^1 \chi_D(x,y)\,d\lambda(y) = 1$$

$$\Longrightarrow \int_0^1\int_0^1 f(x,y)\,d\lambda(y)\,d\mu(x) = 1$$

But

$$\int_0^1 \chi_D(x,y)\,d\mu(x) = 0$$

$$\Longrightarrow \int_0^1\int_0^1 f(x,y)\,d\mu(x)\,d\lambda(y) = 0$$

NOTE: Y, λ is not σ-finite

III. $X = Y = [0,1]$ Lebesgue measure

Continuum hypothesis $\Rightarrow \exists\, j : [0,1] \xrightarrow{1-1} W$ (well-ordered)
s.t. $\forall x \in [0,1]$, $j(x)$ has at most countably many predecessors
Define

$$Q = \{(x,y) : j(x) \text{ precedes } j(y) \text{ in } W\}$$

$$\int_0^1 \chi_Q(x,y)\,dy = 1$$

$\uparrow$

$=1$ except on a countable set

$$\Rightarrow \int_0^1 \int_0^1 \chi_Q(x,y)\,dy\,dx = 1$$

But
$$\int_0^1 \chi_Q(x,y)\,dx = 0$$

$\uparrow$

$=0$ except on a countable set

$$\Rightarrow \int_0^1 \int_0^1 \chi_Q(x,y)\,dx\,dy = 0$$

Note. χ_Q is not $\mathcal{S} \times \mathcal{T}$ measurable

4/28 MEASURE THEORY

THEOREM: If $f, g \in L^1(\mathbb{R})$, then $|f(x-y)\,g(y)| \in L^1(\mathbb{R})$ for almost every x, i.e.

$$\int_{\mathbb{R}} |f(x-y)\,g(y)|\,dy < \infty \quad (\text{almost all } x)$$

For such x, define

$$h(x) := \int_{\mathbb{R}} f(x-y)\,g(y)\,dy \qquad (\text{convolution})$$

Then $\|h\|_1 \le \|f\|_1 \|g\|_1$.

Proof. WLOG f and g are Borel measurable and finite everywhere.

[[Lusin's theorem $\Rightarrow \exists$ continuous f_n s.t. $f_n \to f$ a.e.
Let

$$F := \overline{\lim}\,(\operatorname{Re} f_n) + i\,\overline{\lim}\,(\operatorname{Im} f_n)$$

Then F is Borel measurable and $F = f$ a.e. If either $\overline{\lim}\,(\operatorname{Re} f_n(x)) = \pm\infty$ or $\overline{\lim}\,(\operatorname{Im} f_n(x)) = \pm\infty$, modify F at that x to be 0. We still have $F = f$ a.e., and F is Borel measurable. Moreover, now F is finite everywhere.

Note that the integrands in the theorem are changed only on sets of measure 0.]]

Let $F(x,y) := f(x-y)\, g(y)$. F is measurable w.r.t. $\mathcal{B}_2$, the σ-algebra of Borel sets in $\mathbb{R}^2$. For let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$\varphi(x,y) := x-y$$

Then φ is Borel measurable. Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$\psi(x,y) := y$$

Then ψ is Borel measurable. Then $f\circ\varphi, g\circ\psi$ Borel measurable and

$$F = (f\circ\varphi)(g\circ\psi)$$

so F is Borel measurable
Let $\mathcal{B}_1$ be the Borel sets in $\mathbb{R}$.

Exercise : $\mathcal{B}_2 = \mathcal{B}_1 \times \mathcal{B}_1$

Therefore F is $\mathcal{B}_1 \times \mathcal{B}_1$ measurable. Now notice that

$$\int_{\mathbb{R}} |F(x,y)|\, dx = |g(y)| \int_{\mathbb{R}} |f(x-y)|\, dx = |g(y)|\, \|f\|_1$$

translation invariance of Lebesgue measure

$$\int\int |F(x,y)|\, dx\, dy = \|f\|_1 \int |g(y)|\, dy = \|f\|_1 \|g\|_1$$

Therefore by Fubini (b), $F \in L^1(m_1 \times m_1)$, and from (c), for almost every x, $F_x(y) \in L^1(\mathbb{R})$, i.e.

$$h(x) = \int f(x-y)\, g(y)\, dy \quad \text{exists}$$

and $h \in L^1(\mathbb{R})$. Note

$$\|h\|_1 = \int_{\mathbb{R}} |h(x)|\, dx \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y)\, g(y)|\, dy\, dx$$

$$\underset{\text{Fubini}}{=} \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y)\, g(y)|\, dx\, dy = \|f\|_1 \|g\|_1$$

$\boxed{/\!/}$

EXAMPLE: $(X, \mathcal{S}, \mu)$ $(Y, \mathcal{T}, \lambda)$

Suppose $\exists\, A \in \mathcal{S}$ such that $\mu(A) = 0$ and $A \neq \emptyset$.
Suppose $\exists\, B \subset Y$ such that $B \notin \mathcal{T}$
$\}$ very weak hypotheses

Claim: $\mu \times \lambda$ is not complete, i.e. $(X \times Y, \mathcal{S} \times \mathcal{T}, \mu \times \lambda)$ is not a complete measure space

$A \times B \subset A \times Y$ and

$$(\mu \times \lambda)(A \times Y) = \int \chi_{A \times Y}(x,y)\, d(\mu \times \lambda)$$

$$= \int_X \int_Y \chi_{A\times Y}(x,y)\, d\lambda(y)\, d\mu(x)$$

$$= \int_X \lambda(Y)\, \chi_A(x)\, d\mu(x) = \lambda(Y)\mu(A) = 0$$

If $A\times B \in \mathcal{S}\times\mathcal{T}$, then $(A\times B)_x \in \mathcal{T}$ $\forall x\in X$. But

$$(A\times B)_x = \begin{cases} \emptyset & x\notin A \\ B & x\in A \end{cases}$$

Since $A\neq\emptyset$, $\exists\, x_0\in A$, whence $B = (A\times B)_{x_0} \in \mathcal{T}$ ↯.
Therefore $A\times B \notin \mathcal{S}\times\mathcal{T}$.

<u>THEOREM</u>: m_p be Lebesgue measure on $\mathbb{R}^p$. Then the completion of $m_r \times m_s$ is m_k, where $k = r+s$.

(Recall: $(X,\mathcal{M},\mu)$ $\mathcal{M}^* := \{E : \exists\, A\subset E\subset B,\ A\in\mathcal{M}, B\in\mathcal{M}, \mu(B-A) = 0\}$. For $E\in\mathcal{M}^*$, let $\mu^*(E) = \mu(A)$. $(X,\mathcal{M}^*,\mu^*)$ is the completion of $(X,\mathcal{M},\mu)$.)

Proof. Let $\mathcal{B}_k$ be the Borel sets of $\mathbb{R}^k$ and $\mathcal{M}_k$ the Lebesgue measurable sets. First note

$$\mathcal{B}_k \subset \mathcal{M}_r \times \mathcal{M}_s \subset \mathcal{M}_k$$

Every Euclidean rectangle in $\mathbb{R}^k$ is a measurable rectangle, hence in $\mathcal{M}_r \times \mathcal{M}_s$. Hence $\mathcal{M}_r \times \mathcal{M}_s$ contains all open sets in $\mathbb{R}_k$ and hence all Borel sets.

Suppose $E \in \mathcal{M}_r$. Claim: $E \times \mathbb{R}_s \in \mathcal{M}_k$. Recall $E \in \mathcal{M}_p$ iff $\exists$ F_σ set A, G_δ set B s.t.

$$A \subset E \subset B$$
$$m_p(B-A) = 0$$

Hence $\exists$ F_σ set A in $\mathbb{R}^r$ and a G_δ set B in $\mathbb{R}^r$ s.t. $A \subset E \subset B$ and $m_r(B-A) = 0$. Then

$$\underset{G_\delta}{B \times \mathbb{R}^s} \supset E \times \mathbb{R}^s \supset \underset{F_\sigma}{A \times \mathbb{R}^s}$$

$$(B \times \mathbb{R}^s) - (A \times \mathbb{R}^s) = (B-A) \times \mathbb{R}^s.$$

Thm 2.20 $m_r \times r_s$ $\left\{\begin{array}{l}\text{translation inv.}\\ \text{finite compact sets}\\ \text{defined on}\\ \text{Borel sets}\end{array}\right.$ $\Downarrow$ $m_r \times m_s$ multiple of m_k on Borel sets

$$m_k((B-A)\times \mathbb{R}^s) = (m_r \times m_s)((B-A) \times \mathbb{R}^s)$$

$$= m_r(B-A)\, m_s(\mathbb{R}^s)$$

$$= 0 \cdot \infty = 0$$

Hence $E \times \mathbb{R}^s \in \mathcal{M}_k$. Same argument shows $\mathbb{R}^r \times F \in \mathcal{M}_k$ if $F \in \mathcal{M}_s$. Therefore

$$E \times F = (E \times \mathbb{R}^s) \cap (\mathbb{R}^r \cap F) \in \mathcal{M}_k$$

