Ordered Banach Spaces
(Banach Lattices)

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8/27 ORDERED VECTOR SPACES

Reference - H.H. Schaefer Banach Lattices and Positive Operators

EXAMPLES

1) Let $\mathfrak{X}$ be a topological space and $C(\mathfrak{X})$ all real continuous functions on $\mathfrak{X}$. Can order it by

$$f \leq g \text{ if } f(t) \leq g(t) \quad \forall t \in \mathfrak{X}$$

2) (Ω, Σ, μ) measure space. On $L^p(\Omega, \Sigma, \mu)$ we have

$$f \leq g \text{ if } f(t) \leq g(t) \text{ a.e.}$$

3) $\mathfrak{X}$ compact Hausdorff. $M(\mathfrak{X})$ = all regular signed Borel measures

$$\mu \leq \nu \text{ if } \int f \, d\mu \leq \int f \, d\nu$$

for all $0 \leq f \in C(\mathfrak{X})$

4) ℓ^p, $1 \leq p \leq \infty$, c_0, c

$$x \leq y \text{ if } x_n \leq y_n \quad \forall n$$

5) If E, F are "order vector spaces" that are also normed spaces let $\mathcal{L}(E,F)$ = all continuous linear maps $T: E \to F$.

$$S \leq T \text{ if } Sx \leq Tx \text{ for all } x \geq 0$$

DEFINITION: An ordered vector space is a real vector space E equipped with a partial order $\leq$ (transitive, reflexive, anti-symmetric) such that

i) $x \leq y \Rightarrow x+z \leq y+z \quad \forall z \in E$
ii) $x \leq y \Rightarrow \alpha x \leq \alpha y \quad \forall \alpha \geq 0$

If E is an order vector space, then

$$K := \{ x \in E : x \geq 0 \}$$

is the positive cone.

Properties

(1) $K + K \subset K$
(2) $\alpha K \subset K \quad \forall \alpha \geq 0$
(3) $K \cap (-K) = \{0\}$

On the other hand, if K is any subset of a real vector space E such that (1), (2), and (3) hold, then

$$x \leq y \iff y - x \in K$$

defines a partial order on E for which E is an ordered vector space

Vector Lattices

If A is a subset of an ordered vector space E, then $x_0 \in E$ is the supremum of A if

1) $a \leq x_0 \quad \forall a \in A$
2) $a \leq b \ \forall a \in A \Rightarrow x_0 \leq b$

Definition: An ordered vector space E is a vector lattice if $\sup\{x,y\}$ and $\inf\{x,y\}$ exist for every two-element subset $\{x,y\}$ of E.

Examples

YES 1) $C(\mathcal{X})$, $L^p(\Omega, \Sigma, \mu)$, ℓ^p, c_0, c

NO 2) The following are not vector lattices

Let c = Banach space of convergent sequences $\|x\| = \sup|x_n|$

Define $T_i : c \longrightarrow c$ as follows

$$T_1 x = (x_1, \lim x_n, x_3, \lim x_n, x_5, \lim x_n, \ldots)$$
$$T_2 x = (x_2, \lim x_n, x_4, \lim x_n, x_6, \lim x_n, \ldots)$$

Notice $0 \leq T_1, T_2$ and $\|T_i x\| \leq \|x\| \ \forall x$, so T_i is continuous. Let $T = T_1 - T_2$, so $T \in \mathcal{L}(c,c)$.

Claim: If $S \in \mathcal{L}(c,c)$, $S \geq T$ and $S \geq 0$, then there is an $S_1 \in \mathcal{L}(c,c)$ s.t. $S_1 \geq T$, $S_1 \geq 0$ and $S_1 \leq S$, with $S_1 \neq S$. Hence $\sup\{T, 0\}$ does not exist in $\mathcal{L}(c,c)$, so $\mathcal{L}(c,c)$ is not a vector lattice.

Proof of claim. If $x \geq 0$ in c, then if $e^{(i)} = i^{\text{th}}$ unit vector

$$x \geq x - x_{2n} e^{(2n)} \geq 0$$

Then

$$(Sx)_{2n-1} \geq \left(S(x - x_{2n} e^{(2n)})\right)_{2n-1} \geq \left(T(x - x_{2n} e^{(2n)})\right)_{2n-1}$$

$$= x_{2n-1}$$

In particular, if $x = e = (1, 1, \ldots, 1, \ldots)$, then $(Se)_{2n-1} \geq 1$. Since $Se \in c$, there is an n_0 s.t. $(Se)_{2n_0} > 0$. Now define $S_1 : c \to c$ by

$$(S_1 x)_k = \begin{cases} (Sx)_k & \text{if } k \neq 2n_0 \\ 0 & \text{if } k = 2n_0 \end{cases}$$

Let H be a complex Hilbert space. $T : H \to H$ is self-adjoint if $\langle Tx | y \rangle = \langle x | Ty \rangle$ $\forall x, y \in H$. Let $\mathcal{S}$ denote the set of all self-adjoint operators on H. We say

$$S \leq T \quad \text{if} \quad \langle Sx | x \rangle \leq \langle Tx | x \rangle \; \forall x \in H$$

$\mathcal{S}$ is an ordered vector space. But if $\dim H > 1$, then for $S, T \in \mathcal{S}$, $\sup\{S, T\}$ exists if and only if $S \leq T$ or $T \leq S$. (Kadison)

Lattice Formulas

Most people would agree that trigonometric identities are quite essential to many computations in calculus and yet are quite dull to derive and discuss for their own sake. There are a number of lattice identities and inequalities that play a similar role in vector lattice theory. We will now derive a representative sample of such formulas for future reference.

The fact that the partial order is compatible with addition and scalar multiplication in a vector lattice yields the following basic identities:

(1) $z + x \vee y = (z+x) \vee (z+y)$; $z + x \wedge y = (z+x) \wedge (z+y)$

(2) $z - x \vee y = (z-x) \wedge (z-y)$; $z - x \wedge y = (z-x) \vee (z-y)$

(3) $\alpha(x \vee y) = (\alpha x) \vee (\alpha y)$; $\alpha(x \wedge y) = (\alpha x) \wedge (\alpha y)$ for $\alpha \geq 0$

If we replace z in (2) by $x+y$, we obtain

(4) $x + y = x \vee y + x \wedge y$

The <u>positive part</u> x^+ of an element x of a vector lattice is defined by $x^+ = x \vee 0$, the <u>negative part</u> x^- by $x^- = (-x)^+ = -x \wedge 0$, and the <u>absolute value</u> $|x|$ by $|x| = x \vee (-x)$.

If we set $y = 0$ in (4), we obtain the following important decomposition:

(5) $x = x^+ - x^-$

Since $x + |x| = x + x \vee (-x) \overset{(1)}{=} (2x) \vee 0 \overset{(3)}{=} 2x^+$, it follows from (5) that

(6) $|x| = x^+ + x^-$

(7) $x^+ = \frac{1}{2}(|x| + x)$

(8) $x^- = \frac{1}{2}(|x| - x)$

The lattice operations in a vector lattice satisfy distributive laws. More precisely, if $\{x_\alpha : \alpha \in A\}$ is a subset of a vector lattice E such that $\sup\{x_\alpha : \alpha \in A\}$ exists in E, then

(9) $\sup_\alpha \{x_\alpha \wedge y\} = (\sup_\alpha (x_\alpha)) \wedge y$

for each $y \in E$. Also, if $\inf\{x_\alpha : \alpha \in A\}$ exists in E, then

(10) $\inf_\alpha \{x_\alpha \vee y\} = (\inf_\alpha (x_\alpha)) \vee y$

for each $y \in E$. To prove (9), note that if $x = \sup\{x_\alpha : \alpha \in A\}$, then $x \wedge y \geq x_\alpha \wedge y$ for all $\alpha \in A$ so the right side of (9) dominates the left side. On the other hand, if $z \geq x_\alpha \wedge y \overset{(4)}{=} x_\alpha + y - x_\alpha \vee y$, then $z - y + x_\alpha \vee y \geq x_\alpha$ for all $\alpha \in A$. Hence, since $z - y + x \vee y = z - y + (\sup x_\alpha) \vee y$

$\geq z - y + x_\alpha \vee y$ for all $\alpha \in A$, it follows that $z - y + x \vee y \geq \sup_\alpha x_\alpha = x$, that is, $z \geq x + y - x \vee y \overset{(4)}{=} x \wedge y$. Therefore, the left side of (9) also dominates the right side and so the identity (9) is established. The proof of (10) is similar.

The following special cases of (9) and (10) are commonly referred to as the distributive laws

(11) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$; $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

Since $|x| \geq x, -x$ and $|y| \geq y, -y$ by definition of absolute value, it follows that $|x| + |y| \geq (x+y), -(x+y)$. Therefore the triangle inequality

(12) $|x + y| \leq |x| + |y|$

and its usual companion

(13) $\big|\, |x| - |y| \,\big| \leq |x \pm y|$

are valid in any vector lattice

Two elements x and y of a vector lattice are disjoint if $|x| \wedge |y| = 0$; in this case, we write $x \perp y$. The following descriptions of disjointness are useful:

(14) $x \perp y$ if and only if $|x| \vee |y| = |x| + |y|$

(15) $x \perp y$ if and only if $\big|\, |y| - |x| \,\big| = |y| + |x|$

(16) $x \perp y$ if and only if $|x + y| = |x - y|$

In fact, (14) is an immediate consequence of (4) and the definition of disjointness. To prove (15) note that $|x| \vee |y| \overset{(1)}{=} |x| + (|y| - |x|)^+ \overset{(7)}{=} |x| + \frac{1}{2}\big[\,\big||y| - |x|\big| + |y| - |x|\,\big] = \frac{1}{2}\big[\,\big||y| - |x|\big| + |y| + |x|\,\big]$ and apply (14). Statement (16) follows directly from (18) below:

(17) $|x| \vee |y| = \frac{1}{2}[|x+y| + |x-y|]$

(18) $|x| \wedge |y| = \frac{1}{2}[\,\big||x+y| - |x-y|\big|\,]$

(We will omit the proofs of (17) and (18).)

Note that (16) implies that

(19) $x^+ \perp x^-$ for any x in a vector lattice

Also, note that (7), (8), (15), (16) imply that

(20) If $x \perp y$, then

$$|x+y| = |x| + |y|$$
$$(x+y)^+ = x^+ + y^+$$
$$(x+y)^- = x^- + y^-$$

8/29 BANACH LATTICES

Notation: $x \vee y := \sup\{x,y\}$
$x \wedge y := \inf\{x,y\}$

DEFINITION: Two elements x, y of a vector lattice are disjoint if

$$|x| \wedge |y| = 0$$

Examples

(1) In sequence spaces such as ℓ^p, $p \geq 1$, ℓ^∞, c_0, c, two elements x and y are disjoint if and only if for each $n \in \mathbb{N}$, either $x_n = 0$ or $y_n = 0$

(2) If X is a compact topological space and $C(X)$ is the vector lattice of continuous real functions on X, then for each $h \in C(X)$ define the cozero set N_h of h by

$$N_h := \{x : h(x) \neq 0\}$$

For f, g in $C(X)$, $f \perp g$ iff $N_f \cap N_g = \emptyset$

(3) If (Ω, Σ, μ) is a measure space, then for $f, g \in L^p(\Omega, \Sigma, \mu)$ we have

$$f \perp g \Leftrightarrow \mu(N_f \cap N_g) = 0$$

LEMMA: If a, b, c are elements of the cone in a vector lattice E, then

$$a \wedge (b+c) \leq a\wedge b + a \wedge c$$

Proof. Let $z = a \wedge (b+c)$. Then $z \leq b+c$ and $z \leq a \leq b+a$
Hence ↑ b positive

$$z \leq (b+c) \wedge (b+a) = b + c \wedge a$$

Also, $z \leq a \leq a + c \wedge a$, since a and c are positive. Therefore

$$z \leq (a + c \wedge a) \wedge (b + c \wedge a) = a \wedge b + a \wedge c$$

$\blacksquare$

DECOMPOSITION LEMMA: If $\{x_i : i \leq n\}$ and $\{y_i : i \leq m\}$ are elements in the cone of a vector lattice such that

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{m} y_i$$

then there exist elements $\{z_{ij} : i \leq n, j \leq m\}$ of the positive cone such that

$$x_i = \sum_{j=1}^{m} z_{ij} \qquad y_j = \sum_{i=1}^{n} z_{ij}$$

$$\begin{array}{c|ccccc}
 & y_1 & + & \cdots & + & y_m \\
\hline
x_1 & z_{11} & + & \cdots & + & z_{1m} \\
+ & + & & & & + \\
\vdots & \vdots & & & & \vdots \\
+ & & & & & + \\
x_n & z_{n1} & + & \cdots & + & z_{nm}
\end{array}$$

Proof. By an induction on m, n it suffices to prove the result for $m = n = 2$.

$$\begin{array}{ccc}
 & y_1 & y_2 \\
x_1 & z_{11} & z_{12} \\
x_2 & z_{21} & z_{22}
\end{array}$$

Take $z_{11} = x_1 \wedge y_1$. Then $z_{12} = x_1 - z_{11}$, $z_{21} = y_1 - z_{11}$.
Also $z_{22} = x_2 - z_{21}$.