Hence $\mathfrak{M}_r \times \mathfrak{M}_s \subset \mathfrak{M}_k$

5/1 MEASURE THEORY

COMPLETION OF PROOF

(Have shown $\mathcal{B}_k \subset \mathfrak{M}_r \times \mathfrak{M}_s \subset \mathfrak{M}_k$)

CLAIM: $m_r \times m_s$ coincides with m_k on $\mathfrak{M}_r \times \mathfrak{M}_s$

Suppose $Q \in \mathfrak{M}_r \times \mathfrak{M}_s$. Then $Q \in \mathfrak{M}_k$, so there are F_σ-set A and G_δ-set B s.t.

$$m_k(B-A) = 0$$
$$A \subset Q \subset B$$

Then

$$(m_r \times m_s)(Q-A) \le (m_r \times m_s)(B-A) \underset{\substack{\uparrow \\ \text{Thm } 2.20}}{=} m_k(B-A) = 0$$

and so

$$(m_r \times m_s)(Q) = (m_r \times m_s)(A) = m_k(A) = m_k(Q)$$

We want to show $(\mathbb{R}^k, (\mathfrak{M}_r \times \mathfrak{M}_s)^*, (m_r \times m_s)^*) = (\mathbb{R}^k, \mathfrak{M}_k, m_k)$

Suppose $Q \in (\mathfrak{M}_r \times \mathfrak{M}_s)^*$. By definition $\exists\, A \subset Q \subset B$ where $A, B \in \mathfrak{M}_r \times \mathfrak{M}_s$ and $m_r \times m_s (B-A) = 0$. Therefore $m_k(B-A) = 0$

$A \in \mathfrak{M}_k$, $Q - A \in \mathfrak{M}_k \Rightarrow Q \in \mathfrak{M}_k$ and $m_k(Q) = m_k(A) = (m_r \times m_s)^*(Q)$

Suppose $Q \in \mathcal{M}_k$. $\exists$ Borel sets A, B s.t. $A \subset Q \subset B$ and $m_k(B-A)=0$. But $m_r \times m_s(B-A) = m_k(B-A) = 0$, so $Q \in (\mathcal{M}_r \times \mathcal{M}_s)^*$. Moreover

$$(m_r \times m_s)^*(Q) = (m_r \times m_s)(A) = m_k(A) = m_k(Q)$$

$\boxtimes$

Since $B_2 \subset \mathcal{M}_1 \times \mathcal{M}_1$, to show $F(x,y)$ is measurable, it suffices to show F is Borel measurable (recall composition of Borel measurable functions is measurable)

Differentiation of Measures

Let $m = m_k$ on $\mathbb{R}^k$

__Definition:__ If E_i is a sequence of Borel sets in $\mathbb{R}^k$, $x \in \mathbb{R}^k$, we say E_i shrinks to x nicely if $\exists$ $r_i \downarrow 0$, $\alpha > 0$ s.t.

$$E_i \subset B(x; r_i)$$
$$m(E_i) > \alpha\, m(B(x; r_i))$$

Definition: Suppose μ is a complex Borel measure on $\mathbb{R}^k$. Suppose $x \in \mathbb{R}^k$. If

$$\lim_{i\to\infty} \frac{\mu(E_i)}{m(E_i)} = A$$

for every sequence of Borel sets E_i which shrinks to x nicely, we say the derivative of μ w.r.t. m at x is A, and write

$$D\mu(x) = A$$

PROPOSITION: Suppose Ω is a collection of open balls in $\mathbb{R}_k$. Suppose $t < m(\bigcup_{\Omega} B)$. Then there is a disjoint subcollection $\{B_1, \ldots, B_N\} \subset \Omega$ s.t.

$$\sum_{i=1}^{N} m(B_i) > 3^{-k} t$$

Proof. Since m is regular, there is a compact K s.t. $t < m(K)$ and $K \subset \bigcup_{\Omega} B$. By compactness

$$K = S_1 \cup \ldots \cup S_M$$

where $S_i \in \Omega$ and radius $S_j \geq$ radius S_{j+1}. Let $B_1 = S_1$. Discard all S_j s.t. $S_j \cap S_1 \neq \phi$. Let $B_2 = 1^{st}$ surviving S. Discard all S_j s.t. $S_j \cap S_2 \neq \phi$. Continue until process stops. Arrive at a disjoint collection $B_1, B_2, \ldots, B_N$. The union of all the S_j's $\subset$ the union of balls β_i, where center β_i = center B_i, radius $\beta_i = 3$ radius B_i

$$t < m(K) \leq \sum_{i=1}^{N} m(\beta_i) = 3^k \sum_{i=1}^{N} m(B_i)$$ ▨

LEMMA: μ = positive Borel measure on $\mathbb{R}^k$, finite on compact sets (Recall this implies μ is regular). If $\mu(A)=0$, then $\exists\, A' \subset A$, A' Lebesgue measurable s.t. A Borel measurable

(1) $m(A-A') = 0$
(2) $D\mu(x) = 0 \quad \forall x \in A'$

Proof: Since μ is regular, if $\varepsilon>0$ $\exists$ open $V \supset A$ s.t. $\mu(V) < \varepsilon$. Let

$$A' := \left\{ x \in A : \lim_{r\to 0} \frac{\mu(B(x,r))}{m(B(x,r))} = 0 \right\}$$

Let

$$P_j := \left\{ x \in A : \overline{\lim_{r\to 0}} \frac{\mu(B(x,r))}{m(B(x,r))} \geq 1/j \right\}$$

CLAIM: $m(P_j) = 0$ and $\bigcup_{j=1}^{\infty} P_j = A - A'$

(This proves (1))

If $x \in P_j$, $\exists\, r = r(x)$ s.t. $B(x; r(x)) \subset V$ and $m(B(x, r(x))) \leq j\,\mu(B(x, r(x)))$. Then

$$P_j \subset \bigcup_{x\in P_j} B(x; r(x))$$

By the proposition, if we could find $t < m\left(\bigcup_{x \in P_j} B(x, r(x))\right)$, then

$\exists \{B_1, \dots, B_N\}$ s.t
(disjoint)

$$t < 3^{-k} \sum_{k=1}^{N} m(B(x_i ; r_i(x_i))) < j\, 3^{-k} \sum_{k=1}^{N} \mu(B(x_i ; r_i(x_i)))$$

$$< j\, 3^{-k} \mu(V) < j\, 3^{-k} \varepsilon$$

By ε was arbitrary, no such t exists, so $m\left(\bigcup_{x \in P_j} B(x, r(x))\right) = 0$
Therefore $m(P_j) = 0$.

If $x \in A'$ and (E_j) shrinks to x nicely, then

$$\frac{\mu(E_i)}{m(E_i)} \leq \frac{\mu(B(x, r_i))}{\alpha m(B(x, r_i))} \to 0$$

$\blacksquare$

5/3 MEASURE THEORY

THEOREM: Suppose μ is a complex Borel measure on $\mathbb{R}^k$.
(a) $D\mu(x)$ exists a.e. $[m]$
(b) $D\mu(x) \in L^1(\mathbb{R}^k, m)$
(c) $\exists$ complex μ_s with $\mu_s \perp m$ and $D\mu_s(x) = 0$ a.e. $[m]$.

and moreover

$$(*) \qquad \mu(E) = \mu_s(E) + \int_E D\mu(x)\, dm(x)$$

for every Borel set E. (This gives the Lebesgue decomposition of μ w.r.t m and shows that the Radon-Nykodym derivative of μ is $D\mu$)

COROLLARY: μ complex Borel measure on $\mathbb{R}^k$
(i) $\mu \perp m$ iff $D\mu(x) = 0$ a.e. $[m]$
(ii) $\mu \ll m$ iff $\mu(E) = \int_E D\mu(x)\, dm(x)$ $\forall$ Borel set E

Proof of Corollary: Recall $\mu = \mu_1 + \mu_2$ (uniquely) where $\mu_1 \perp m$ and $\mu_2 \ll m$.

(i) If $\mu \perp m$, then $\mu = \mu_s$, and so $D\mu(x) = D\mu_s(x) = 0$ a.e. $[m]$. On the other hand, if $D\mu = 0$ a.e., then from $(*)$ $\mu = \mu_s$ and so $\mu \perp m$.

(ii) If $\mu(E) = \int_E D\mu(x)\, dm(x)$, then certainly $\mu \ll m$. If $\mu \ll m$, then by uniqueness $\mu_s = 0$ and so $\mu(E) = \int_E D\mu(x)\, dm(x)$

Proof of theorem: It is sufficient to prove separately for the cases $\mu \perp m$ and $\mu \ll m$. For in general, $\mu = \mu_1 + \mu_2$

where $\mu_1 \perp m$ and $\mu_2 << m$. Suppose theorem holds for μ_1 and μ_2. Then we know $D\mu_1$ exists a.e. and $D\mu_1 \in L^1(\mathbb{R}^k, m)$. Moreover (c) says $D\mu_1 = D\mu_s = 0$ a.e. $[m]$. Also $D\mu_2$ exists a.e. and $D\mu_2 \in L^1(\mathbb{R}^k, m)$. Then $D\mu = D\mu_1 + D\mu_2 \in L^1(\mathbb{R}^k, m)$ and

$$\mu(E) = \mu_1(E) + \mu_2(E) = \mu_1(E) + \int_E D\mu_2(x)\, dm(x)$$

$$= \mu_1(E) + \int_E D\mu(x)\, dm(x) \qquad [[D\mu = D\mu_2 \text{ a.e.}]]$$

It is also sufficient to prove for the real and imaginary parts of μ separately

CASE I: μ real, $\mu \perp m$

$\mu^+ = \frac{1}{2}(|\mu| + \mu) \perp m$, so $\exists$ Borel set A s.t. m is concentrated on A and μ^+ is concentrated on $\mathbb{R}^k - A$

$$m(\mathbb{R}^k - A) = 0 = \mu^+(A)$$

The previous lemma $\Rightarrow \exists A' \subset A$ s.t. $m(A - A') = 0$ and $D\mu^+(x) = 0$ everywhere on A'. Hence $D\mu^+ = 0$ a.e. $[m]$. Similarly $D\mu^- = 0$ a.e. $[m]$, so $D\mu = 0$ a.e. $[m]$. Then (a), (b), (c) are satisfied

CASE II: μ real, $\mu << m$

Radon-Nikodym theorem $\Rightarrow \exists$ Borel measurable

$f \in L^1(\mathbb{R}^k, m)$ s.t.

$$\mu(E) = \int_E f \, dm \quad \forall \text{Borel } E$$

It is sufficient to show $f(x) = D\mu(x)$ a.e.

For $r \in \mathbb{Q}$, let

rationals in $\mathbb{C}$

$$A_r := \{ x : f(x) < r \}$$
$$B_r := \{ x : f(x) \geq r \}$$

(Borel sets)

For $r \in \mathbb{Q}$, define a positive measure λ_r on the Borel sets by

$$\lambda_r(E) := \int_{E \cap B_r} (f(x) - r) \, dm(x)$$

Note that $\lambda_r(A_r) = 0$ since $A_r \cap B_r = \phi$. By the lemma $\exists A_r' \subset A_r$ s.t. $m(A_r - A_r') = 0$ and $D\lambda_r(x) = 0$ on A_r'. Let

$$Y = \bigcup_{r \in \mathbb{Q}} (A_r - A_r')$$

Then Y is Lebesgue measurable with $m(Y) = 0$. Suppose $x \notin Y$. Sufficient to show $D\mu(x) = f(x)$. Consider a sequence of Borel sets E_i shrinking to x nicely. Consider $r \in \mathbb{Q}$ with $r > f(x)$. Then $x \in A_r$. But $x \notin Y$, so we must have $x \in A_r'$. Therefore $D\lambda_r(x) = 0$.

$$\mu(E_i) - r\,m(E_i) = \int_{E_i} (f(x)-r)\, dm(x)$$

$$\Longrightarrow \quad \frac{\mu(E_i)}{m(E_i)} - r = \frac{1}{m(E_i)} \int_{E_i} (f(t)-r)\, dm(t)$$

$$\leq \frac{1}{m(E_i)} \int_{E_i \cap B_r} (f(t)-r)\, dm(t)$$

$$= \frac{\lambda_r(E_i)}{m(E_i)} \xrightarrow[i\to\infty]{} D\lambda_r(x) = 0$$

Hence $\overline{\lim} \dfrac{\mu(E_i)}{m(E_i)} \leq r \implies \overline{\lim} \dfrac{\mu(E_i)}{m(E_i)} \leq f(x)$

Now consider $-\mu$. Its R-N derivative is $-f$. Applying result just obtained, we get

$$\overline{\lim} \frac{-\mu(E_i)}{m(E_i)} \leq -f(x)$$

$$\shortparallel$$

$$-\underline{\lim} \frac{\mu(E_i)}{m(E_i)}$$

Hence $\underline{\lim} \dfrac{\mu(E_i)}{m(E_i)} \geq f(x)$. Therefore $f(x) = \lim\limits_{i\to\infty} \dfrac{\mu(E_i)}{m(E_i)} = D\mu(x)$. ▨

5/5 ANALYSIS

Remark: Suppose $f \in L^1(\mathbb{R}^k, m)$. Define

$$\mu(E) := \int_E f(x)\,dm(x) \quad \forall \text{Borel } E$$

D.C.T. $\Rightarrow$ μ complex Borel measure. Moreover, $\mu \ll m$. By (c) of the last theorem,

$$\mu(E) = \int_E D\mu(x)\,dm(x) \quad \forall \text{Borel } E$$

Therefore $D\mu(x) = f(x)$ a.e. on $[m]$. Suppose x_0 is such that $f(x_0) = D\mu(x_0)$. Consider a seq. of Borel sets E_i shrinking nicely to x_0.

$$\frac{\mu(E_i)}{m(E_i)} - f(x_0) = \frac{1}{m(E_i)} \int_{E_i} [f(x) - f(x_0)]\,dm(x)$$

As $i \to \infty$, LHS tends to $D\mu(x_0) - f(x_0) = 0$. Hence

$$\lim_{i \to \infty} \frac{1}{m(E_i)} \int_{E_i} [f(x) - f(x_0)]\,dm(x) = 0$$

Specifically, take the case $f = \chi_Q$, where $m(Q) < \infty$. Then $\mu(E) = m(E \cap Q)$. For almost every x_0, for every (E_i) Borel sets shrinking to x_0 nicely, we have

$$\frac{m(Q \cap E_i)}{m(E_i)} \longrightarrow \chi_Q(x_0)$$

(density of $Q = \chi_Q$ a.e.)

<u>THEOREM</u>: Suppose $f \in L^1(\mathbb{R}^k)$. Let L_f (the Lebesgue set of f) be the set of all $x_0 \in \mathbb{R}^k$ such that

$$\lim_{i \to \infty} \frac{1}{m(E_i)} \int_{E_i} |f(x) - f(x_0)| \, dm(x) = 0$$

for every sequence of Borel sets E_i shrinking nicely to x_0. Then

$$m(\mathbb{R}^k - L_f) = 0$$

Proof. Sufficient to show $m(B(0,1) - L_f) = 0$. For $r \in \mathbb{Q}$, define for E Borel

$$\mu_r(E) = \int_E |f(x) - r| \, \chi_{B(0,2)}(x) \, dm(x)$$

Then μ_r is a complex Borel measure on $\mathbb{R}^k$. As before

$$D\mu_r(x) = |f(x) - r| \, \chi_{B(0,2)}(x) \quad \text{a.e. } [m]$$

Let $Y_r = \{x \in B(0;1) : D\mu_r(x) \neq |f(x) - r|\}$. Then $m(Y_r) = 0$.
Let $Y = \bigcup_{r \in \mathbb{Q}} Y_r$. Then $m(Y) = 0$.