$z_{11} > 0$ by definition
$z_{12} = x_1 - x_1 \wedge y_1 \geq 0$
$z_{21} = y_1 - x_1 \wedge y_1 \geq 0$

Notice $z_{12} \wedge z_{21} = (x_1 - z_{11}) \wedge (y_1 - z_{11}) = x_1 \wedge y_1 - z_{11} = 0$. Also

$$z_{12} + x_2 = x_1 - z_{11} + x_2 = y_1 - z_{11} + y_2 = z_{21} + y_2$$

$$\Rightarrow y_2 = z_{12} + x_2 - z_{21} = z_{12} + z_{22}$$

Finally $z_{21} = x_2 - y_2 + z_{12} \leq x_2 + z_{12} \Rightarrow z_{21} = z_{21} \wedge (z_{12} + x_2)$

$$\leq (z_{21} \wedge z_{12}) + z_{21} \wedge x_2 = z_{21} \wedge x_2$$

lemma

Hence $z_{21} \leq x_2 \Rightarrow z_{22} = x_2 - z_{21} \geq 0$

$\blacksquare$

Notation: $[x,y] := \{z \in E : x \leq z \leq y\}$ order interval

COROLLARY: If x, y are elements of the cone in a vector lattice E, then

$$[0,x] + [0,y] = [0, x+y]$$

Proof. Suppose $0 \leq z \leq x+y$. Then $\exists w \geq 0$ s.t. $z+w = x+y$
By decomposition lemma,

	x	y
z	s	t
w	r	u

where $0 \leq s \leq x$, $0 \leq t \leq y$ and $z = s+t$.

$\blacksquare$

DEFINITION: A vector lattice E is

(i) order complete if every subset A of E with an upper bound in E has a sup in E.

(ii) σ-order complete if every countable subset A of E with an upper bound in E has a supremum in E

Example: Suppose (Ω, Σ, μ) is a finite measure space and that $\dot{M}(\Omega, \Sigma, \mu)$ is the vector space of equivalence classes of measurable functions mod null functions.

$$\dot{f} \leq \dot{g} \iff f(\omega) \leq g(\omega) \text{ a.e.}$$

Then $\dot{M}(\Omega, \Sigma, \mu)$ is a vector lattice which is order complete

Proof. It would suffice to show that $\sup\{\dot{f}_\alpha\}$ exists if
(1) $(\dot{f}_\alpha)$ has an upper bound in $\dot{M}$
(2) for α_1, α_2, $\exists\, \alpha_3$ s.t. $\dot{f}_{\alpha_3} \geq \dot{f}_{\alpha_1}$, $\dot{f}_{\alpha_3} \geq \dot{f}_{\alpha_2}$

Special Case There is a bounded measurable function g_0 s.t. $\dot{f}_\alpha \leq \dot{g}_0$ $\forall \alpha$. Then

$$\int f_\alpha \, d\mu \leq \int g_0 \, d\mu < \infty$$

Let $M = \sup_\alpha \int f_\alpha \, d\mu$ and choose $\alpha_1, \alpha_2, \ldots$ s.t. $\dot{f}_{\alpha_1} \leq \dot{f}_{\alpha_2} \leq \ldots$ and

$$\int f_{\alpha_n} \, d\mu \nearrow M$$

Let $f_0 := \sup f_{\alpha_n}$. Then f_0 is measurable. Claim $\dot{f}_\alpha \leq \dot{f}_0$ $\forall \alpha$. Suppose not, so suppose $\exists\, \alpha_0$ s.t.

$$A = \{ f_{\alpha_0}(t) > f_0(t) \}$$

has positive measure. Choose α_n' s.t. $f_{\alpha_n'} \geq f_{\alpha_0}, f_{\alpha_n}$, and let

$$f_0' := \sup f_{\alpha_n}'$$

Then $\int f_0' \geq \int f_0$, but $\int f_0' = M = \int f_0$, so $f_0' = f_0$ a.e.
(already know $f_0' \geq f_0$)

Now let

$$\dot{f}_{\alpha n} = \dot{f}_\alpha \wedge n 1_\Omega$$

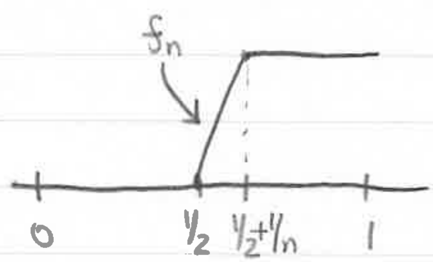
By the above, $g_n = \sup \dot{f}_{\alpha n}$ exists. Then $\sup \dot{g}_n = \sup \dot{f}_\alpha$.

8/31 BANACH LATTICES

$C(X)$ vector lattice of all real continuous functions on a topological space X under order

$$f \le g \iff f(t) \le g(t) \ \forall t \in X$$

1) $C[0,1]$, however, is not σ-order complete



$\sup\{f_n\}$ does not exist

2) $C(\mathbb{N})$ ($\mathbb{N}$ natural numbers in the discrete topology) is order complete

3) $C(\alpha\mathbb{N}) = c$ (convergent sequences) is not σ-order complete
↑ one point compactification

$f_1 \quad (1,-1,1,1,\ldots)$
$f_2 \quad (1,-1,1,-1,1,1,\ldots)$
etc.

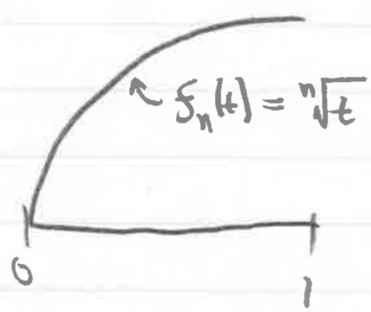
$\inf\{f_n\}$ does not exist

4) $C(\beta\mathbb{N}) = \ell_\infty$ is order complete
↑ Stone-Čech compactification

PROBLEM: Characterize the order completeness and σ-order completeness of $C(X)$ in terms of topological properties of X

Observe that $A \subset C(X)$ may have a supremum different than the pointwise supremum.

e.g.



$\sup(f_n) = 1_{[0,1]}$

but $\sup(f_n(0)) = 0$

Suppose f is any bounded function on X.

$$\bar{f}(x) := \inf_{V \in \eta(x)} \sup_{x \in U} f(x) = \overline{\lim_{y \to x}} f(y)$$

(upper envelope)

Note: $\bar{f}$ is upper semicontinuous

PROPOSITION: Suppose A is a majorized subset of $C(X)$ and that $f(x) :=$ point sup A. If $\bar{f}$ is continuous, then $\sup A$ exists in $C(X)$ and $\bar{f} = \sup A$.

If X is completely regular and if $\bar{f}$ is not continuous then $\sup A$ does not exist in $C(X)$.

Proof. If $\bar{f}$ is continuous, then $\bar{f}$ is an upper bound for A in $C(X)$. If g is any other upper bound of A, then $g \geq f$, so $g = \bar{g} \geq \bar{f}$

Hence $\bar{f} = \sup A$

Suppose $\bar{f}$ is not continuous and that X is completely regular. We will show that if $g \in C(X)$ is an upper bound for A, there is a $g' \in C(X)$ that is also an upper bound with $g' \leq g$, $g' \neq g$.

Since $\bar{f}$ is u.s.c. and $\bar{f} \geq f$, we have that $g - \bar{f}$ is l.s.c and $g - \bar{f} \geq 0$. Since $g - \bar{f} \neq 0$ there is an $x_0 \in X$ s.t. $(g-\bar{f})(x_0) > 0$. Since $g - \bar{f}$ is l.s.c. there is a neighborhood U of x_0 and a $\delta > 0$ s.t.

$$(g - \bar{f})(x) > \delta \quad \forall x \in U$$

By complete regularity, there is a $k \in C(X)$ s.t. $0 \leq k(x) \leq \delta$ $\forall x \in X$ and $k(x) = 0$ for $x \notin U$, $k(x_0) = \delta$. Let $g' = g - k$ ▨

Proposition: Suppose A is a subset of $C(X)$ (completely regular) with $\sup A = g$ and pt.$\sup A = f$. Then $\{x \in X : f(x) \neq g(x)\}$ is of first category.

Proof. Note $g \geq f$. For each $n \in \mathbb{N}$ let $B_n = \{x : g(x) - f(x) \geq 1/n\}$

Then

$$\{x : f(x) \neq g(x)\} = \bigcup_{n=1}^{\infty} B_n$$

Claim: B_n is nowhere dense. First note that B_n is closed since f is l.s.c. (as sup of cont functions (pointwise)). Suppose $x_0 \in \operatorname{int} B_n$. Then there is a neighborhood U of x_0 s.t. $g(x) - f(x) \geq 1/n$ $\forall x \in U$. Then $\exists\, k \in C(X)$ with

$$0 \leq k(x) \leq 1/n \ \forall x, \quad k(x) = 0 \text{ for } x \notin U \quad k(x_0) = 1/n$$

Then $g' := g - k$ is an upper bound of A ↯. Hence $\mathrm{int}\, B_n = \emptyset$. ▨

DEFINITION: Suppose X is a Hausdorff topological space.

i) X is Stonean (extremally disconnected) if the closure of every open set is open

ii) X is σ-Stonian (ω-extremally disconnected) if the closure of every open F_σ-set is open.

Examples and Remarks

(1) If X is Stonian and if V, U are disjoint open sets in X, then $\overline{U}$ and $\overline{V}$ are disjoint

Proof. If $U \cap V = \emptyset$, then $U \subset V^c \Rightarrow \overline{U} \subset V^c \Rightarrow V \subset \overline{U}^c$
$\Rightarrow \overline{V} \subset \overline{U}^c \Rightarrow \overline{V} \cap \overline{U} = \emptyset$

Since X Stonian

(2) Definition: X is totally disconnected if the singletons are the only connected subsets of X

(3) Every Stonian space is totally disconnected

Proof. Suppose $x \neq y$ and that x, y belong to a connected subset C of X. Choose open sets U, V s.t. $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Then $\{C \cap \overline{U}, C \cap \overline{U}^c\}$ is a separation of C ↯

9/5 BANACH LATTICES

__DEFINITION:__ A space is zero-dimensional if the clopen sets form a base for the topology

Fact: Zero-dimensional Hausdorff spaces are totally disconnected

Proof. If $x \neq y$ choose a clopen set U such that $x \in U$ and $y \notin U$. If C is a connected subset and if $x, y \in C$, then $\{C \cap U, C \cap U^c\}$ is a separation of C ↯.

Fact: Compact totally disconnected spaces are zero dimensional.

__Example__ Let $\mathbb{Q}$ be the rationals in the induced topology. For $x, y \notin \mathbb{Q}$, let $(x,y)_{\mathbb{Q}} = \{z \in \mathbb{Q} : x < z < y\}$. $(x,y)_{\mathbb{Q}}$ is clopen in $\mathbb{Q}$ and form a base, so $\mathbb{Q}$ is zero dimensional $\Rightarrow$ $\mathbb{Q}$ is totally disconnected. However, $\mathbb{Q}$ is not Stonean:

$$U = (-\infty, 1/2) \cap \mathbb{Q} \qquad V = (1/2, \infty) \cap \mathbb{Q}$$

U, V open but

$$U \cap V = \emptyset \qquad \overline{U} \cap \overline{V} = \{1/2\}$$

__Fact:__ A topological space X is Stonean iff βX is Stonean.

Clearly any discrete topological space is Stonean.

THEOREM: Suppose X is a Hausdorff space.

(1) If X is Stonean, then $C(X)$ is order complete. The converse is true if X is completely regular.

(2) If X is σ-Stonean, then $C(X)$ is σ-order complete. The converse is true if X is normal.

Proof. (1) Suppose X is completely regular and that $C(X)$ is order complete. Let U be an open set in X. For each $y \in U$, choose $f_y \in C(X)$ s.t. $0 \le f_y \le 1$, $f_y(y) = 0$, $f_y(x) = 1$ for each $x \in U^c$. The set $\{f_y : y \in U\}$ has an infimum g in $C(X)$. Then $g(y) = 0 \ \forall y \in U$ and so $g(y) = 0 \ \forall y \in \bar{U}$. Let $x \in \bar{U}^c$. Choose $h_x \in C(X)$ s.t. $h_x(x) = 1$, $h_x(y) = 0 \ \forall y \in \bar{U}$. Then

$$h_x \le f_y \quad \forall y \in U \quad \forall x \in \bar{U}^c$$

Hence $h_x \le g \ \forall x \in \bar{U}^c$, whence $g(x) = 1 \ \forall x \in \bar{U}^c$. Therefore $g = \chi_{\bar{U}^c}$. Hence $\bar{U}^c$ is closed, so $\bar{U}$ is open.

Suppose X is Stonean. It would suffice to show that any family $\{f_\alpha : \alpha \in A\}$ of non-negative functions has an infimum of $C(X)$. For each $r > 0$, let

$$G_{\alpha r} := \{x \in X : f_\alpha(x) < r\} \quad (\text{open})$$

$$G_r := \bigcup_{\alpha \in A} G_{\alpha r} \quad (\text{open})$$

Then $\bar{G}_r$ is open. As $r \uparrow$, so does $\bar{G}_r$. Also, $\bigcup_{r>0} \bar{G}_r = X$. Therefore, for each $x \in X$, either

(1) $$x \in \bigcap_{r>0} \overline{G_r}$$

or (2) for some r, $x \in \overline{G}_{r-\varepsilon}$ and $x \notin \overline{G}_{r+\varepsilon}$ for all $0<\varepsilon<r$. Set

$$g(x) := \begin{cases} 0 & \text{if (1) holds for } x \\ r & \text{if (2) holds for } x \end{cases}$$

CLAIM: $g = \inf\{f_\alpha : \alpha \in A\}$

(i) $g \in C(X)$. If $0 \le r_1 < r_2$, it suffices to show that $g^{-1}(r_1,r_2)$ is open.

$$g^{-1}(r_1,r_2) = g^{-1}(-\infty,r_2) \setminus g^{-1}(-\infty,r_1]$$

Now

$$g^{-1}(-\infty,r_2) = \bigcup_{0<\varepsilon<r_2} \overline{G}_{r_2-\varepsilon} \quad \text{(open)}$$

$$g^{-1}(-\infty,r_1] = \bigcap_{\varepsilon>0} \overline{G}_{r+\varepsilon} \quad \text{(closed)}$$

Hence $g^{-1}(r_1,r_2)$ is open, so g is continuous.

(ii) $g \le f_\alpha \quad \forall \alpha \in A$

Proof. Assume not. Then $\exists\, x_0 \in X$, $\alpha_0 \in A$, r_0, ε_0, s.t.

$$f_{\alpha_0}(x_0) < r_0 - \varepsilon_0 < r_0 < g(x_0)$$

Hence $x_0 \in \overline{G}_{r_0-\varepsilon_0}$, so $g(x_0) \leq r_0$ ↯.