If $x_0 \in B(0,1) - Y$, we will show $x_0 \in L_f$. Given $\varepsilon > 0$, $\exists \tilde{r} \in \mathbb{Q}^2$ s.t. $|f(x_0) - \tilde{r}| < \varepsilon$. Then if (E_i) shrinks nicely to x_0

$$\frac{1}{m(E_i)} \int_{E_i} |f(x) - f(x_0)| \, dm(x)$$

$$\leq \frac{1}{m(E_i)} \int_{E_i} \left(|f(x) - \tilde{r}| + |\tilde{r} - f(x_0)| \right) dm(x)$$

$$\leq \varepsilon + \frac{1}{m(E_i)} \int_{E_i} |f(x) - \tilde{r}| \, dm(x)$$

$$= \varepsilon + \underbrace{\frac{\mu_{\tilde{r}}(E_i)}{m(E_i)}}_{\to\, |f(x_0) - \tilde{r}| \text{ since } x_0 \notin Y_{\tilde{r}}} < 2\varepsilon \quad \text{if } i \text{ large}$$

$\blacksquare$

Functions of Bounded Variation

Definition: $f : \mathbb{R} \to \mathbb{C}$. $x_0 < x_1 < \dots < x_N = x$. (partition)

$$T_f(x) := \sup_{\substack{\text{all such} \\ \text{partitions}}} \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})|$$

If $\lim_{x \to \infty} T_f(x) < \infty$, say $f \in BV$

normalized

DEFINITION: $f \in NBV$ if
a) $f \in BV$
b) $\lim_{x \to -\infty} f(x) = 0$
c) f is left continuous everywhere (i.e. $\forall x_0$, $\lim_{x \uparrow x_0} f(x) = f(x_0)$)

PROPOSITION: $f \in NBV \Rightarrow T_f \in NBV$

Proof: $f \in BV \Rightarrow T_f$ is bounded and non-decreasing, so $T_f \in BV$
Select $x, \varepsilon > 0$. $\exists$ $x_0 < x_1 < \dots < x_n = x$ s.t.

$$\sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| \geq T_f(x) - \varepsilon$$

Suppose $t_0 < t_1 < \dots < t_M = x_0$.

$$\sum_{j=1}^{M} |f(t_j) - f(t_{j-1})| + \underbrace{\sum_{i=1}^{N} |f(x_i) - f(x_{i-1})|}_{\geq\, T_f(x) - \varepsilon} \leq T_f(x)$$

Hence $\sum_{j=1}^{M} |f(t_j) - f(t_{j-1})| \leq \varepsilon$, and so $T_f(x_0) \leq \varepsilon$

Therefore $\lim_{x \to -\infty} T_f(x) = 0$

Consider the same x_i's

$$x_0 < x_1 < \dots < x_{N-1} < t < x_N = x$$

$$\sum_{i=1}^{N-1} |f(x_i) - f(x_{i-1})| + |f(t) - f(x_{N-1})|$$

$$\leq T_f(t) \leq T_f(x-) \leq T_f(x)$$

Let $t \uparrow x$. Since f is left continuous

$$T_f(x) - \varepsilon \leq \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| \leq T_f(x-) \leq T_f(x)$$

choice of x_i's

Therefore $T_f(x-) = T_f(x) \Rightarrow T_f$ is left continuous

$\boxtimes$

THEOREM: (a) Suppose μ is a complex Borel measure on $\mathbb{R}$. Then $\exists\, f : \mathbb{R} \to \mathbb{C}$ s.t. $f(x) = \mu(-\infty, x)$ and $f \in NBV$

(b) If $f \in NBV$, then $\exists!$ complex Borel measure μ on $\mathbb{R}$ s.t. $f(x) = \mu(-\infty, x)$ and $|\mu|(-\infty, x) = T_f(x)$ $\forall x \in \mathbb{R}$

Proof. (a) Show $f \in BV$. Consider $x_0 < x_1 < \dots < x_N = x$

$$\sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{N} |\mu([x_{i-1}, x_i))|$$

$$\leq |\mu|(-\infty, x) \leq |\mu|(\mathbb{R}) < \infty$$

Therefore $T_f(x) \le |\mu|(\mathbb{R}) \ \forall x$, so $f \in BV$.

5/8 MEASURE THEORY (241 10 A.M. FRI)

Proof of previous theorem

(a) Showed $T_f(x) \le |\mu|(-\infty, x)$.

f is left continuous: Suppose $x_n \uparrow x$. Then

$$f(x_n) = \mu(-\infty, x_n) \longrightarrow \mu(-\infty, x) = f(x)$$

write $\mu = Re\,\mu^+ - Re\,\mu^- + i\,(Im\,\mu^+ - Im\,\mu^-)$
and use results on positive measures

Now suppose $x_n \downarrow -\infty$. Then $\bigcap_{n=1}^{\infty} (-\infty, x_n) = \phi$, and so

$$0 = |\mu|(\phi) = \lim_{n \to \infty} |\mu|(-\infty, x_n) \implies$$

$$|f(x)| = |\mu(-\infty, x)| \le |\mu|(-\infty, x) \xrightarrow[x \to -\infty]{} 0$$

(b) Suppose f real. Write $f = f_1 - f_2$ where f_j is strictly increasing, positive and bounded. WARNING: Also assume f_j continuous. For E Borel, define

$$\mu_j(E) = m(f_j(E))$$

(f_j is a homeomorphism of $\mathbb{R}$ onto $(0, \alpha)$, so $f_j(E)$ is Borel)
Then f_j 1-1 $\implies$ μ_j is a Borel measure. Define

$$\mu = \mu_1 - \mu_2$$

Note that $\mu_j(-\infty, x) = m(f_j(-\infty, x)) = m(0, f_j(x)) = f_j(x)$
Then

$$\mu(-\infty, x) = f_1(x) - f_2(x) = f(x)$$

Now for f complex, work with real and imaginary parts separately.
uniqueness: Suppose λ is a complex Borel measure s.t.

$$\lambda(-\infty, x) = f(x)$$

We know λ, μ are regular (by Th^m 2.18). Since

$$\lambda(-\infty, x) = \mu(-\infty, x) \quad \forall x$$

Then $\lambda[\alpha, \beta) = \mu[\alpha, \beta) \quad \forall \alpha < \beta$, and so μ and λ agree on all open intervals $\Rightarrow$ on all open sets. Now suppose E is Borel. By regularity, $\exists$ open $(V_n) \supset E$ s.t. $V_{n+1} \subset V_n$

$$|\mu|(V_n) < |\mu|(E) + 1/n$$
$$|\lambda|(V_n) < |\lambda|(E) + 1/n$$

Let $V = \bigcap V_n \supset E$. Then $|\mu|(V - E) = 0 = |\lambda|(V - E)$
Hence

$$\lambda(E) = \lambda(V) \quad ; \quad \mu(E) = \mu(V)$$

But

$$\mu(V) = \lim_{n\to\infty} \mu(V_n) \overset{\substack{V_n \text{ open} \\ \downarrow}}{=} \lim_{n\to\infty} \lambda(V_n) = \lambda(V)$$

and so $\lambda(E) = \mu(E)$. Hence $\lambda = \mu$

From (a), $T_f(x) \leq |\mu|(-\infty, x)$. $f \in NBV \Rightarrow T_f \in NBV$ (last time) Hence there is a complex Borel measure λ such that for every x

$$\lambda(-\infty, x) = T_f(x)$$

Since $|f(\alpha) - f(\beta)| \leq T_f(\beta) - T_f(\alpha)$ for $\alpha < \beta$, we have

$$|\mu[\alpha, \beta)| \leq \lambda[\alpha, \beta)$$

Therefore $|\mu(E)| \leq \lambda(E)$ for all Borel sets E. Hence $|\mu|(E) \leq \lambda(E)$ and so

$$|\mu|(-\infty, x) \leq \lambda(-\infty, x) = T_f(x)$$

Therefore $T_f(x) = |\mu|(-\infty, x)$

$\blacksquare$

DEFINITION: $f : \mathbb{R} \to \mathbb{C}$ is absolutely continuous if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if the intervals (a_i, b_i), $1 \le i \le N$, are disjoint and $\sum_{i=1}^{N} (b_i - a_i) < \delta$, then

$$\sum_{i=1}^{N} |f(b_i) - f(a_i)| < \varepsilon$$

THEOREM: Suppose $f \in NBV$. Then f is absolutely continuous if and only if the unique complex Borel measure μ associated with f is absolutely continuous w.r.t. Lebesgue measure.

Proof. Suppose $\mu \ll m$. Given $\varepsilon > 0$ $\exists \delta > 0$ s.t. if $m(E) < \delta$, then $|\mu|(E) < \varepsilon$. Suppose (a_i, b_i), $1 \le i \le N$ are disjoint and $\sum (b_i - a_i) < \delta$. Let

$$E = \bigcup_{i=1}^{N} [a_i, b_i)$$

(disjoint union). Then

$$\sum_{i=1}^{N} |f(b_i) - f(a_i)| = \sum_{i=1}^{N} |\mu[a_i, b_i)|$$

$$\le |\mu|(E) < \varepsilon$$

since $m(E) < \delta$.