(3) If h is any lower bound of $\{f_\alpha\}$, then

$$h^{-1}(-\infty, r-\varepsilon) \supset G_{r-\varepsilon}$$

By the continuity of h,

$$h^{-1}(-\infty, r-\varepsilon) \supset \overline{G}_{r-\varepsilon} \quad \forall r>0 \ \forall\, 0<\varepsilon<r$$

Hence

$$h^{-1}(-\infty, r) \supset \bigcup_{0<\varepsilon<r} G_{r-\varepsilon} = g^{-1}(-\infty, r) \Rightarrow h \leq g$$

For σ-stonian we have $\{f_n\}$ and same proof for ($\Rightarrow$)

Since

$$G_{nr} = \{x \in X : f_n(x) < r\} = \bigcup_{k=1}^{\infty} \{x : f_n(x) < r - 1/k\}$$

so G_r would be F_σ.

For the converse, let $C(X)$ be σ-order complete and X normal. Let G be an open F_σ

$$G = \bigcup_{n=1}^{\infty} F_n$$

where F_n is closed and $G^c \cap F_n = \phi$. For each n, choose $f_n \in C(X)$ s.t.

$$0 \leq f_n \leq 1$$
$$f_n(x) = 1 \quad \forall x \in F_n$$
$$f_n(x) = 0 \quad \forall x \in G^c$$

Then $f := \sup f_n \in C(X)$. As before $f = \chi_{\overline{G}}$, so $\overline{G}$ is open.

9/7 BANACH LATTICES

HAHN-BANACH (Usual case) $p: E \to \mathbb{R}$ sublinear, M subspace of E, and $f: M \to \mathbb{R}$ linear with $f(x) \le p(x)$, then there exists an extension $\hat{f}$ of f to E such that $\hat{f}(x) \le p(x)$ $\forall x \in E$.

Proof. By Zorn's lemma there is a maximal element of

$$\mathcal{E} = \{g : g|_M = f,\ g \text{ linear},\ g(x) \le p(x)\ \forall x \in D(g)\}$$

say g_0. If $x_0 \notin D(g_0)$, consider

$$\hat{M} := \mathrm{sp}\{x_0\} \oplus D(g_0)$$

Every $x \in \hat{M}$ is of the form

$$x = \alpha x_0 + y \qquad \alpha \in \mathbb{R},\ y \in D(g_0)$$

For any real c, $g_1(x) := c\alpha + g_0(y)$ defines a linear extension of g_0 to $\hat{M}$. For $y_1, y_2 \in D(g_0)$

$$g_1(y_1) - g_1(y_2) = g_1(y_1 - y_2) \le p(y_1 - y_2) \le p(y_1 + x_0) + p(-y_2 - x_0)$$

$$\Rightarrow -p(-y_2 - x_0) - g_1(y_2) \le p(y_1 + x_0) - g_1(y_1)$$

$$\Rightarrow \exists c \text{ s.t. } \sup(-p(-y_2 - x_0) - g_1(y_2)) \le c \le \inf(p(y_1 + x_0) - g_1(y_1))$$

NOTE: This proof works if $\mathbb{R}$ is replaced by any order complete vector lattice.

COROLLARY (Hahn-Banach): If f is a continuous linear functional defined on a subspace M of a normed space E, then f has a continuous extension $\hat{f}$ to E s.t. $\|\hat{f}\| = \|f\|_M$.

Proof. Take $p(x) := \|f\|_M \|x\|$

Now suppose E is an order complete Banach lattice

$$\begin{tikzcd}
\text{B-space} \arrow[r] & F \arrow[dr, dashed, "\hat{T}"] & \\
& M \arrow[u, hook] \arrow[r, "T"] & E \\
& \text{subspace} &
\end{tikzcd}$$

When does $\hat{T}$ exist with $\|\hat{T}\| = \|T\|$?

i) $E = C(X)$ X Stonean compact Hausdorff. Take $p(x) = \|T\|_M \|x\| \chi_X$. Then $Tx \leq p(x)$ $\forall x \in M$. Note

unit ball in $C(X) = \{f : -\chi_X \leq f \leq \chi_X\}$

By Hahn-Banach $\exists\, \hat{T} : F \longrightarrow C(X)$ s.t. $\hat{T}x \leq p(x)$ $\forall x \in F$ and $\hat{T}|_M = T$.
Then

$$\hat{T}(-x) \leq p(-x) = p(x)$$

$$\Rightarrow \quad -p(x) \leq \hat{T}(x) \leq p(x) \qquad \forall x \in F$$

$$\Rightarrow \quad \|\hat{T}\| = \|T\|$$

<u>THEOREM</u>: (Goodner, Nachbin, Kelley) Suppose E is a Banach space. TFAE

(1) E is isometric to $C(X)$ for X Stonean compact Hausdorff

(2) For each Banach space F, each linear subspace M of F, and each bounded linear operator $T: M \to E$, there is a bounded linear norm-preserving extension to F

Proof Suppose E has property (2). Let Y be the weak* closure of the set of extreme points of the unit ball S^* of E^*. Equip Y with the induced w^* topology. Y is a compact Hausdorff space. Let

$$\mathcal{O} = \{G \subset Y : G \text{ open}, (-G) \cap G = \emptyset\}$$

Order $\mathcal{O}$ by inclusion. If $(G_\alpha : \alpha \in A)$ is a totally ordered subset of $\mathcal{O}$, then $\cup G_\alpha$ is an upper bound, so $\mathcal{O}$ has a maximal element G_0

Note $G_0 \cup -G_0$ is dense in Y, for if not there is an open $U \subset Y$ s.t. $U \cap (-G_0 \cup G_0) = \emptyset$. Let $G_1 = G_0 \cup U$. Then $G_1 \in \mathcal{O}$ and $G_0 \neq G_1$, $\lightning$ maximality of G_0.

Lemma: Suppose U, V are disjoint open sets in Y s.t. $U \cup V \in \mathcal{O}$ and $-(U \cup V) \cap (U \cap V)$ is dense in Y. Let

$$Z := (\{0\} \times \bar{U}) \cup \{(1) \times \bar{V}\}$$

be the topological disjoint union of $\bar{U}$ and $\bar{V}$. Define

$$H : E \to C(Z)$$

by

$$[Hx](0, u) = u(x) \qquad x \in E,\ u \in \bar{U}$$

$$[Hx](1, v) = v(x) \qquad x \in E,\ v \in \bar{V}$$

Then H is an isometry of E onto $C(Z)$. The sets $\bar{U} \cap \bar{V}$, $(-\bar{U} \cap -\bar{V}) \cap (\bar{U} \cup \bar{V})$ are empty, and H^* maps the point evaluations in $C(Z)^*$ homeomorphically onto $\bar{U} \cup \bar{V}$ (for respective weak*-topologies.

Proof of theorem from lemma: Let $U = G_0$ Take $U = G_0$ and $V = \phi$ The lemma applies

$$-\bar{G_0} \cap \bar{G_0} = \phi$$

We know $-\bar{G_0} \cup \bar{G_0} = \overline{(-G_0) \cup G} = \mathcal{Y} \implies G^{-1}$ is clopen
Lemma also says E_1 is isometric to $C(\bar{G_0})^*$

Claim: G_0 is Stonean

Let U be open in $\bar{G_0}$. Let $V = \bar{G_0} \setminus \bar{U}$. U and V are disjoint open sets; $U \cup V \in G$, since $V \cup U \subset \overline{G_0}$ and $-\overline{G_0} \cap G_0 = \emptyset$

$$\overline{U} \cup \overline{V} = \overline{U} \cup (\overline{G} \cap \overline{V}^c) = \overline{U} \cup \left(\overline{G_0 - \overline{U}^c}\right) \supset \overline{U} \cup (\overline{G_0} \cap \overline{U}^c) = \overline{G_0}$$

$$\Longrightarrow \overline{U} \cup \overline{V} = \overline{G_0}$$

Then by the lemma $\overline{U} \cap \overline{V} = \emptyset$. But $\overline{U} \cup \overline{V} = G_0$, so $\overline{U}$ is open

9/10 BANACH LATTICES

E has the Hahn-Banach extension property. Let Y be the weak* closure of ext (S^*) equipped with the relative topology. Let

$$\mathcal{O} = \{ G \subset Y : G \text{ open and } G \cap (-G) = \phi \}$$

By Zorn's lemma, $\mathcal{O}$ has a maximal element G_0. $(-G_0) \cup G_0$ is dense in Y.

LEMMA: Let U, V be disjoint open sets in Y such that $U \cup V \in \mathcal{O}$ and $-(U \cup V) \cup (U \cup V)$ is dense in Y. Define

$$Z := (\{0\} \times \overline{U}) \cup (\{1\} \times \overline{V})$$

and $H : E \longrightarrow C(Z)$ by

$$Hx(0,u) = u(x) \qquad u \in \overline{U}$$
$$Hx(1,v) = v(x) \qquad x \in \overline{V}$$

Then H is an isometry onto $C(Z)$; $\overline{U}, \overline{V}$ are disjoint; $-(\overline{U} \cup \overline{V}) \cup (U \cup V)$ are disjoint; H^* maps the point evaluations in $C(Z)^*$ homeomorphically onto $\overline{U} \cup \overline{V}$.

Proof. 1) H is an isometry.

since $\overline{U} \cup \overline{V} \subset S^*$

$$\| Hx \| = \sup_{(a,b) \in Z} | Hx(a,b) | = \max \{ \sup |u(x)|, \sup |v(x)| \} \leq \|x\|$$

Let $x \in E$ and define

$$F_x := \{ \mu \in E^* : \|\mu\| \le 1, \ \mu(x) = \|x\| \}$$

By Hahn-Banach $F_x \neq \emptyset$. Also F_x is weak* closed, convex, and a face of S^*, i.e. if $w \in F_x$ and $w = \frac{1}{2}s + \frac{1}{2}t$ for $s, t \in S^*$, then $s, t \in F_x$. Therefore there exists an extreme point μ of S^* that lies in F_x. In particular, there is a $\mu \in F_x \cap Y$. But then either μ or $-\mu$ belongs to $\bar{U} \cup \bar{V}$, so

$$\| Hx \| = \sup_{(a,b) \in Z} | Hx(a,b) | \ge |\mu(x)| = \|x\|$$

2) $H^* : C(Z)^* \longrightarrow E^*$ maps $\varepsilon_{(0,\mu)}$ into μ and $\varepsilon_{(1,\nu)}$ into ν.

$$\langle x, H^* \varepsilon_{(0,\mu)} \rangle = \langle Hx, \varepsilon_{(0,\mu)} \rangle = Hx(0,\mu) = \mu(x)$$

3) If $\mu \in U$ and μ is an extreme point of S^* and if T^* is the dual unit ball in $C(Z)^*$, then

$$(H^*)^{-1}(\mu) \cap T^* = \varepsilon_{(0,\mu)}$$

(also follows from (2))

Observe that $(H^*)^{-1}(\mu) \cap T^* \neq \emptyset$ since H is an isometry. This set is also weak*-closed and convex. It is also a face of T^* since μ is an extreme point of S^* and H^* has norm 1. Therefore either $(H^*)^{-1}(\mu) \cap T^* = \{\varepsilon_{(0,\mu)}\}$ by (2) or $(H^*)^{-1}(\mu) \cap T^*$ contains at least two extreme points of T^*. Now

$$\text{Ext}(T^*) = \{\text{point evaluations and their negatives}\}$$

So it is not possible for there to be two extreme points of T^* since $u \notin \overline{V}$

4) E has the Hahn-Banach extension property and $H(E) \subset C(Z)$

$$\begin{tikzcd} C(Z) \arrow[drr, dashed, "G"] & & \\ \cup & & \\ H(E) \arrow[rr, "H^{-1}"] & & E \end{tikzcd}$$

So $\exists\, G : C(Z) \to E$ with $\|G\| = 1$. Claim: $GH = I_E$ by extension property. Hence $H^*G^* = I_{E^*}$. Since

$$H^* \varepsilon_{(0,u)} = u$$
$$H^* \varepsilon_{(1,v)} = v$$

we see that if $u \in U$ is an extreme point of S^*, then $G^*u = \varepsilon_{(0,u)}$. Similarly $G^*v = \varepsilon_{(1,v)}$ if $v \in V$ is an extreme point of S^*. But points u, v of this sort are dense in U and V, so by definition of Y, G^* maps a dense subset of Y onto a weak* dense subset of $(-P) \cup P$, where P = point evaluations on $C(Z)$. These sets $(-P) \cup P$ and Y are w^*-compact, so $G^*(Y) = (-P) \cup P$

5) $H^*|_{P \cup (-P)}$ and $G^*|_Y$ are inverse homeomorphisms (for the weak* topology.)

We already know $H^*G^* = I_{E^*}$ and that if $u \in U$ is an extreme point of S^*, then

$$G^* H^* (\varepsilon_{0,u}) = \varepsilon_{0,u}$$

Similarly, $G^* H^* (1,v) = \varepsilon_{(1,v)}$ for $v \in V$ extreme in S^*. Therefore $G^* H^*$ is the identity on a dense subset of $(-P \cup P)$.

6) $H(E) = C(Z)$

We know $G^*(S^*)$ is convex and weak* compact subset of T^*. Also, $G^*(S^*)$ contains all the extreme points of T^*. Therefore $G^*(S^*) = T^* \Rightarrow G^*(E^*) = C(Z)^*$. Also $H^* G^* = 1_{E^*}$ $\Rightarrow H^*$ is 1-1 $\Rightarrow T$ is onto.