Now suppose f is absolutely continuous. Suppose E is Borel and $m(E) = 0$. Given $\varepsilon > 0$ $\exists \delta > 0$ s.t. definition of δ is satisfied. Will show $|\mu(E)| \leq \varepsilon$. m regular $\Rightarrow$ $\exists$ open $O \supset E$ s.t. $m(O) < \delta$. Since μ is regular, $\exists$ open $V_n \supset E$ s.t. $|\mu|(V_n) < |\mu|(E) + 1/n$. Let $W_n = O \cap V_n$. Then $m(W_n) < \delta$. WLOG $W_{n+1} \subset W_n$ $\forall n$. Let

$$W := \bigcap_{n=1}^{\infty} W_n$$

Then $|\mu|(W-E) = 0 \Rightarrow \mu(W) = \mu(E)$, and so $\mu(W_n) \to \mu(E)$. W_n open, so we can write

$$W_n = \bigcup_k I_{nk} \quad \text{(disjoint, closed on left, open on right)}$$

Sufficient to show $|\mu(W_n)| \leq \varepsilon$. But

$$|\mu(W_k)| \leq \sum_k^{\infty} |\mu(I_{nk})| = \lim_{N\to\infty} \sum_{k=1}^{N} |\mu(I_{nk})|$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} |f(b_{kn}) - f(a_{kn})| \quad [[\, I_{nk} = [a_{kn}, b_{kn}) \,]]$$

$$\leq \varepsilon$$

each of these $\leq \varepsilon$

since $m(\sum b_k - a_k) \leq m(\bigcup_{k=1}^{\infty} I_{nk}) < \delta$

5/10 MEASURE THEORY

REVIEW

__THEOREM I__: μ complex Borel measure on $\mathbb{R}^k$

1) $D\mu$ exists a.e. $[m]$
2) $D\mu \in L^1(\mathbb{R}^k, m)$
3) $\exists$ complex Borel measure $\mu_s \perp m$, $D\mu_s = 0$ a.e. $[m]$ s.t. $\forall$ Borel E

$$\mu(E) = \mu_s(E) + \int_E D\mu(x)\, dm(x)$$

__THEOREM II__:

a) μ complex Borel measure on $\mathbb{R} \Rightarrow f(x) := \mu(-\infty, x) \in NBV$
b) $f \in NBV \Rightarrow \exists!$ complex Borel measure μ s.t.

$$\mu(-\infty, x) = f(x) \quad \forall x$$

Also $|\mu|(-\infty, x) = T_f(x)$

__THEOREM III__: Suppose $f \in NBV$. f is absolutely continuous iff the unique μ from theorem IIb is such that $\mu \ll m$

<u>THEOREM</u>: Suppose $g \in L^1(\mathbb{R})$. Then

$$F(x) := \int_{-\infty}^{x} g(t)\,dt$$

satisfies $F \in NBV$, F is absolutely continuous, and $F'(x) = g(x)$ a.e. $[m]$

Proof. Define μ complex Borel measure μ by

$$\mu(E) := \int_E g(t)\,dt$$

for every Borel E

Then by IIa, $F(x) = \mu(-\infty, x)$ is in NBV. Clearly $\mu \ll m$, so III $\Rightarrow$ F is absolutely continuous

By Theorem I and the uniqueness of the Lebesgue decomposition

$$\int g(t)\,dt = \mu(E) = \int_E (D\mu)(t)\,dm(t)$$

and so $g(x) = D\mu(x)$ a.e $[m]$. Select x_0 s.t. $D\mu(x_0)$ exists.

Claim: $F'(x_0) = D\mu(x_0)$. Take $h > 0$.

$$\frac{F(x_0+h) - F(x_0)}{h} = \frac{\mu([x_0, x_0+h))}{m([x_0, x_0+h))} \to D\mu(x_0)$$

(As $h \to 0$, $[x_0, x_0+h)$ shrinks nicely to x_0). This shows claim.

Hence $F' = g$ a.e. $[m]$.

$\boxtimes$

THEOREM: Suppose $f \in NBV$

1) f' exists a.e.
2) $f' \in L^1(\mathbb{R})$
3) $\exists f_s$ s.t. $f_s' = 0$ a.e. and

$$f(x) = f_s(x) + \int_{-\infty}^{x} f'(t)\,dt$$

$\forall x \in \mathbb{R}$. Furthermore, $f_s = 0$ if and only if f is absolutely continuous. If f is real and non-decreasing, then f_s is real and non-decreasing.

Proof. Apply Theorem IIb to get a complex Borel measure μ s.t. $\mu(-\infty, x) = f(x)$ $\forall x$. By Theorem I, $D\mu$ exists a.e. and $D\mu \in L^1(\mathbb{R})$. The claim of the previous proof $\Rightarrow f' = D\mu$ wherever $D\mu$ exists. Hence f' exists a.e. and $f' \in L^1(\mathbb{R})$

By Theorem I, $\exists \mu_s \perp m$ with $D\mu_s' = 0$ a.e. and

$$\mu(E) = \mu_s(E) + \int_E D\mu(x)\,dm(x)$$

Define

$$f_s(x) := \mu_s(-\infty, x)$$

As before, $f_s' = D\mu_s' = 0$ a.e. $[m]$. Moreover

$$f(x) = \mu(-\infty, x) = \mu_s(-\infty, x) + \int_{-\infty}^{x} D\mu(t)\,dt = f_s(x) + \int_{-\infty}^{x} f'(t)\,dt$$

By theorem III, f is absolutely continuous $\Leftrightarrow \mu \ll m \Leftrightarrow \mu_S = 0$
But $\mu_S = 0 \Rightarrow f_S = 0$. If $f_S = 0$, then uniqueness of theorem IIb $\Rightarrow \mu_S = 0$

uniqueness of Lebesgue decomposition

Suppose f real, non-decreasing (Recall –

$$\lim_{x \to -\infty} f(x) = 0 \;)$$

So $f \geq 0$. Claim: $\mu \geq 0$. $f \uparrow \Rightarrow$

$$|\mu|(-\infty, x) = T_f(x) = f(x) = \mu(-\infty, x)$$

Suppose $E \subset (-\infty, x)$, $\mu(E) \not\geq 0$. Then

$$\mu(-\infty, x) = \mu(E) + \mu((-\infty, x) - E)$$

and so

$$\mu(-\infty, x) < |\mu(E)| + |\mu((-\infty, x) - E)| \leq |\mu|(-\infty, x) \;\lightning$$

Hence $\mu(E) \geq 0$.

Claim. $\mu_S \geq 0$. Suppose E Borel set with $\mu_S(E) < 0$
$\mu_S \perp m \Rightarrow \exists$ Borel set A s.t. $\mu_S(E) = \mu_S(E \cap A)$ and $m(A) = 0$
Then

$$0 \leq \mu(E \cap A) = \mu_S(E \cap A) + \int_{E \cap A} D\mu(t)\, dm(t) \underset{\substack{\uparrow \\ m(E\cap A) = 0}}{=} \mu_S(E \cap A) = \mu_S(E)$$

But now $\mu_s \geq 0 \Rightarrow f_s \geq 0$ and if $a<b$

$$f_s(b) - f_s(a) = \mu_s([a,b)) \geq 0$$

$\blacksquare$

Suppose $a<b$. Then

$$f(b) = f_s(b) + \int_{-\infty}^{b} f'(t)\,dt$$

$$f(a) = f_s(a) + \int_{-\infty}^{a} f'(t)\,dt$$

$$\Rightarrow \int_a^b f' = [f(b)-f(a)] - [f_s(b) - f_s(a)] \leq f(b) - f(a)$$

<u>THEOREM</u>: If f' exists everywhere on $[a,b]$ and is integrable, then

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

Proof uses Vitali - Carathéodory

5/12 MEASURE THEORY

X compact T_2-space
$C_{\mathbb{R}}(X)$ continuous real functions on X } Banach spaces
$C(X)$ continuous complex-valued functions on X } with sup norm

<u>STONE-WEIERSTRASS THEOREM</u>: A subspace A of $C_{\mathbb{R}}(X)$ is dense in $C_{\mathbb{R}}(X)$ if

a) A is an algebra (i.e. $f_1, f_2 \in A \Rightarrow f_1 f_2 \in A$)
b) A contains (real) constants
c) A separates points of X, i.e. if $x \neq y$ in X, then $\exists f \in A$ s.t. $f(x) \neq f(y)$)

<u>COROLLARY</u>: A subspace A of $C(X)$ is dense in $C(X)$ if

a) A is an algebra
b) A contains complex constants
c) A separates points of X
d) A is closed under conjugation (i.e. $f \in A \Rightarrow \bar{f} \in A$)

Remark – ① Recall from 441 that Weierstrass' theorem says that the real polynomials are dense in $C_{\mathbb{R}}(X)$, where $X = [a,b]$. This is a special case of the S-W theorem. Note that polynomials with complex coefficients are dense in $C[a,b]$.