Left to show $\bar{U} \cap \bar{V} = \phi$, $[-(\bar{U} \cup \bar{V}) \cap (\bar{U} \cup \bar{V})] = \phi$, $H^*(P) = \bar{U} \cup \bar{V}$

9/12 BANACH LATTICES

Linear mappings on vector lattices

Let E, F be vector lattices. Let $T: E \longrightarrow F$ be linear

i) T is positive if $Tx \geq 0$ for all $x \geq 0$
ii) T is order bounded if T maps order intervals in E into subsets of order intervals in F

Remarks
(1) Every positive linear map is order bounded, since

$$T[x,y] \subset [Tx, Ty] \quad \forall x \leq y \in E$$

(2) The set K of all positive linear maps of E into F is a cone in the space $L(E,F)$ of linear maps of E into F.

Notation: $L^+(E,F)$ = all differences of positive linear maps from E into F
$L^b(E,F)$ = all order bounded linear maps of E into F.

Both $L^+(E,F)$ and $L^b(E,F)$ are linear subspaces of $L(E,F)$. Moreover $L^+(E,F) \subset L^b(E,F)$.

Example If $E = C(X)$, $F = C(Y)$ for X, Y compact, then

$$L^b(E,F) = \text{all continuous linear maps of } E \text{ into } F = \mathcal{L}(E,F)$$

because the unit ball is an order interval and every order interval is norm bounded

<u>PROPOSITION</u>: Suppose E, F are vector lattices and that F is order complete. Then $L^b(E,F) = L^+(E,F)$ is an order complete vector lattice for the order determined by the cone of positive linear maps. Moreover, the lattice operations in $L^b(E,F)$ are defined for $x \geq 0$ in E by

$$(1) \quad T^+ x = \sup\{Tz : 0 \leq z \leq x\}$$
$$(2) \quad |T|\, x = \sup\{Tz : |z| \leq x\}$$
$$(3) \quad (S \vee T)\, x = \sup\{Sz + Ty : y+z = x,\ y \geq 0, z \geq 0\}$$
$$(4) \quad (S \wedge T)\, x = \inf\{Sz + Ty : y+z = x,\ y \geq 0,\ z \geq 0\}$$

If $\{T_\alpha : \alpha \in A\}$ is a subset of $L^b(E,F)$ that is bounded above (below) then

$$(i) \quad \left(\sup_{\alpha\in A} T_\alpha\right) x = \sup\left\{\sum_{i=1}^n T_{\alpha_i} x_i : \alpha_i \in A,\ x_i \geq 0,\ \sum_{i=1}^n x_i = x\right\}$$

$$(ii) \quad \left(\inf_{\alpha\in A} T_\alpha\right) x = \inf\left\{\sum_{i=1}^n T_{\alpha_i} x_i : \alpha_i \in A,\ x_i \geq 0,\ \sum_{i=1}^n x_i = x\right\}$$

If F is not necessarily order complete but if for some $T \in L(E,F)$ one of the sups in (1) or (2) exists $\forall x \geq 0$, then both sups exist and define T^+ and $|T|$.

Proof. Suppose E, F are vector lattices (but not that F is order complete) and suppose that for some $T : E \longrightarrow F$, the sup in (1) exists for all $x \geq 0$. Define S on the cone by

$$Sx := \sup\{Tz : 0 \leq z \leq x\}$$

Then S is positive homogeneous. S is superadditive (i.e. $S(x_1+x_2) \geq Sx_1 + Sx_2$) S is also subadditive since $[0, x_1+x_2] = [0,x_1] + [0,x_2]$ (corollary of decomposition theorem) Therefore S is additive on the cone of E.
Extend S to all of E by

$$Sx = Sx_1 - Sx_2$$

where $x = x_1 - x_2$ for $x^+, x^- \geq 0$. This is independent of the choice of x_1, x_2, for if $w = x_1 - x_2 = x_1' - x_2'$, then

$$x_1 + x_2' = x_1' + x_2 \implies Sx_1 + Sx_2' = S(x_1+x_2') = S(x_1'+x_2)$$
$$= Sx_1' + Sx_2$$

$$\implies Sx_1 - Sx_2 = Sx_1' - Sx_2'$$

Then $S : E \to F$ is linear. $Sx \geq Tx$ $\forall x \geq 0$ by definition of S. Hence $S \geq T$. Also, $Sx \geq 0$ $\forall x \geq 0$, so $S \geq 0$.

Let $R \geq T, R \geq 0$ where $R : E \to F$ is linear. If $x \geq 0$ and $0 \leq z \leq x$, then

$$Tz \leq Rz \leq Rx$$

and so $Sx \leq Rx$. Therefore $S = T^+$

If the sup in (1) exists for a given T and all $x \geq 0$, want to show sup in (2) exists. Notice

$$-Tx + \sup\{Tz : 0 \leq z \leq 2x\}$$

$$= \sup\{-Tx + Tz : 0 \leq z \leq 2x\}$$

$$(*) \qquad = \sup\{T(-x+z) : 0 \leq z \leq 2x\}$$

$$= \underbrace{\sup}_{\text{sup in (2)}}\{Tz_1 : -x \leq z_1 \leq x\}$$

Similarly existence of sup in (2) implies sup in (1) exists.

If the sup in (2) exists for some $T: E \to F$ and all $x \geq 0$, then T^+ exists and is given by (1). Define

$$Rx = \sup\{Tz : |z| \leq x\}$$

Then $(*)$ shows that $Rx = -Tx + 2T^+x$. In particular, R is additive and positive homogeneous on the cone, so R has a unique linear extension to E. Note for $x \geq 0$

$$Rx \geq -Tx + 2Tx = Tx$$

so $R \geq T$. Also $Rx \geq -Tx$, so $R \geq -T$. Therefore $R \geq T, -T$. Suppose $R_1 \geq T, -T$, then $R_1 + T \geq 2T$, and $R_1 + T \geq 0$, so

$$R_1 + T \geq 2T^+$$

$$\Rightarrow R_1 \geq -T + 2T^+ = R$$

Therefore $R = |T|$.

If F is order complete and if $T \in L^b(E,F)$, then the sups in (1) and (2) exist for all $x \geq 0$. Hence $L^b(E,F)$ is a vector lattice with (1) and (2) defining T^+ and $|T|$ for every $T \in L^b(E,F)$. In particular, $L^b(E,F) = L^+(E,F)$ since $T = T^+ - T^-$ for $T \in L^b(E,F)$.

Suppose S, T are in $L^b(E,F)$. Then for $x \geq 0$

$$(S \vee T)x = [(S-T)^+ + T]x = \sup_{0 \leq z \leq x}\{(S-T)z\} + Tx$$

$$= \sup_{0 \leq z \leq x}\{(S-T)z + Tx\}$$

$$= \sup_{0 \leq z \leq x}\{Sz + T(x-z)\}$$

$$= \sup_{\substack{y+z=x \\ y,z \geq 0}}\{Sz + Ty\}$$

Suppose $\{T_\alpha : \alpha \in A\}$ is bounded above by $S \in L^b[E,F]$. If $x \geq 0$ and $x = x_1 + \dots + x_n$, $x_i \geq 0$, and if $x_i \in A$ $i = 1, \dots, n$, then

$$\sum_{i=1}^{n} T_{\alpha_i} x_i \leq \sum_{i=1}^{n} S x_i = Sx$$

Hence the sup in (i) exists.

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(Proof continued)

Note that $\sup\{\sum_{i=1}^{n} T_{\alpha_i} x_i : x = \sum_{i=1}^{n} x_i,\ x_i \geq 0,\ \alpha_i \in A\}$ exists since

$$(*) \qquad \sum T_{\alpha_i} x_i \leq \sum_{i=1}^{n} S x_i = Sx$$

Define Rx to be this sup. Then R is positively homogeneous on the cone and R is superadditive and, by the decomposition lemma, it is also subadditive on the cone of E. Therefore R is additive and positive homogeneous on the cone, so R extends uniquely to a linear map on E. Note $R \geq T_\alpha$ $\forall \alpha \in A$, and by the argument $(*)$, $R \leq S$ for any upper bound S of $(T_\alpha : \alpha \in A)$. Hence $R = \sup\{T_\alpha : \alpha \in A\}$.

If $\{T_\alpha : \alpha \in A\}$ is bounded below in $L^b(E,F)$ and if $x \geq 0$ then

$$\inf\{T_\alpha : \alpha \in A\} = \inf\{\sum_{i=1}^{n} T_{\alpha_i} x_i : x = \sum_{i=1}^{n} x_i,\ x_i \geq 0,\ \alpha_i \in A\}$$

$\blacksquare$

We say that a set D in an ordered vector space E is directed $(\leq)$ if $d_1, d_2 \in D \Rightarrow \exists\, d_3 \in D$ with $d_3 \geq d_1, d_2$. If $\{T_\alpha : \alpha \in A\}$ is a directed family in $L^b(E,F)$ that is bounded above in $L^b(E,F)$, then for $x \geq 0$

$$\left(\sup_{\alpha\in A} T_\alpha\right)x = \sup_{\alpha\in A}\left(T_\alpha x\right)$$

The direction ($\geq$) is clear. For ($\leq$), note

$$\sum_{i=1}^{n} T_{\alpha_i} x_i \leq \sum_{i=1}^{n} T_{\alpha_0} x_i = T_{\alpha_0} x$$

where $T_{\alpha_0} \geq T_{\alpha_i}$ for $1 \leq i \leq n$.

COROLLARY: If E is a vector lattice, then the vector space E^b of all order bounded linear functionals on E is an order complete vector lattice. (E^b = order dual of E)

Sublattices and Ideals in Vector Lattices

Suppose E is a vector lattice and M is a linear subspace of E.

(1) M is a sublattice of E iff $x \vee y \in M$ $\forall x, y \in M$ (or $x^+ \in M$ $\forall x \in M$, $|x| \in M$ $\forall x \in M$ etc.)

(2) M is a lattice ideal if $x \in M$ whenever $|x| \leq |y|$ and $y \in M$

Remarks and examples

(1) Every lattice ideal is a sublattice (since $x \in M \Rightarrow |x| \in M$)

(2) c is a sublattice of ℓ^∞ but not a lattice ideal

(3) c_0 is a lattice ideal in c and ℓ^∞

(4) $\dot{M}(X,\Sigma,\mu)$ = all eq. classes mod null functions of measurable functions on a σ-finite (X,Σ,μ). This is an order complete vector lattice

$L^p(X,\Sigma,\mu)$ $p \geq 1$, $L^\infty(X,\Sigma,\mu)$ are lattice ideals in $\dot{M}(X,\Sigma,\mu)$

(5) If M is a lattice ideal in E and E is order complete, then M is order complete.

Proof. Let $(x_\alpha) \subset M$, $x_\alpha \geq 0$. Then $\inf x_\alpha$ exists and

$$0 \leq \inf x_\alpha \leq x_\alpha \in M$$

$$\Longrightarrow \inf x_\alpha \in M$$

M lattice ideal

Hence $L^p(X,\Sigma,\mu)$ and $L^\infty(X,\Sigma,\mu)$ are order complete vector lattices.

(6) Let X be a compact Hausdorff space. $C(X)$ is a vector lattice and a Banach algebra ($\|fg\| \leq \|f\|\|g\|$).

algebraic ideal $I \subset C(X)$: $\forall f \in I, g \in C(X) \Rightarrow fg \in I$

Any lattice ideal is an algebraic ideal

Fact: The closed algebraic ideals in $C(X)$ are just those ideals I s.t.

there is a closed set F in X for which

$$I = \{ f : f(x) = 0 \quad \forall x \in F \}$$

Any such ideal is a lattice ideal. Hence the closed lattice ideals are the same as the closed algebraic ideals in $C(X)$

<u>PROPOSITION</u>: Suppose that E is a vector lattice and that M is a lattice ideal in E^b. For each $x \in E$, define $\hat{x} \in M^b$ by

$$\hat{x}(f) := f(x)$$

The mapping $J : E \longrightarrow M^b$ defined by $Jx = \hat{x}$ preserves the lattice operations (e.g. $Jx^+ = (Jx)^+$). J is 1-1 iff M separates points of E.

Proof. We want to show $\hat{x}^+(f) = f(x^+)$ $\forall f \geq 0$ in M. We know that

$$\hat{x}^+(f) = \sup \{ g(x) : 0 \leq g \leq f \} \quad \forall f \geq 0$$

If $0 \leq g \leq f$, then

$$f(x^+) = \widehat{x^+}(f) \geq g(x^+) \geq g(x)$$

$$\Rightarrow f(x^+) \geq \hat{x}^+(f)$$

For $0 \leq f \leq M$, define h_f on the cone of E by

$$h_f(y) := \sup\{f(z) : 0 \leq z \leq y,\ z \leq rx^+ \text{ for some } r \geq 0\}$$

h_f is positively homogeneous and additive on the cone, so h_f extends to a linear functional h_f on E. Hence $h_f \geq 0$, $h_f \leq f$, so $h_f \in M$ by ideal property. Note that $h_f(x^-) = 0$ and $h_f(x^+) = f(x^+)$. Hence

$$f(x^+) = h_f(x^+) = h_f(x) - h_f(x^-) = h_f(x) \leq \hat{x}^+(f)$$

$\blacksquare$

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COROLLARY: If E is a vector lattice and if f is a positive linear functional on E, then

$$f(x^+) = \sup\{g(x) : 0 \le g \le f\}$$

$$f(|x|) = \sup\{g(x) : |g| \le f\}$$

DEFINITION: (1) If E is a vector lattice and p is a seminorm [norm] on E, then p is a lattice seminorm [lattice norm] if $|x| \le |y| \Rightarrow p(x) \le p(y)$ $\forall x, y \in E$.

(2) If E is a normed space and a vector lattice, and if the norm on E is a lattice norm, then E is called a normed lattice space; in addition, if E is a Banach space, E is called a Banach lattice.

(3) If $E(\tau)$ is a locally convex space and a vector lattice and if there is a family $\{p_\alpha : \alpha \in A\}$ of seminorms generating τ that are lattice seminorms, then $E(\tau)$ is a locally convex lattice.