② The trigonometric polynomials are dense in $C(T)$ (consequence of Féjer's theorem) This is also a trivial consequence of S-W. Note that the real-valued trigonometric polynomials are dense in $C_{\mathbb{R}}(T)$ (The real part of a trig. polynomial is a trig. polynomial

$$\text{Re } a_n e^{in\theta} = \frac{1}{2} a_n e^{in\theta} + \frac{1}{2} \overline{a_n} e^{-in\theta} \Big)$$

Examples: 1) $X = [-1,1]$, A = even real polynomials. A is not dense (can't approx odd polynomials) Note c) fails

2) $X = [-1,1]$ A = real polynomials with $p(0) = 0$. A is not dense. Note b) fails

3) $X = \mathbb{R}$, A = real polynomials. $\|p(x) - e^x\|_\infty = \infty$ $\forall$ real polynomials

only locally compact, not compact

Notation: $f_1, f_2 \in C_{\mathbb{R}}(X)$, let

$$\left.\begin{array}{l} f_1 \wedge f_2 := \min(f_1, f_2) \\ f_1 \vee f_2 := \max(f_1, f_2) \end{array}\right\} \in C_{\mathbb{R}}(X)$$

<u>DEFINITION:</u> $L \subset C_{\mathbb{R}}(X)$ is a lattice if $f_1, f_2 \in L \Longrightarrow$ $f_1 \wedge f_2 \in L$ and $f_1 \vee f_2 \in L$

Proof of Theorem: Throughout X is a compact T_2-space

<u>LEMMA 1</u>: Suppose $L \subset C_{\mathbb{R}}(X)$ is a lattice. Let $g = \inf\limits_{f \in L} f$. If g is continuous, then $\forall \varepsilon > 0$ $\exists f \in L$ s.t.

$$0 \leq f - g < \varepsilon \quad \text{everywhere on } X$$

[g need not always be continuous: $L = \{x^n : n \in \mathbb{N}\}$, $X = [0,1]$

$$g(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$
]

Proof. $\forall x \in X$, $\exists f_x \in L$ such that

$$0 \leq f_x(x) - g(x) < \varepsilon/3$$

Since both f_x and g are continuous, $\exists$ open $\mathcal{O}_x$ containing x such that

$$y \in \mathcal{O}_x \implies \begin{cases} |f_x(x) - f_x(y)| < \varepsilon/3 \\ |g(x) - g(y)| < \varepsilon/3 \end{cases}$$

Then $y \in \mathcal{O}_x \Rightarrow |f_x(y) - g(y)| < \varepsilon$

Since X is compact, there is a finite subset $F \subset X$ such that

$$X = \bigcup_{x \in F} \mathcal{O}_x$$

Let $f := \bigwedge_{x \in F} f_x \in L$ (since Lattice). If $y \in X$, then $y \in \mathcal{O}_x$ for some $x \in F$, and so

$$0 \leq f(y) - g(y) \leq f_x(y) - g(y) \underset{\text{since } y \in \mathcal{O}_x}{<} \varepsilon \qquad \blacksquare$$

LEMMA 2: If $\mathcal{F} \subset C_{\mathbb{R}}(X)$ satisfies

(i) $\mathcal{F}$ separates points

(ii) $f \in \mathcal{F}, c \in \mathbb{R} \Rightarrow cf \in \mathcal{F}$ and $c+f \in \mathcal{F}$

then if $x \neq y \in X$ and $a, b \in \mathbb{R}$, then $\exists f \in \mathcal{F}$ such that

$$f(x) = a, \quad f(y) = b$$

Proof: Suppose $x \neq y$. $\exists g \in \mathcal{F}$ s.t. $g(x) \neq g(y)$ (by (i))

Define

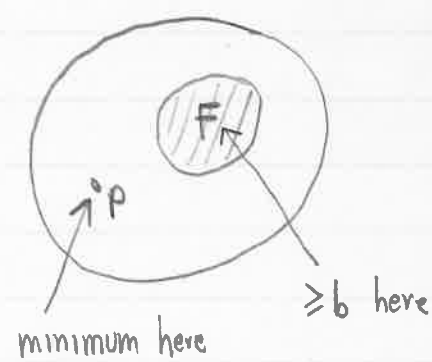
$$f := \frac{a-b}{g(x)-g(y)}\, g + \frac{b\,g(x) - a\,g(y)}{g(x)-g(y)}$$

Then $f \in \mathcal{F}$ by (ii).

$\blacksquare$

LEMMA 3: Suppose $L \subset C_{\mathbb{R}}(X)$ is a lattice which has properties (i) and (ii) above. Suppose F is a closed subset of X, and $p \in X - F$. If $a < b$ in $\mathbb{R}$, then $\exists f \in L$ s.t.

$$f(x) \geq a \quad \forall x \in X$$
$$f(x) \geq b \quad \forall x \in F$$
$$f(p) = a$$



Proof. Note lemma 2 applies to L. So $\forall x \in F$, $\exists f_x \in L$ s.t. $f_x(p) = a$ and $f_x(x) = b+1$. Let

$$\mathcal{O}_x := \{ y \in X : f(y) > b \}$$

Then $\mathcal{O}_x$ is open and $x \in \mathcal{O}_x$. F compact $\Rightarrow$

$$F \subset \bigcup_{x \in A} \mathcal{O}_x$$

where $A \subset F$ is finite. Let

$$g := \bigvee_{x \in A} f_x \in L$$

Then $g(p) = a$ and $g(x) > b$ $\forall x \in F$. Now let

$$f = g \vee a \in L$$

L contains $0 \Rightarrow$ contains all constants (property (ii))

$\boxtimes$

LEMMA 4: Suppose L is a lattice which separates points and has property $c \in \mathbb{R}$, $f \in L \Rightarrow cf \in L$ and $c + f \in L$. $\forall g \in C_{\mathbb{R}}(X)$ and $\forall \varepsilon > 0$ $\exists f \in L$ s.t.

$$0 \leq f(x) - g(x) < \varepsilon \quad \forall x \in X$$

Proof. Let $L' \subset L$ be given by

$$L' = \{ f \in L : f(x) \geq g(x) \ \forall x \in X \}$$

Then L' is a lattice. It is sufficient to show $g = \inf_{f \in L'} f$ by lemma 1.

Select $p \in X$ and $\eta > 0$. The set

$$F := \{ x \in X : g(x) \geq g(p) + \eta \}$$

is closed. Certainly $\exists M > g(p) + \eta$ s.t. $g(x) \leq M \ \forall x \in X$. By lemma 3, $\exists f_0 \in L$ s.t.

$$f_0(p) = g(p) + \eta$$
$$f_0(x) \geq M \quad \forall x \in F$$
$$f_0(x) \geq g(p) + \eta \quad \forall x \in X$$

Then $f_0 \in L'$, and so

$$\inf_{f \in L'} f(p) \leq f_0(p) = g(p) + \eta$$

But clearly $g(p) \leq \inf_{f \in L'} f(p)$ by definition of L'. Hence $g(p) = \inf_{p \in L'} f(p)$

Recall: Weierstrass's Thm $\Rightarrow$ $\forall \varepsilon > 0$ $\exists$ real poly. P s.t.
$|P(x) - |x|| < \varepsilon$ for $-1 \leq x \leq 1$

Proof of theorem: Note by lemma 4 that it is sufficient to show $\overline{A}$ is a lattice (closure in sup topology)

It is clear that $\overline{A}$ is an algebra. Suppose $f \in \overline{A}$ and $\|f\|_\infty \leq 1$. Then

$$|P(f(x)) - |f(x)|| < \varepsilon \quad \forall x \in X$$

(P from above remark) $\overline{A}$ an algebra $\Rightarrow$ $P(f) \in \overline{A}$. Also $\overline{A}$ closed $\Rightarrow$ $|f| \in \overline{A}$

Suppose $f \in \overline{A}$. Then $f / \|f\|_\infty \in \overline{A}$, and so by above paragraph, $|f| / \|f\|_\infty \in \overline{A} \Rightarrow |f| \in \overline{A}$.

Suppose $f, g \in \overline{A}$; then

$$f \wedge g = \frac{1}{2}\left((f+g) - |f-g|\right) \in \overline{A}$$

$$f \vee g = \frac{1}{2}\left((f+g) + |f-g|\right) \in \overline{A}$$

Hence $\overline{A}$ is a lattice.

Now lemma 4 says $\overline{A}$ is dense in $C_{\mathbb{R}}(X)$, and so A is dense ($\overline{A}$ is closed, so actually $\overline{A}$ dense $\Rightarrow$ $\overline{A} = C_{\mathbb{R}}(X)$)

$\blacksquare$

Proof of corollary:

$$f \in A \Rightarrow \bar{f} \in A \Rightarrow \operatorname{Re} f = \frac{1}{2}(f + \bar{f}) \in A$$

Let $A' = \{ \mathrm{Re}\, f : f \in A \} \subset C_{\mathbb{R}}(X)$. Then A' satisfies (a), (b), and (c) of S-W theorem, and so A' is dense in $C_{\mathbb{R}}(X)$. But $A' \subset A$, so given $g \in C(X)$ we can approximate $\mathrm{Re}\, g$ and $\mathrm{Im}\, g$ by members of A, and thus can approximate g by a member of A.

$\blacksquare$

Fourier Analysis

$$H := L^2\left([-\pi,\pi], \frac{d\theta}{2\pi}\right)$$

normalized Lebesgue measure

$$T := \{z \in \mathbb{C} : |z| = 1\} \quad \text{(circle group)}$$

$C(T)$:= continuous complex-valued functions on T

(The elements of $C(T)$ can be identified with the continuous periodic complex-valued functions on $\mathbb{R}$ with period 2π)

<u>Proposition 1</u> (p105): $\{e^{int} : n \in \mathbb{Z}\}$ is an orthonormal family in H.

Proof.

$$(e^{int} \mid e^{imt}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)t}\, dt = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

$\boxtimes$

Remember that $L^2([-\pi,\pi])$ is actually a space of equivalence classes. If $f \in L^2([-\pi,\pi])$, we can define

$$f_0(t) := \begin{cases} f(t) & t \neq \pi \\ f(-\pi) & t = \pi \end{cases}$$

Then

$$\int_{-\pi}^{\pi} (f - f_0)^2 (t)\, dt = 0$$

so f and f_0 both represent the same "element" in $L^2([-\pi,\pi])$. Therefore we may consider the functions in $L^2([-\pi,\pi])$ as periodic functions on $\mathbb{R}$ with period 2π, or equivalently, as elements of $L^2(T)$.

DEFINITION: If $f \in L^2([-\pi,\pi])$, its n^{th} Fourier coefficient is

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} f(t)\, dt$$

for every $n \in \mathbb{Z}$. The Fourier series of f is

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{int}$$

DEFINITION: For $f \in L^2([-\pi,\pi])$ and $N \in \mathbb{N}$, define

$$S_N(x, f) = S_N(x) := \sum_{k=-N}^{N} \hat{f}(k) e^{ikt}$$

Then S_N is the N^{th} partial sum of the Fourier series for f.

<u>Fejér's Theorem</u> (p110) Suppose $f \in C(T)$. Let

$$\sigma_N(x,f) = \sigma_N(x) := \frac{1}{N+1} \sum_{k=0}^{N} S_k(x)$$

Then $\sigma_N \to f$ uniformly on $[-\pi,\pi]$.

Proof later

Note that

$$\sigma_N(x) = \sum_{k=-N}^{N} c_k e^{ikx}$$

for some choice of c_k; σ_N is a trigonometric polynomial of degree N.

[If (X,M,μ) is a measure space where μ has the properties of the conclusion of the Riesz Representation theorem, then $C_c(X)$ is dense in $L^p(\mu)$ for $1 \le p < \infty$. (p 84)]

We know $C(T)$ is dense in $L^2([-\pi,\pi])$. By Fejér's theorem the set of finite linear combinations of $\{e^{ikx} : k \in \mathbb{Z}\}$ (i.e. the trigonometric polynomials) is dense in $C(T)$, and therefore are dense in $L^2([-\pi,\pi])$. Hence $\{e^{inx} : n \in \mathbb{Z}\}$ is a maximal orthonormal family

[Suppose H is a Hilbert space and $(u_\alpha : \alpha \in A)$ is an orthonormal family in H. TFAE

i) $(u_\alpha : \alpha \in A)$ is a maximal orthonormal family

ii) The set of finite linear combinations of members of this family is dense in H

iii) PARSEVAL'S THEOREM: $\forall x \in H$

$$\|x\|^2 = \sum_{\alpha \in A} |\hat{x}(\alpha)|^2$$

iv) $$(x|y) = \sum_{\alpha \in A} \hat{x}(\alpha)\,\overline{\hat{y}(\alpha)} \qquad \forall x,y \in H$$

(p103)

Suppose $f \in L^2([-\pi,\pi])$. By Parseval's theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\,dt = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2$$

Also, if $f, g \in L^2([-\pi,\pi])$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\overline{g(t)}\,dt = \sum_{n \in \mathbb{Z}} \hat{f}(n)\,\overline{\hat{g}(n)}$$

Suppose $f \in L^2([-\pi,\pi])$. Then for each $N \in \mathbb{N}$

$$\widehat{f - S_N}(k) = \begin{cases} \hat{f}(k) & |k| > N \\ 0 & |k| \le N \end{cases}$$

Therefore, by Parseval's theorem

$$\| f - S_N \|_2^2 = \sum_{|k|>N} |\hat{f}(k)|^2 \longrightarrow 0 \quad \text{as } N \to \infty$$

Hence S_N converges to f in the L^2-norm, and so there is a subsequence (S_{N_j}) of (S_N) such that $S_{N_j}(x) \to f(x)$ almost everywhere.

Note that S_N is the trigonometric polynomial of degree N which best approximates f in the L^2 sense

[Suppose F is a finite orthonormal family in H. For every $x \in H$

$$\| x - \sum_{u \in F} (x|u) u \| \leq \| x - \sum_{u \in F} \lambda_u u \|$$

for any family $(\lambda_u : u \in F)$ of scalers. Equality holds if and only if $\lambda_u = (x|u) \; \forall u \in F$ (p98)]

Thus

$$\| f - S_N \|_2 \leq \| f - \sum_{k=-N}^{N} c_k e^{ikx} \|_2$$

for any family $(c_k : -N \leq k \leq N)$ of scalers.

Definition: Dirichlet Kernel

$$D_m(x) := \sum_{k=-m}^{m} e^{ikx} \qquad m \in \mathbb{N}$$

Fejér Kernel

$$K_n(x) := \frac{1}{n+1} \sum_{m=0}^{n} D_m(x) \qquad n \in \mathbb{N}$$

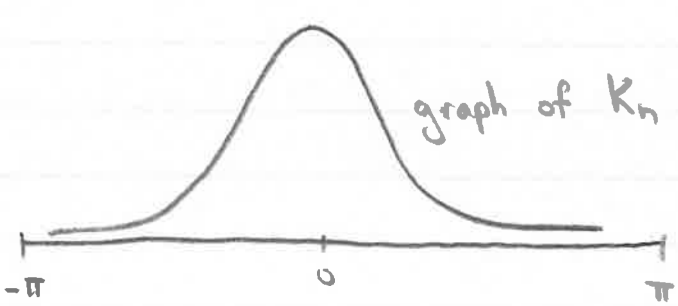
Proposition:

(1) $$D_m(x) = \frac{\sin(m+1/2)x}{\sin \frac{1}{2}x}$$

(2) $$K_n(x) = \frac{1}{n+1} \frac{1-\cos(n+1)x}{1-\cos x}$$

(3) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_m(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1$$

(4) $0 \leq K_n(x)$ $\forall x$ and $K_n(x) \leq \frac{2}{n+1} \frac{1}{1-\cos\delta}$ for $\delta \leq |x| \leq \pi$



Proof. (1) Observe that

$$(*) \quad (e^{ix}-1)\,D_m(x) = \sum_{k=-m}^{m} e^{i(k+1)x} - \sum_{k=-m}^{m} e^{ikx} = e^{i(m+1)x} - e^{-imx}$$

Multiplying by $e^{-ix/2}$

$$(e^{ix/2} - e^{-ix/2})\,D_m(x) = e^{i(m+1/2)x} - e^{-i(m+1/2)x}$$

$$(2i \sin \tfrac{1}{2}x)\,D_m(x) = 2i \sin (m+1/2)x$$

$$D_m(x) = \frac{\sin (m+1/2)x}{\sin 1/2\, x}$$

(2) $(*)$ also implies that

$$(n+1)(e^{ix}-1)\,K_n(x) = \sum_{m=0}^{n} \left(e^{i(m+1)x} - e^{-imx}\right) = \sum_{k=-n}^{n+1} c_k e^{ikx}$$

where

$$c_k = \begin{cases} 1 & 1 \le k \le n+1 \\ -1 & -n \le k \le 0 \end{cases}$$

Hence

$$(n+1)(e^{ix}-1)(e^{-ix}-1)\,K_n(x) = \sum_{k=-n}^{n+1} c_k e^{i(k-1)x} - \sum_{k=-n}^{n+1} c_k e^{ikx}$$

$$= -e^{-i(n+1)x} - e^{i(n+1)x} + 2$$

$$= 2 - 2\cos(n+1)x$$

Therefore

$$(n+1)K_n(x) = \frac{2-2\cos(n+1)x}{2-2\cos x} = \frac{1-\cos(n+1)x}{1-\cos x}$$