Examples

(1) $C(X)$ X compact, T_2
$L^p(X, \Sigma, \mu)$ $1 \le p \le \infty$ — Banach lattices
c, c_0, ℓ^p $1 \le p \le \infty$

(2) $L^p(X, \Sigma, \mu)$ $1 \le p < \infty$. If $g \in L^q(X, \Sigma, \mu)$, define

$$p_g(f) := \int_X |fg|\, d\mu \qquad \forall f \in L^p(\mu)$$

Then $\{p_g : g \in L^q(\mu)\}$ generates a locally convex lattice topology on $L^p(\mu)$

Remarks

(1) If p is a lattice norm or seminorm, then p has the following property

$$(*) \qquad 0 \le x \le y \Rightarrow p(x) \le p(y)$$

(2) If $E(\tau)$ is a locally convex space and an ordered vector space such that there is a generating family $\{p_\alpha : \alpha \in A\}$ of seminorms for τ satisfying $(*)$, then we say that the cone in $E(\tau)$ is normal.

(3) The cone in any Banach lattice, norm lattice, or locally convex lattice is normal

Example: Consider ℓ^p, $1 \le p < \infty$, and its weak topology. Every element of ℓ^q is the difference of positive elements, so the weak topology is generated by

$$\mathcal{P} = \{p_u : 0 \le u \in \ell^q\}$$

where $p_u(x) = \left|\sum_{n=1}^{\infty} x_n u_n\right|$. Clearly $0 \le x \le y \Rightarrow p_u(x) \le p_u(y)$ for $u \ge 0$

Therefore the cone in ℓ^p (weak) is normal

PROPOSITION: If $E(\tau)$ is an ordered locally convex space with a normal cone, then every continuous linear functional on E is the difference of positive continuous linear functionals.

Proof. Suppose f is a continuous linear functional on $E(\tau)$. Since the cone is normal, there is a neighborhood V of 0 such that

$$0 \le y \le x \in V \Rightarrow y \in V \quad \text{(holds for any basic nbhd)}$$

and $|f(z)| \le 1 \ \forall z \in V$. But then for each $x \ge 0$, the set $\{f(y) : 0 \le y \le x\}$ is bounded above. Define

$$p(x) = \sup\{f(y) : 0 \le y \le x\}$$

for each $x \ge 0$. Then p is positively homogeneous and superadditive on the cone. Also p is continuous at 0 on the cone ($0 \le p(x) \le 1$ for all $0 \le x \le V$)

Define $E_1 := E \times \mathbb{R}$ (product topology). Let

$$C = \{(t,x) : x \ge 0,\ 0 \le t \le p(x)\}$$

Then C is a cone in E_1. Note $(1,0) \notin \overline{C}$ [for if $(t_\alpha, x_\alpha) \in C$ and $(t_\alpha, x_\alpha) \to (1,0)$, then $t_\alpha \to 1$ and $x_\alpha \to 0$. But then $p(x_\alpha) \to 0$ while $0 \le t_\alpha \le p(x_\alpha)$ ↯] Choose a continuous linear functional F

on E, s.t.

$$F(1,0) < \inf \{F(t,x) : (t,x) \in \overline{C}\}$$

Therefore $F(t,x) \geq 0 \ \forall (t,x) \in C$ (since C is a cone), and so $F(1,0) < 0$.

Notice

$$F(t,x) = t\,F(1,0) + \underbrace{F(0,x)}_{g(x)}$$

g is continuous on $E(\tau)$. $\forall\, x \geq 0$, then $p(x) \geq 0$ so $(0,x) \in C$

Hence

$$g(x) = F(0,x) \geq 0$$

Therefore $g \geq 0$. Also,

$$(p(x), x) \in C \quad \forall x \geq 0 \implies F(p(x), x) \geq 0$$

Since $F(p(x),x) = p(x)\, F(1,0) + g(x)$, we have

$$p(x) \leq \frac{-g(x)}{F(1,0)} =: f_1(x)$$

Then $f_1 \geq 0$, $f_1 \geq f$, and f_1 is continuous. Now write $f = f_1 - (f_1 - f)$

since $f(x) \leq p(x) \leq f_1(x)$

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<u>COROLLARY</u>: If the cone in an ordered locally convex space $E(\tau)$ is normal, then it is also normal for the weak topology

Proof. The cone in $E(\tau)$ is normal $\Rightarrow \mathcal{P} = \{p_u : 0 \leq u \in E^*\}$ generates the weak topology, where

$$p_u(x) = |u(x)|$$

Since $u \geq 0$, it follows that $0 \leq x \leq y \Rightarrow p_u(x) \leq p_u(y)$.

<u>Remarks</u>

(1) If $E(\tau)$ is a locally convex lattice, then the maps $x \mapsto x^+$, $x \to |x|$, $x \to x^-$ are all continuous on E. Also, the maps $(x,y) \mapsto x \vee y$ and $(x,y) \to x \wedge y$ are continuous from $E \times E$ to E.

(2) The cone K in any locally convex lattice is a closed set since $K = \{x \in E : x^- = 0\}$

<u>PROPOSITION</u>: Every positive linear map from a Banach lattice E into a norm lattice F is continuous.

Proof. Let $T: E \to F$ be positive but not continuous. Then T is not bounded on the unit ball. Therefore T is not bounded on the positive part of the unit ball, so there exist $x_n \geq 0$, $\|x_n\| \leq 1$ such that $\|Tx_n\| \geq n^3$. Then

$$\left(\sum_{n=1}^{k} \frac{x_n}{n^2} : k \in \mathbb{N} \right)$$

is a Cauchy sequence in E since $\|x_n\| \leq 1$, so $\exists\, z = \sum_{n=1}^{\infty} \frac{x_n}{n^2}$.
Since the cone in E is closed,

$$\sum_{n=1}^{k} \frac{x_n}{n^2} - \frac{x_p}{p^2} \geq 0 \quad \forall k \geq p \Rightarrow z - \frac{x_p}{p^2} \geq 0$$

Hence

$$z \geq \frac{x_n}{n^2} \geq 0 \;\; \forall n \Rightarrow \|Tz\| \geq \left\| \frac{Tx_n}{n^2} \right\| \geq n \;\; \forall n \;\; \lightning$$

$\blacksquare$

<u>COROLLARY</u>: If E is a Banach lattice, the norm dual E^* of E coincides with the order dual E^b of E; in particular, the norm dual of E is an order complete vector lattice.

Fact: If E is a complete metrizable ordered locally convex space with a closed generating cone K ($E = K - K$) and if F is an ordered locally convex space with a normal cone, then every positive linear map of E into F is continuous.

Extension of Positive Linear Functionals

Let E be a vector space

$\tau_\omega :=$ finest locally convex topology on E = topology generated by all seminorms on E

$E(\tau_\omega)^* =$ algebraic dual E' (since $\sigma(E,E') \leq \tau_\omega$)

PROPOSITION: Suppose that f is a linear functional defined on a linear subspace M of an ordered vector space E. Then f has a positive linear extension $\hat{f}$ to E if and only if there is a convex radial, circled set U such that f is bounded above on $M \cap (U-K)$ where K is the cone in E

Proof. Note

$$M \cap (U-K) = \{x \in M : x \leq u \text{ for some } u \in U\}$$

If there is such an extension $\hat{f}$ of f, define

$$U = \{x \in E : |\hat{f}(x)| < 1\}$$

If $x \in M \cap (U-K)$, then $x \leq u$ for some $u \in U$, so

$$f(x) = \hat{f}(x) \leq \hat{f}(u) < 1$$

Therefore f is bounded above by 1 on $M \cap (U-K)$.

Conversely, suppose there is a set U with the required property. Choose $\gamma > 0$ s.t.

$$x \in M \cap (U-K) \Rightarrow f(x) < \gamma$$

Define $H_f := \{x \in M : f(x) = \gamma\}$. This is a hyperplane in M and a linear manifold in E. Note

$$H_f \cap (U-K) = \emptyset$$

U is a 0-nbhd for τ_ω, so $U-K$ is a convex 0-nbhd. By Hahn-Banach there is a closed hyperplane $H \supset H_f$ in E s.t. H misses the interior of $U-K$. $0 \notin H$, so we can choose $\hat{f} \in E^*$ such that

$$H = \{x \in E : \hat{f}(x) = \gamma\}$$

Then $H \cap M \supset H_f$. But $H \cap M$ is an manifold and H_f is a manifold, so $H \cap M = H_f \Rightarrow \hat{f}$ is an extension of f.

If $x \leq 0$, then since $0 \in \text{int}(U-K)$, it follows that $f(x) = \gamma$. Since any positive multiply of x also satisfies $z \leq 0$, it follows that $\hat{f}(x) \leq 0$, so $\hat{f}$ is positive

$\boxtimes$

<u>COROLLARY</u> If f is a positive continuous linear functional defined on a sublattice M of a normed lattice E, then f has a

norm preserving positive linear extension $\hat{\xi}$ to all of E.

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PROPOSITION: If f is a linear functional defined on a sublattice M of a normed lattice E which is positive and continuous, then f has a positive linear extension $\hat{f}$ to E of the same norm

Proof. WLOG $\|f\|_M = 1$. Let U = unit ball of E (convex, radial circled set) If

$$x \in M \cap (U - K) \iff x \in M \text{ and } 0 \le x \le u \in U$$

$$\Rightarrow 0 \le x^+ \le u^+ \in U \text{ and } x \in M$$

$$\Rightarrow x^+ \in U \cap M$$

$$\Rightarrow \underbrace{f(x) \le f(x^+)}_{f \ge 0} \underbrace{\le 1}_{\|f\| = 1}$$

Preceding proposition $\Rightarrow \exists$ extension $\hat{f} \ge 0$ of f to E. A look at the proof would show that $H = \{x \in E : \hat{f}(x) = 1\}$ does not intersect the interior of U. Therefore $|\hat{f}(x)| < 1$ for $\|x\| \le 1 \Rightarrow \|\hat{f}\| \le 1$. But $\hat{f}$ is an extension of f, so $\|\hat{f}\| = \|f\|$.

$\boxtimes$

Suppose E is a vector lattice and that A is a subset of E. The lattice ideal $I(A)$ generated by A is the smallest lattice ideal in E containing A. Then

$$I(A) = \{ y \in E : |y| \le n \sum_{i=1}^{n} |x_i| \quad x_i \in A \}.$$

If A is a sublattice of E, then

$$I(A) = \{ y \in E : |y| \le |x| \text{ for some } x \in A \}.$$

COROLLARY 2: Suppose that M is a sublattice of a vector lattice E such that $I(M) = E$. Then every positive linear functional on M has a positive linear extension $\hat{f}$ to E.

Proof. For each $x \in E$, there is a $y \in M$ such that $|x| \le y$ (since $I(M) = E$). Define

$$p(x) = \inf \{ f(y) : y \ge |x|, y \in M \}$$

Then p is a lattice seminorm on E. Also, for $y \in M$

$$f(y) \le f(|y|) = p(y)$$

Let $U = \{ x \in E : p(x) < 1 \}$. Then U is convex, radial, and circled

$$y \in M \cap (U - K) \Rightarrow y = u - k \quad p(u) < 1,\ k \ge 0$$

Then

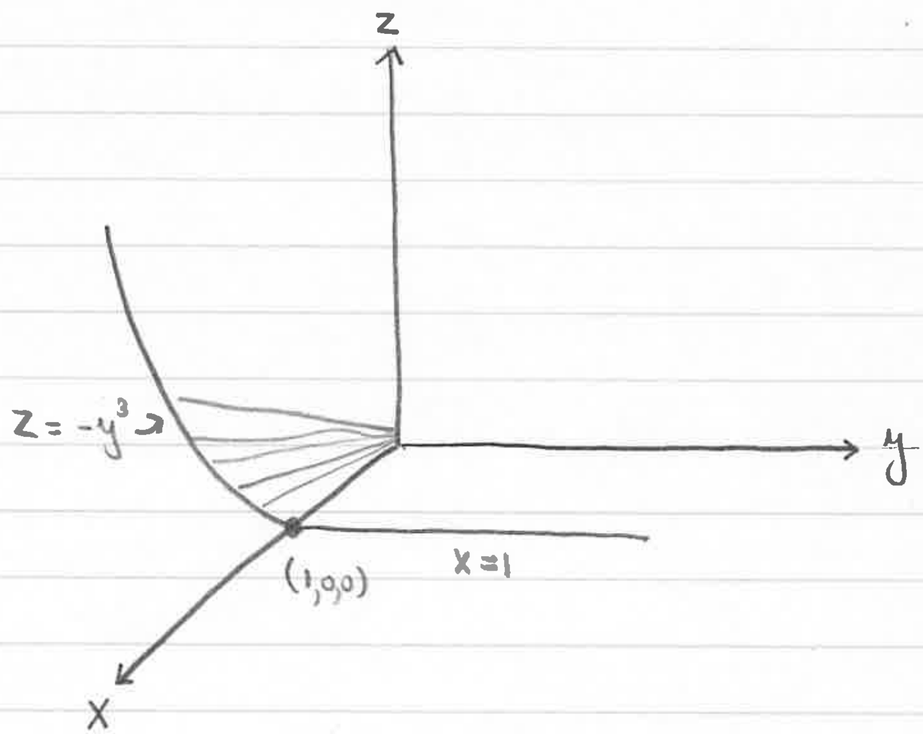
$$0 \leq y^+ \leq u + y^-$$

$$\Rightarrow f(y) = f(y^+) - f(y^-) = p(y^+) - p(y^-)$$

$$\leq p(u + y^-) - p(y^-) \leq p(u) < 1$$

Hence f is bounded above on $M \cap (U - K)$, so f has a positive linear extension to E.

Ex.



generate a cone with this curve and $(0,0,0)$

$M =$ xy-plane Define $f(x,y,0) = y$ on M

$f \geq 0$

If $\hat{f}$ is a positive linear extension of f, then the hyperplane

$H = \{(x,y,z) : \hat{f}(x,y,z) = 0\}$. Has to miss the cone and contain the positive x-axis. Hence $H = M$ ↳ since $\hat{f}$ extends f.

<u>PROPOSITION</u>: If $E(\tau)$ is a locally convex lattice, then the dual E^* of E is a lattice ideal in E^b. E^* is a locally convex lattice for the strong topology $\beta(E^*, E)$ and the canonical map $J : E \longrightarrow E^{**}$ preserves lattice operations.

Proof. The cone in $E(\tau)$ is normal, so $E^* \subset E^b$. Suppose $g \in E^b$, $f \in E^*$, $|g| \leq |f|$. Suppose $x_\alpha \longrightarrow 0$ in τ (so $|x_\alpha| \longrightarrow 0, \tau$) Moreover, if $|y_\alpha| \leq |x_\alpha|$ $\forall \alpha$, then $y_\alpha \longrightarrow 0$. Since

$$|f|x = \sup\{fy : |y| \leq x\}$$

for $x \geq 0$ it follows that for each $\varepsilon > 0$ and each α, there is a y_α with $|y_\alpha| \leq |x_\alpha|$ s.t.

$$0 \leq |f|(|x_\alpha|) \leq f(y_\alpha) + \varepsilon$$

But $f(y_\alpha) \longrightarrow 0$, so $|f|(|x_\alpha|) \leq 2\varepsilon$ $\forall \alpha \geq \alpha_0$ for some α_0. But then

$$|g(x_\alpha)| \leq |g|(|x_\alpha|) \leq |f|(|x_\alpha|) \leq 2\varepsilon$$

$$\Rightarrow \quad g(x_\alpha) \longrightarrow 0$$

Hence $g \in E^*$. Therefore E^* is a lattice ideal in E^b

[$\forall$ bdd, convex, radial, circled B, let $p_B(f) = \sup\{|f(x)| : x \in B\}$
Then $\beta(E^*, E)$ is the topology generated by these seminorms. This is the norm topology for normed spaces]

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(Proof continued)

[If B is a subset of a vector lattice, then the solid hull $|B|$ of B is

$$|B| = \{ y \in E : |y| \leq |x| \text{ for some } x \in B \}$$

If $E(\tau)$ is a locally convex lattice, the solid hull $|B|$ of a bounded set is still bounded. Also $B \subset |B|$ for any bounded set B. Therefore the topology $\beta(E^*, E)$ = topology on E^* of uniform convergence on all bounded sets in $E(\tau)$ is generated by the seminorms

$$\mathcal{P} = \{ p_{|B|} : B \text{ bounded in } E(\tau) \}$$

where $p_{|B|}(f) = \sup_{x \in |B|} |f(x)|$]

If $|f| \leq |g|$ for $f, g \in E^*$, then

$$p_{|B|}(f) = \sup_{x \in |B|} |f(x)| \leq \sup_{x \in |B|} |f|(x) \leq \sup_{x \in |B|} |g|(|x|)$$

$$= \sup_{x \in |B|} \sup_{|y| \leq |x|} g(y) = \sup_{y \in |B|} g(y) = p_{|B|}(g)$$

Therefore $p_{|B|}$ is a lattice seminorm, so $E^*(\beta(E^*, E))$ is a locally convex lattice ▨

COROLLARY (To proof): The norm dual of a normed lattice is a Banach lattice.

In particular $E^{**} = E^{*}(\beta(E^{*},E))^{*}$ is an ideal in $(E^{*})^{b}$. Also E^{*} is a lattice ideal in E^{b}. Therefore the evaluation map $J: E \to E^{**}$ preserves the lattice operations

The Convergence of Monotone or Directed Families

If D is a direction and $x \in D$, we define

$$S_x = \{ y \in D : y \geq x \}$$

Then $\{S_x : x \in D\}$ is a filter base for the filter $\mathcal{F}(D)$ of sections of D.

PROPOSITION: If $E(\tau)$ is a locally convex space ordered by a closed cone K and if D is a directed subset of E such that x_0 is a cluster point of the filter $\mathcal{F}(D)$ of sections of D, then $x_0 = \sup D$

Proof. Let $x \in D$. Then $S_k \subset x + K$. Since x_0 is a cluster point

$$x_0 \in \overline{S_x} \subset \overline{x+K} = x + \overline{K} = x + K$$

Therefore $x_0 \geq x \ \forall x \in D$. Suppose $z \geq x \ \forall x \in D$. Then $D \subset z - K$

Hence

$$x_0 \in \overline{D} \subset \overline{z-K} = z-\overline{K} = z-K$$

and so $x_0 \leq z$.

▨

PROPOSITION: Suppose that $E(\tau)$ is a locally convex space ordered by a normal cone K. If the filter $\mathcal{F}(D)$ of sections of a directed set converges weakly, it also converges for τ to the same limit

Proof: WLOG, assume D is directed $(\geq)$ and that $\mathcal{F}(D) \to 0$ weakly.
directed down
We can also assume that K is closed

[[The closure $\overline{K}$ of a normal cone K is normal — Notice

$$0 \leq x \leq y \implies p(x) \leq p(y)$$

$$\Updownarrow$$

$$0 \leq s,t \implies p(s+t) \geq p(s)$$

So if $0 \leq s,t$ in $\overline{K}$, then $s_\alpha \to s$, $t_\alpha \to t$ for $s_\alpha, t_\alpha \in K$ and

$$p(s) \leftarrow p(s_\alpha) \leq p(s_\alpha + t_\alpha) \to p(s+t)$$]]

Suppose $\mathcal{F}(D) \not\to 0$ for τ. Then there is an open convex 0-nbhd W such that

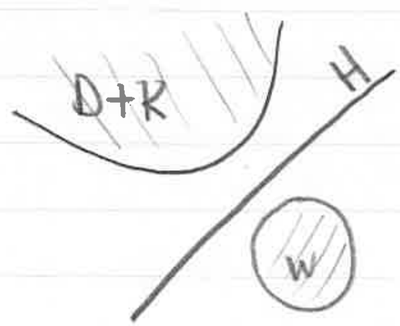
$$0 \le y \le x \in W \Rightarrow y \in W$$

$$S_x \not\subset W \text{ for any } x \in D$$

Hence $D \cap W = \emptyset$. For if $x \in D \cap W$, then $S_x \subset W$

$$\Rightarrow \underbrace{(D+K)}_{\text{open convex}} \cap W = \emptyset$$

Now $D+K = \bigcup_{d \in D} \underbrace{(d+K)}_{\text{convex}}$ is convex since D is directed. Therefore there is a hyperplane H separating $D+K$ and W. One side of the hyperplane is a weak neighborhood of 0 that does not contain a section of $\mathcal{F}(D)$



since $S_x \subset D+K$ $\forall x$. This is a contradiction ▨

Applications : Let X be a compact Hausdorff space. If $(f_n) \subset C(X)$ and $f_n(x) \uparrow f_0(x)$ for some $f_0 \in C(X)$, then $f_n \to f_0$ uniformly

$\Downarrow$ $f_n \to f_0$ weakly and (f_n) directed upward

COROLLARY: If $E(\tau)$ is a locally convex space ordered by a normal closed cone K and if $E(\tau)$ is weakly sequentially complete, then each monotone increasing sequence (x_n) in E s.t.

$$\sup_n f(x_n) < \infty$$

for each positive continuous linear functional on E, then $x_0 = \sup x_n$ exist and $x_n \longrightarrow x_0$ for τ.

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Bands in vector lattices

Suppose that E is a vector lattice and that M is a lattice ideal in E. Then M is a band if $\sup A \in M$ whenever $A \subset M$ such that $\sup A$ exists in E.

Example: ① c_0 is a lattice ideal in ℓ^∞, but it is not a band

$$x^{(n)} := (1,0,1,0,1,\ldots,\underset{\substack{\uparrow \\ 2n^{th} \text{ place}}}{1},0,0,0,\ldots)$$

Then $\sup_n x^{(n)} = (1,0,1,0,1,\ldots) = x^{(0)} \notin c_0$

② Suppose $B(X)$ is the Banach lattice of all bounded functions on a set X (sup norm and pointwise order) For any subset Y of X let

$$M_Y := \{ f \in B(X) : f(x) = 0 \ \forall x \in Y \}$$

Then M_Y is a band in $B(X)$ for any $Y \subset X$.

E itself is a band and the intersection of bands in a band. For any subset N of E, define the band generated by N as the intersection of all bands containing N; denote this by $B(N)$

For any subset N of E, define
$$N^{\perp} = \{y \in E : y \perp x \ \ \forall x \in N\}$$

PROPOSITION: $N^{\perp}$ is a band in E

Proof. (1) $N^{\perp}$ is a linear subspace. Let $y_1, y_2 \in N^{\perp}$; $\alpha, \beta \in \mathbb{R}$
Then

$$|\alpha y_1 + \beta y_2| \wedge |x| \leq (|\alpha y_1| + |\beta y_2|) \wedge |x|$$
$$\leq |\alpha y_1| \wedge |x| + |\beta y_2| \wedge |x| = 0$$

Π $|y| \wedge |x| = 0 \Rightarrow |\gamma y| \wedge |x| = 0 \ \ \forall \gamma$. For if $|\gamma| \leq 1$, then
$$0 \leq |\gamma y| \wedge |x| \leq |y| \wedge |x| = 0$$

If $|\gamma| > 1$, and $s = |\gamma y| \wedge |x|$, then
$$|y| \geq \frac{1}{|\gamma|} s$$
$$|x| \geq s \geq \frac{1}{|\gamma|} s$$

and so $s = 0$ $\square$

(2) $N^{\perp}$ is an ideal. If $y \perp x$ and $|z| \leq |y|$, then

$$0 \le |z| \wedge |x| \le |y| \wedge |x| = 0$$

and so $z \in N^\perp$

(3) <u>$N^\perp$ is a band</u> WLOG let $A \subset M$ consist of positive elements. If $x \in N$

$$|(\sup A)| \wedge |x| = \sup A \wedge |x| = \sup\{a \wedge |x| : a \in A\} = 0$$

Hence $\sup A \in N^\perp$.

$\boxtimes$

Note that $(N^\perp)^\perp = N^{\perp\perp} \supset N$ and $N^{\perp\perp}$ is a band. Therefore $N^{\perp\perp} \supset B(N)$

<u>DEFINITION</u>: A vector lattice E is Archimedean if for any $x \ge 0$, $y \ge 0$ in E, $nx \le y$ for all $n \in \mathbb{N}$ implies $x = 0$

1) Any σ-order complete (or order complete) vector lattice is Archimedean.

Proof: $nx \le y \ \forall n \Rightarrow z := \sup_n nx \le y$. If $x \ne 0$, then

$$z \ge z - x \ge (n+1)x \quad \forall n$$

$$\Rightarrow z - x \ge nx \quad \forall n \quad \lightning$$

since $z = \sup nx$.

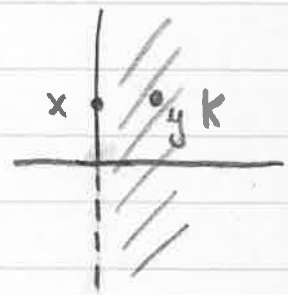
2) If there is a topology on E such that the cone is closed and scaler multiplication is continuous at 0, then E is Archimedean

$$0 \leq nx \leq y$$

$$\Rightarrow 0 \leq x \leq \tfrac{1}{n} y \qquad \text{let } n \to \infty \text{ and use cone-closed}$$

$$\Rightarrow 0 \leq x \leq 0 \implies x = 0$$

3) Consider $\mathbb{R}^2$ with lexiographic ordering



Note $nx \leq y \quad \forall n$

PROPOSITION: If N is any subset of an archimedean vector lattice, then $B(N) = N^{\perp\perp}$.

Proof. Suppose $u \in N^{\perp\perp} \setminus B(N)$. WLOG $u \geq 0$. If $I(N)$ = lattice ideal generated by N, then u is an upper bound of $\{v \in I(N) : 0 \leq v \leq u\}$, but it is not the supremum of this set since $I(N) \subset B(N)$. Thus there is another upper bound w of this set such that $w < u$. Since $0 \leq w < u$ and $u \in N^{\perp\perp}$, it follows that $w \in N^{\perp\perp}$. Hence $u - w \in N^{\perp\perp}$. Suppose

$$(u-w)\wedge z = 0 \quad \forall z \in I(N),\ z \geq 0$$

Then $u-w \in I(N)^{\perp} = N^{\perp}$. Hence $u-w \in N^{\perp\perp} \cap N^{\perp} = \{0\}$, i.e. $u = w$ ↯. Therefore there must be a $z_1 \in I(N)$ s.t. $z_1 \geq 0$ and $0 < z_1 \leq u-w$ (Take inf)

For each $v \in I(N)$, $0 \leq v \leq u$, we have

$$0 \leq z_1 + v \leq z_1 + w \leq u - w + w = u$$

$$\Rightarrow z_1 + v \leq u \quad \forall v \in I(N) \text{ with } 0 \leq v \leq u$$

By using the same argument with $z_1 + v$ in place of v, we conclude

$$0 \leq 2z_1 + v \leq u$$

In general, $0 \leq nz_1 + v \leq u$. Hence $nz_1 \leq u \ \forall n \Rightarrow z_1 = 0$ ↯

▨

Order direct sum: $E = N \oplus M$ where $u \geq 0 \Leftrightarrow v \geq 0, w \geq 0$ (for $u = v + w$)

<u>PROPOSITION</u> Suppose M is a linear subspace of a vector lattice E and that $E = M + M^{\perp}$. Then $M = M^{\perp\perp}$ and E is the order direct sum of M and $M^{\perp}$. If $z \in E$, and if $z = x + y$ where $x \in M$, $y \in M^{\perp}$, then $|z| = |x| + |y|$, $z^{+} = x^{+} + y^{+}$, and $z^{-} = x^{-} + y^{-}$

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Proof. (a) $M \subset M^{\perp\perp}$ in general. If $z \in M^{\perp\perp}$ and $z = x+y$ where $x \in M$, $y \in M^{\perp}$ then for each $w \in M^{\perp}$ we have

$$0 = |w| \wedge |z| = |w| \wedge |x+y| = |x-y| \wedge |w|$$

Hence $x-y \in M^{\perp}$. But $x+y \in M^{\perp\perp} \Rightarrow y = \frac{1}{2}((x+y)-(x-y)) \in M^{\perp\perp}$.
Hence $y=0$, so $z = x \in M$, i.e. $M^{\perp\perp} \subset M$. [Also: $x \in M \subset M^{\perp\perp}, z \in M^{\perp\perp} \Rightarrow y = z - x \in M^{\perp\perp}$]

(b) $M \cap M^{\perp} = \{0\}$, so representations in $M+M^{\perp}$ are unique.
If $z \in E$ and $z = x+y$ where $x \in M, y \in M^{\perp}$, then $|z| = |x+y| = |x|+|y|$
Therefore, if $z \geq 0$, then $z = |z|$, so $x = |z| \geq 0$ and $y = |y| \geq 0$.