(3)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_m(x)\,dx = \frac{1}{2\pi}\sum_{k=-m}^{m}\int_{-\pi}^{\pi} e^{ikx}\,dx$$

$$= \frac{1}{2\pi}\sum_{\substack{k=-m \\ k\neq 0}}^{m} \frac{1}{ik}\left(e^{ik\pi} - e^{-ik\pi}\right) + 1$$

$$= 0+1 = 1$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = \frac{1}{n+1}\left(\sum_{m=0}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi} D_m(x)\,dx\right)$$

$$= \frac{1}{n+1}\sum_{m=0}^{n} 1 = 1$$

(4) It is clear from (2) that $K_n(x) \geq 0$ $\forall x$. Also, if $\delta \leq |x| \leq \pi$, then $1-\cos x \geq 1-\cos\delta$, so that

$$K_n(x) \leq \frac{2}{n+1}\,\frac{1}{1-\cos x} \leq \frac{2}{n+1}\,\frac{1}{1-\cos\delta}$$

$\blacksquare$

FEJÉR'S THEOREM: Suppose $f \in C(\mathbb{T})$. Let

$$S_N(x) := \sum_{k=-N}^{N} \hat{f}(k)\, e^{ikx}$$

and

$$\sigma_N(x) := \frac{1}{N+1} \sum_{k=0}^{N} S_k(x)$$

Then $\sigma_N \to f$ uniformly on $[-\pi,\pi]$.

Proof. Observe that

$$S_N(x) = \sum_{k=-N}^{N} \hat{f}(k)\, e^{ikx} = \sum_{k=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt}\, dt \right) e^{ikx}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{k=-N}^{N} e^{ik(x-t)}\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, D_N(x-t)\, dt$$

$$= \frac{1}{2\pi} \int_{x+\pi}^{x-\pi} f(x-u)\, D_N(u)\, (-du)$$

($f(x-u) D_N(u)$ has period 2π, so we may replace x by 0 in limits)

$$= \frac{1}{2\pi} \int_{\pi}^{-\pi} f(x-u)\, D_N(u)\, (-du)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)\, D_N(t)\, dt$$

Then

$$\sigma_n(x) = \frac{1}{n+1} \sum_{N=0}^{n} S_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) \left[\frac{1}{n+1} \sum_{N=0}^{n} D_N(t) \right] dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)\, K_n(t)\, dt$$

Because $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(t)\, dt = 1$, we have

$$\sigma_n(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[f(x-t) - f(t) \right] K_n(t)\, dt$$

and so

$$| \sigma_n(x) - f(x) | \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} | f(x-t) - f(t) |\, K_n(t)\, dt$$

(since $K_n(t) \geq 0$!) Now f is continuous, and so uniformly continuous. Therefore $\exists\, M > 0$ such that

$$|f(y)| < M \qquad \forall y \in [-\pi, \pi]$$

and, given $\varepsilon > 0$, $\exists\, 0 < \delta < \pi$ such that

$$|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon/2$$

For $\delta \leq |t| \leq \pi$, we have

$$K_n(t) \leq \frac{2}{n+1} \frac{1}{1-\cos\delta}$$

and so we can find $L \in \mathbb{N}$ such that $\forall n \geq L$

$$\delta \leq |t| \leq \pi \Rightarrow K_n(t) \leq \frac{\varepsilon}{4M}$$

Therefore, $\forall n \geq L$

$$|\sigma_n(x) - f(x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)|\, K_n(t)\, dt$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\pi}\right)|f(x-t) - f(x)|\, K_n(t)\, dt$$

$$\leq \frac{1}{2\pi}\left(2M \cdot \frac{\varepsilon}{4M} \cdot 2\pi + 2\pi \cdot \frac{\varepsilon}{2}\right) = \varepsilon$$

$\blacksquare$

Note that we must use $K_n(x)$ instead of $D_m(x)$ since

$$|S_N(x) - f(x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)|\, |D_N(t)|\, dt$$

and

$$\int_{-\pi}^{\pi} |D_N(t)|\, dt > c \log N$$

so we can not get a good estimate on $|S_N(x) - f(x)|$.

RIESZ-FISCHER THEOREM: Let H be a Hilbert Space and $(u_\alpha : \alpha \in A)$ an orthonormal family. Given $\varphi \in \ell^2(A)$, there exists $x \in H$ such that $\hat{x} = \varphi$ (p101)

PROPOSITION: If $(c_n : n \in \mathbb{Z})$ satisfies

$$\sum_{n\in\mathbb{Z}} |c_n|^2 < \infty$$

then there is an $f \in L^2[-\pi,\pi]$ such that $\forall n \in \mathbb{Z}$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$$

Recall that $S_n \longrightarrow f$ in the L^2-norm (for $f \in C(T)$) and so there is some subsequence S_{n_k} which converges to f a.e.

QUESTION: If $f \in C(T)$, does $S_n(x,f) \to f(x)$ for every $x \in [-\pi,\pi]$?

Define $\Lambda_n : C(T) \longrightarrow \mathbb{C}$ by

$$\Lambda_n(f) := S_n(0,f)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,D_n(t)\,dt$$

By Hölder's inequality

$$|\Lambda_n f| \leq \|f\|_\infty \|D_n\|_1$$

and so $\|\Lambda_n\| \leq \|D_n\|_1$.

Define for each $n \in \mathbb{N}$

$$g_n(t) := \begin{cases} +1 & \text{if } D_n(t) \geq 0 \\ -1 & \text{if } D_n(t) < 0 \end{cases}$$

Then g_n is a step function, and we can find a sequence $(f_j) \subset C(T)$ with $\|f_j\|_\infty = 1$ and

$$\lim_{j\to\infty} f_j(t) = g_n(t) \quad \forall t \in [-\pi, \pi] \text{ a.e.}$$

By the Dominated Convergence theorem

$$\lim \Lambda_n f_j = \lim \frac{1}{2\pi} \int_{-\pi}^{\pi} f_j(t) D_n(t)\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(t) D_n(t)\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt$$

$$= \|D_n\|_1$$

Because $\|f_j\| = 1$ $\forall j \in \mathbb{N}$, we have $\|\Lambda_n\| \geq \|D_n\|_1$. Therefore $\|\Lambda_n\| = \|D_n\|_1$ $\forall n \in \mathbb{N}$.

Claim: $\|D_n\|_1 \to \infty$

$$\|D_n\|_1 = \frac{1}{\pi}\int_0^{\pi} \frac{|\sin(n+1/2)t|}{\sin t/2}\,dt$$

$$\geq \frac{2}{\pi}\int_0^{\pi} \frac{|\sin(n+1/2)t|}{t}\,dt \qquad [\![\sin x \leq x \quad \forall x \geq 0]\!]$$

$$= \frac{2}{\pi}\int_0^{(n+1/2)\pi} \frac{|\sin u|}{u}\,du$$

$$\geq \frac{2}{\pi}\sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi} \frac{|\sin u|}{u}\,du$$

$$\geq \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi} |\sin u|\,du$$

$$= \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k} \longrightarrow \infty \quad \text{as } n\to\infty$$

Hence $\|\Lambda_n\| \to \infty$ as $n \to \infty$

[[UNIFORM BOUNDEDNESS THEOREM: Suppose X is a Banach space, and Y is a normed linear space. Suppose $\{\Lambda_\alpha : \alpha \in A\} \subset \mathcal{B}(X,Y)$. Then one of the following alternatives must occur:

(1) $\exists M > 0$ s.t. $\|\Lambda_\alpha\| \leq M \quad \forall \alpha \in A$

(2) $\sup_{\alpha\in A} \|\Lambda_\alpha x\| = \infty$ for a dense G_δ-set in X

(p114)]]

Since $\|\Lambda_n\| \to \infty$, the Uniform Boundedness principle says there is a dense G_δ-set $E \subset C(\mathbb{T})$ such that

$$\sup_{n\in\mathbb{N}} |S_n(f,0)| = +\infty \quad \forall f \in E$$

and so $S_n(f,0)$ does not converge.

There is nothing special about 0. For every $x \in [-\pi,\pi]$ there exists a dense G_δ-set $E_x \subset C(\mathbb{T})$ such that

$$\sup_{n\in\mathbb{N}} |S_n(f,x)| = \infty \quad \forall f \in E_x$$