(c) We know from the proof of (b) that

$$z = x+y \Rightarrow |z| = |x|+|y|$$

for $x \in M, y \in M^{\perp}$. For the others

$$z^+ = \frac{1}{2}(|z|+z) = \frac{1}{2}(|x|+|y|+x+y) = x^+ + y^+$$

▨

PROPOSITION: If M is a lattice ideal $\subset$ vector lattice E then the following are equivalent
(1) $E = M + M^{\perp}$
(2) $\forall u \geq 0$ in E, $\sup([0,u] \cap M)$ exists and belongs to M

(3) There is a projection P of E onto M such that $0 \le P \le I$.

Proof. (1) $\Rightarrow$ (2) Last proposition implies E is the order direct sum of M and $M^{\perp\perp}$. If $u \ge 0$ in E, then $u = u_1 + u_2$ where $u_1 \ge 0$ in M and $u_2 \ge 0$ in $M^{\perp}$. If $v \in [0,u] \cap M$

$$v = v \wedge u = v \wedge (u_1 + u_2) \le v \wedge u_1 + v \wedge u_2$$

$$= v \wedge u_1 \le v \wedge u$$

Therefore $v = v \wedge u = v \wedge u_1$, so $v \le u_1$. On the other hand, $u_1 \in [0,u] \cap M$. Therefore $u_1 = \sup([0,u] \cap M)$

(2) $\Rightarrow$ (3). For each $u \ge 0$ in E define $Pu = \sup([0,u] \cap M)$. P is positively homogeneous and additive on the cone of E, so P has a linear extension to E. If $0 \le u \in E$, then $0 \le Pu \le u$. Therefore $0 \le P \le I$. Note $Pu \in M$ $\forall u \ge 0$ in E, so $P(E) \subset M$. If $0 \le u \in M$, then $Pu = u$. Hence $P(E) = M$.

(3) $\Rightarrow$ (1) Let $N = (I-P)(E)$. N is a linear subspace of E and E is the order direct sum of M and N, since $0 \le P \le I$. If $0 \le u \le v \in N$ and $u = u_1 + u_2$ where $u_1, u_2 \in N$, then

$$0 \le u_1 = Pu \le Pv = 0 \Rightarrow u = u_2 \in N$$

If $x \in E$, then $x^+ \ge x$, $x^+ \ge 0 \Rightarrow (I-P)x^+ \ge (I-P)x, 0$ and so $(I-P)x^+ \ge [(I-P)x]^+$. Hence, if $x \in N$, then $x^+ \in N$

Therefore N is a lattice ideal in E.

We also know that M is a lattice ideal. Claim: $N \subset M^\perp$.

For if $x \in N$ and $y \in M$, then

$$|x| \wedge |y| \leq |x|, \quad |x| \wedge |y| \leq |y|$$

$$\Rightarrow |x| \wedge |y| \in M \cap N = \{0\}$$

$$\Rightarrow N \subset M^\perp$$

If $0 \leq z \in M^\perp$, then $z = x + y$ where $0 \leq x \in M$, $0 \leq y \in N$. Then $x \in M^\perp$ since $0 \leq x \leq z$. Hence $x = 0$, so $z = y \in N$, i.e. $M^\perp \subset N$. Hence $N = M^\perp$.

$\blacksquare$

<u>COROLLARY</u>: If E is an order complete vector lattice and if M is a band in E, then E is the order direct sum of M and $M^\perp$.

Proof. In this case (2) holds, so $E = M + M^\perp$. Use the penultimate proposition.

Decomposition of bounded additive set functions (Yosida-Hewitt)

(X, Σ) measurable space (with Σ algebra). Let $ba(X, \Sigma)$ denote the space of all bounded additive set functions on (X, Σ).

<u>**THEOREM**</u>: If $\mu \in ba(X,\Sigma)$ and $\mu \geq 0$, then

$$\mu = \mu_c + \mu_{pa}$$

where $0 \leq \mu_c$ is countably additive and $0 \leq \mu_{pa}$ is purely finitely additive.

$$\left(\begin{array}{c} 0 \leq \nu \leq \mu_{pa}, \ \nu \text{ countable add} \\ \Rightarrow \nu = 0 \end{array} \right)$$

Setting: $ba(X,\Sigma)$ is a vector lattice by the Jordan Decomposition theorem. If $\{\mu_\alpha : \alpha \in A\} \subset ba(X,\Sigma)$, WLOG $\mathcal{D}$ is directed ($\leq$) and $\mu_\alpha \geq 0$. If $\mathcal{D}$ is majorized, then define

$$\mu(E) = \sup_\alpha \mu_\alpha(E)$$

for all $E \in \Sigma$. μ is bounded and additive, i.e. $\mu \in ba(X,\Sigma)$. Hence $ba(X,\Sigma)$ is order complete.

Let $ca(X,\Sigma)$ be the countably additive measures.

10/1 BANACH LATTICES

DEFINITION: A band M in a vector lattice E is a projection band if there is a projection P on E with range M such that $0 \leq P \leq I$

The last proposition implies that M is a projection band

$\Leftrightarrow$ $\forall u \geq 0$ in E, $\sup([0,u] \cap M)$ exists and belongs to M

$\Leftrightarrow$ $E = M + M^{\perp}$

PROPOSITION: If E is an order complete vector lattice, then every band in E is a projection band.

Remark: If M is a projection band then $M = M^{\perp\perp}$ (since if $E = M + M^{\perp}$, then $M = M^{\perp\perp}$ by earlier proposition)

Let $B(E)$ = set of all projection bands in E.
$P(E)$ = set of all projections such that $0 \leq P \leq I$

Boolean Algebras

A lattice B is a Boolean algebra if it satisfies

(a) B is distributive ($a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$)

(b) $\exists$ zero element 0 s.t. $0 \vee a = a$, $0 \wedge a = 0$ $\forall a \in B$

(c) $\exists$ unit element $1 \in B$ s.t. $1 \vee a = 1$, $1 \wedge a = a$ $\forall a \in B$
(d) $\forall a \in B$ $\exists a' \in B$ s.t. $a \wedge a' = 0$ and $a \vee a' = 1$

If B_1 and B_2 are Boolean algebras and $\varphi : B_1 \longrightarrow B_2$ is a 1-1, onto map such that

$$\varphi(a) \vee_2 \varphi(b) = \varphi(a \vee_1 b)$$
$$\varphi(a) \wedge_2 \varphi(b) = \varphi(a \wedge_1 b)$$
$$\varphi(a)' = \varphi(a')$$

$\forall a, b \in B_1$, then φ is a Boolean isomorphism and B_1, B_2 are Boolean isomorphic

<u>Proposition</u>: If E is a vector lattice, then the collection $B(E)$ of projection bands is a Boolean algebra with

$$M \vee N = M + N$$
$$M \wedge N = M \cap N$$
$$M' = M^{\perp}$$

The collection $P(E)$ is a Boolean algebra with

$$P \vee Q = P + Q - PQ$$
$$P \wedge Q = PQ$$
$$P' = I - P$$

Moreover, the map $\varphi(M) = P_M$ where P_M is the projection onto M s.t.

$0 \le P_M \le I$ is a Boolean isomorphism of $\mathcal{B}(E)$ onto $\mathcal{P}(E)$.

Proof. 1) If M and N are projection bands with associated band projections P_M, P_N, then $M \cap N$ is a projection band with associated band projection $P_M P_N$.

(If $0 \le u \in E$, then $0 \le P_N u \in N$ and $0 \le P_M P_N u \le P_N u \in N$. But certainly $P_M P_N u \in M$, so $P_M P_N u \in M \cap N$. On the other hand, if $x \in M \cap N$, then

$$X = P_M x = P_M P_N x$$

and so $P_M P_N(E) = M \cap N$.

If $0 \le u$ in E, then $u - P_M P_N u \in (M \cap N)^\perp$ for if $v \in M \cap N$ and $0 \le v \le u - P_M P_N u$, then

$$v = P_M v \le P_M u - P_M P_N u \le u - P_N u$$

$$v = P_N v \le P_N(u - P_N u) = P_N u - P_N u = 0$$

$$\Rightarrow u - P_M P_N u \in (M \cap N)^\perp$$

$$\Rightarrow (I - P_M P_N)(E) \subset (M \cap N)^\perp$$

$$\Rightarrow E = (M \cap N) + (M \cap N)^\perp$$

$\Rightarrow$ $M \cap N$ is a band in E

and $M \cap N$ is a projection band. Now

$$x = P_M P_N x + (I - P_M P_N) x$$
$$\in M \cap N \qquad \in (M \cap N)^\perp$$

Hence $P_M P_N = P_{M \cap N}$)

(2) If M is a projection band with associated projection P_M. Then $M = M^{\perp\perp}$ and $E = M + M^\perp = M^\perp + M^{\perp\perp}$. Hence $M^\perp$ is a projection band and $P_{M^\perp} = (I - P_M)$

(3) If M, N are projection bands and if

$$F = (M^\perp \wedge N^\perp)^\perp$$

then $F = M \vee N$ and $P_{M \vee N} = P_M + P_N - P_M P_N$,

$$\left(M^\perp \wedge N^\perp \leq M^\perp, N^\perp \Rightarrow F = (M^\perp \wedge N^\perp)^\perp \begin{array}{l} \supset M^{\perp\perp} = M \\ \supset N^{\perp\perp} = N \end{array} \right.$$

$$\Rightarrow F \geq M, N$$

If $G \geq M, N$, then $G^\perp \leq M^\perp, N^\perp$, so $G = G^{\perp\perp} \supset (M^\perp \wedge N^\perp)^\perp = F$
Hence $M \vee N = (M^\perp \wedge N^\perp)^\perp$. Also

$$P_{M \vee N} = P_{(M^\perp \wedge N^\perp)^\perp} = I - P_{M^\perp \wedge N^\perp}$$

$$= I - P_{M^\perp} P_{N^\perp} = I - (I - P_M)(I - P_N)$$

$$= P_M + P_N - P_M P_N$$

PROPOSITION: If I is a lattice ideal in a vector lattice E, then $u \geq 0$ belongs to the band $B(I)$ generated by I iff

$$u = \sup \{ v \in I : 0 \leq v \leq u \}$$

$$\left(= \sup \left\{ u_\alpha : \begin{array}{l} \{u_\alpha\} \text{ directed } (\leq) \\ u_\alpha \geq 0,\ u_\alpha \in I \end{array} \right\} \right)$$

for some $\{v_\alpha\}$

10/3 BANACH LATTICES

Proof. If $v = \sup\{[0,u] \cap I\}$, then $u \in B(I)$ by definition of a band.

Let $B = \{u \in E : u = \sup\{u_\alpha\}$ where $\{u_\alpha\}$ is directed $(\leq)$ and $0 \leq u_\alpha \in I\}$. Then $I \subset B - B \subset B(I)$. Also

$$B = \{u \in E : u = \sup\{[0,u] \cap I\}\}$$

and B is a cone in E. B is the set of positive elements in $B-B$. For if $u \geq 0$ in $B-B$, then $u = u_1 - u_2$ where $u_1, u_2 \in B$
$\Rightarrow 0 \leq u \leq u_1 = \sup\{u_\alpha\} \Rightarrow u \in B$ $\quad (u = \sup[u_\alpha \wedge u])$
B is a lattice ideal in E. For if $|v| \leq |u|$, $u \in B-B$, then $u = u_1 - u_2$, for $u_1, u_2 \in B$. Then

$$0 \leq v^+, v^- \leq |v| \leq |u| \leq u_1 + u_2$$

$$\Rightarrow v^+, v^- \in B$$

$$\Rightarrow v \in B - B$$

Finally, $B-B$ is a band in E. For if $\{u_\alpha\}$ is a directed $(\leq)$ set of positive elements of $B-B$, i.e. $\{u_\alpha\} \subset B$, such that $\sup_\alpha u_\alpha = u$ exists in E. Then

$$u = \sup_\alpha u_\alpha = \sup_\alpha \sup\{[0, u_\alpha] \cap I\}$$

$$= \sup\{v \in I : 0 \le v \le u_\alpha \text{ for some } \alpha\} \in B$$

directed ($\le$) set of positive elements of I.

Hence $B - B = B(I)$. Hence if $u \ge 0$ and $u \in B(I)$, then u is a positive element of $B-B$, so $u \in B$. Hence

$$u = \sup\{[0,u] \cap I\}$$

$\boxtimes$

PROPOSITION: Suppose that A is a subset of a vector lattice E. Then the band $B(A)$ generated by A is a projection band if and only if the supremum

$$\sup\left\{u \wedge n\sum_{i=1}^{m}|x_i| : x_i \in A, n, m \in \mathbb{N}\right\}$$

exists for each $u \ge 0$ in E. If $B(A)$ is a projection band, the associated band projection P is defined for $u \ge 0$ by

$$Pu = \sup\left\{u \wedge n\sum_{i=1}^{m}|x_i| : x_i \in A, n, m \in \mathbb{N}\right\}$$

Proof. $B(A)$ is a projection band if and only if for each $u \ge 0$, $\sup\{[0,u] \cap B(A)\}$ exists in E. If $B(A)$ is a projection band, the associated band projection P is defined for $u \ge 0$ by

$$P u = \sup \{ [0,u] \cap B(A) \}$$

(Earlier proposition) We have also seen

$$I(A) = \{ y \in E : |y| \leq n \sum_{i=1}^{m} |x_i| , x_i \in A \}$$

Note

$$Pu = \sup \{ [0,u] \cap B(A) \} \geq \sup \{ [0,u] \cap I(A) \}$$

if this sup exists

By the last proposition,

$$Pu = \sup \{ [0, Pu] \cap I(A) \} \leq \sup \{ [0,u] \cap I(A) \}.$$

Hence $Pu = \sup \{ [0,u] \cap I(A) \}$, so so this exists

$$Pu = \sup \{ v \in E : 0 \leq v \leq u , v \leq n \sum_{i=1}^{m} |x_i| , x_i \in A \}$$

$$= \sup \{ u \wedge n \sum_{i=1}^{m} |x_i| : x_i \in A \}$$

($\Leftarrow$) On the other hand, if $\sup \{ [0,u] \cap I(A) \}$ exists for each $u \geq 0$, then so does $\sup \{ [0,u] \cap B(A) \}$ and these are equal. To see this

$$\sup \{ [0,u] \cap I(A) \} = \sup \{ x \in I(A) ; 0 \leq x \leq u \}$$

$$= \sup \{ \sup (w \in I(A) : 0 \leq w \leq v) : 0 \leq v \leq u \}$$

$$= \sup\{v \in B(A) : 0 \le v \le u\} = \sup\{[0,u] \cap B(A)\}$$

$\blacksquare$

Important special case: $A = \{u\}$. $B(A)$ is called a principal band since it is generated by one element. Suppose $u \ge 0$ and write $B(u)$ for $B(A)$. Then

COROLLARY: $B(u)$ is a projection band $\Leftrightarrow$ for each $x \ge 0$, the supremum

$$(*) \qquad \sup\{x \wedge nu : n \in \mathbb{N}\}$$

exists. If $B(u)$ is a projection band, $(*)$ defines Px for $x \ge 0$.

If $u \ge 0$ in a vector lattice E, then u is a weak order unit if $u \wedge |x| = 0 \Rightarrow x = 0$ for any $x \in E$

Examples:

$C(X)$ $\quad f(x) > 0 \ \forall x \in X \Rightarrow f$ w.o.u.

ℓ^p, c_0 $\quad$ if u has strictly positive coordinates

$L^p(X, \Sigma, \mu)$ $\quad f > 0$ a.e. $\Rightarrow f$ w.o.u.

(If (X, Σ, μ) is not σ-finite, $L_1(X, \Sigma, \mu)$ has no weak order units)

Remarks:

(1) If $u \geq 0$, then u is a weak order unit in $B(u)$

If $x \in B(u)$ and $u \wedge |x| = 0$, then $x \in \{u\}^\perp$. But $B(u) \subset \{u\}^{\perp\perp}$
so $x \in \{u\}^\perp \cap \{u\}^{\perp\perp} \Rightarrow x = 0$

(2) If v is a weak order unit in a band M in an Archimedean vector lattice E, then $B(v) = M$.

$v \in M \Rightarrow B(v) \subset M \Rightarrow B(v)^\perp \supset M^\perp$. Suppose $x \in B(v)^\perp$ but $x \notin M^\perp$. Then $\exists\, y \in M$ s.t. $|x| \wedge |y| = z > 0$. Now $z \in M$ so $z \wedge v > 0$ since v is a w.o.u. Then

$$0 = |x| \wedge v \geq z \wedge v > 0 \qquad \lightning$$

Hence $B(v)^\perp = M^\perp \Rightarrow M = M^{\perp\perp} = B(v)^{\perp\perp} = B(v)$

Archimedean

10/5 Banach Lattices

Remarks:

3) If E is an order complete vector lattice containing a weak order unit u, then every band in E is a principal band.

Proof. Suppose M is a band in E. Since E is order complete there is a band projection P_M ($0 \le P_M \le I$) with range M. Let $v = P_M u$. Claim: $B(v) = M$. Note that v is a weak order unit in M. For if $x \in M$ and $|x| \wedge v = 0$, then let $y = |x| \wedge u \geq 0$. Notice $0 \le y \le |x| \Rightarrow y \in M$, so $y = P_M y \le P_M u = v$. Hence $y \le |x| \wedge v = 0$. Hence $|x| \wedge u = 0 \Rightarrow x = 0$. By remark 2, $M = B(v)$. ▨

Order Convergence and Order Continuity

DEFINITION: If $\{x_\alpha : \alpha \in D\}$ is a net in a vector lattice E that is order bounded (i.e. contained in an order interval). Then

a) $x_\alpha \downarrow x_0$ means $(x_\alpha : \alpha \in D)$ is directed ($\le$) by D and $x_0 = \inf \{x_\alpha : \alpha \in D\}$.

b) $x_\alpha \xrightarrow{o} x_0$ if there is a $y_\alpha \downarrow 0$ such that

$$|x_\alpha - x_0| \le y_\alpha$$

for all α.

Examples

(1) Let E be a sequence space such as c, c_0, ℓ_p $p \geq 1$, ℓ_∞

$$(x_\alpha) \downarrow 0 \text{ in } E \text{ means } (x_\alpha)_n \downarrow 0 \text{ for each } n$$

and (x_α) is order bounded.

(2) (X, Σ, μ) σ-finite. $\dot{M}(X, \Sigma, \mu)$ = all equivalence classes modulo μ-null functions of measurable functions

$\dot{f}_\alpha \xrightarrow{o} \dot{f}_0$ means $f_\alpha \longrightarrow f_0$ a.e. and $\exists\, \dot{h}, \dot{g} \in \dot{M}$ with $\dot{h} \leq \dot{f}_\alpha \leq \dot{g}$ $\forall \alpha$

DEFINITION: If E, F are vector lattices and if $T \in L^b(E, F)$, then

(a) T is order continuous if $Tx_\alpha \xrightarrow{o} Tx_0$ whenever $x_\alpha \xrightarrow{o} x_0$

(b) T is order sequentially continuous when $Tx_n \xrightarrow{o} Tx_0$ whenever $x_k \xrightarrow{o} x_0$.

Example: E = all bounded measurable functions on $[0,1]$. Order E by $f \leq g$ if $f(x) \leq g(x)$ $\forall x \in [0,1]$. Define

$$\varphi(f) = \int_0^1 f(t)\,dt$$

φ is order sequentially continuous. Then $\{\chi_F : F^c \text{ finite}\} \downarrow 0$ but $\varphi(\chi_F) = 1$ for all F. Hence φ is not order continuous.

Example: Let E be any one of the spaces c_0, l_p, l_∞. Then $f \in E^* (= E^b)$ is order sequentially continuous iff there is a $u = (u_n)$ such that

$$\sum_{n=1}^{\infty} |x_n u_n| < +\infty$$

for all $x = (x_n) \in E$ and

$$f(x) = \sum_{n=1}^{\infty} x_n u_n$$

(Application: 1) order sequentially cont = order bounded = continuous on c_0, l_p
2) For l_∞, order seq. cont. = w^*-cont functionals = l_1)

Proof. Suppose $f \in E^*$ is sequentially order continuous. Let $u_n = f(e^{(n)})$. If $x \in E$, then

$$x(\leq n) := (x_1, \dots, x_n, 0, 0, \dots)$$

Then $x(\leq n) \xrightarrow{o} x$.

$$f(x(\leq n)) = \sum_{k=1}^{n} x_k u_k \longrightarrow f(x)$$

Hence $\sum_{k=1}^{\infty} x_k u_k$ converges for all $x = (x_n) \in E$, so the convergence is absolute since

$$\left((\operatorname{sign} x_k u_k) x_k \right) \in E \quad \text{when } (x_k) \in E$$

Hence $f(x) = \sum_{n=1}^{\infty} x_n u_n$

Now suppose that there is such a $u = (u_n)$. WLOG $f \geq 0$. Then $u_n \geq 0 \ \forall n$. It would be enough to show that if $y^{(n)} \downarrow 0$ then $f(y^{(n)}) \longrightarrow 0$ $\left(x^{(n)} \xrightarrow{o} x^{(0)} \Rightarrow |x^{(n)} - x^{(0)}| \leq y^{(n)} \downarrow 0 \right)$

$$f(y^{(n)}) = \sum_{k=1}^{m} y_k^{(n)} u_k + \sum_{k=m+1}^{\infty} y_k^{(n)} u_k$$

choose m suff. large so this $< \varepsilon/2$ for $n=1$
$\Rightarrow < \varepsilon/2$ for all n

With m fixed, choose n suff large so this $< \varepsilon/2$

$$\left[\!\left[\; |f(x^{(n)}) - f(x^{(0)})| \leq f(|x^{(n)} - x^{(0)}|) \leq f(y^{(n)}) \longrightarrow 0 \; \right]\!\right]$$

10/8 BANACH LATTICES

NEW DEFINITION: $y_\alpha \downarrow 0$ means $y_{\alpha_1} \le y_{\alpha_2}$ if $\alpha_1 \ge \alpha_2$ and $\inf y_\alpha = 0$

$L^o(E,F)$ = all order continuous linear maps of E into F.
$L^{so}(E,F)$ = all order sequentially continuous linear maps

PROPOSITION: If E, F are vector lattices and F is order complete, then $L^o(E,F)$ and $L^{so}(E,F)$ are bands in $L^b(E,F)$.

Proof. (For $L^{so}(E,F)$) Suppose $T \in L^{so}(E,F)$ and suppose $x_n \downarrow 0$. If $y_n = x_1 - x_n$, then $y_n \uparrow x_1$. The sequence $\{T^+ y_n\}$ is increasing with upper bound $T^+ x_1$. For any z such that $0 \le z \le x_1$, then $0 \le z \wedge y_n \le y_n$ and $z \wedge y_n \uparrow z$, so

$$\begin{array}{c} T(z \wedge y_n) \le T^+ y_n \le T^+ x_1 \\ 0 \big\downarrow \\ Tz \end{array}$$

i.e. $Tz \le \sup_n T^+ y_n \le T^+ x_1$. Therefore $T^+ x_1 = \sup_n T^+ y_n$
Hence

$$T^+ x_1 = \sup_n \{T^+ y_n\} = T^+ x_1 - \inf \{T^+ x_n\}$$

$$\Rightarrow \inf \{T^+ x_n\} = 0$$

Hence $T^+ \in L^{so}(E,F)$. Therefore $L^{so}(E,F)$ is a sublattice of $L^b(E,F)$. Clearly $L^{so}(E,F)$ is a lattice ideal in $L^b(E,F)$, for $\forall\ 0 \le S \le T \in L^{so}(E,F)$

$$x_n \xrightarrow{o} x_0 \Rightarrow |x_n - x_0| \le y_n \downarrow 0$$

$$\Rightarrow |Sx_n - Sx_0| \le S|x_n - x_0| \le T|x_n - x_0| \le Ty_n \downarrow 0$$

Now suppose $\{T_\alpha : \alpha \in A\}$ is a directed $(\le)$ set of positive maps in $L^{so}(E,F)$ with a supremum T_0 in $L^b(E,F)$. Let $x_n \downarrow 0$. Then $x_1 - x_n \uparrow x_1$. For each $\alpha \in A$, $n \in \mathbb{N}$

$$T_\alpha(x_1 - x_n) \le T_0(x_1 - x_n) \Rightarrow T_0 x_n - T_\alpha x_n \le T_0 x_1 - T_\alpha x_1$$

Fix α and let $n \to \infty$.

$$0 \le \inf T_0 x_n \le T_0 x_1 - T_\alpha x_1$$

But $T_0 x_1 = \sup_\alpha T_\alpha x_1$, so we have $\inf T_0 x_n = 0$. Hence $T_0 \in L^{so}(E,F)$.

DEFINITION: A Banach lattice E has an order continuous norm if $x_\alpha \xrightarrow{o} x_0$ implies $x_\alpha \longrightarrow x_0$ in norm.

Examples: (a) $x^{(n)} = (1,1,\dots,1,0,0,\dots) \in \ell^\infty$

$\uparrow$ nth place

$$x^{(n)} \uparrow x^0 = (1,1,1,1,\dots) \in \ell^\infty$$

Hence $x^{(n)} - x^0 \downarrow 0$, but $\|x^{(n)} - x^0\|_\infty = 1 \quad \forall n$

(b) $\ell^p \ (1 \le p < \infty)$ Suppose $x^{(\alpha)} \downarrow 0$ and $\varepsilon > 0$. Choose $\alpha_0 \in A$ m_ε s.t.

$$\sum_{k=m_\varepsilon+1}^{\infty} |x_k^{(\alpha_0)}|^p \le \left(\frac{\varepsilon}{2}\right)^p$$

$$\Rightarrow \sum_{k=m_\varepsilon+1}^{\infty} |x_k^{(\alpha)}|^p \le \left(\frac{\varepsilon}{2}\right)^p$$

for all $\alpha \ge \alpha_0$. Choose $\alpha_1 \ge \alpha_0$ s.t.

$$\sum_{k=1}^{m_\varepsilon} |x_k^{(\alpha)}|^p \le \left(\frac{\varepsilon}{2}\right)^p$$

for all $\alpha \ge \alpha_1$. Then $\|x^{(\alpha)}\| \le \varepsilon \quad \forall \alpha \ge \alpha_1$. Therefore ℓ^p for $1 \le p < \infty$ has an order continuous norm.

PROPOSITION: TFAE for a Banach lattice E

(1) E is order complete and each continuous linear functional is order continuous

(2) Each directed $(\leq)$ subset D which is bounded above has a filter of sections that converges weakly

(3) E has an order continuous norm

(4) E is σ-order complete and each decreasing sequence with infimum 0 converges to 0 in norm.

(5) Every continuous linear functional is order continuous.

(6) The canonical map $J: E \longrightarrow E^{**}$ maps E onto a lattice ideal in E^{**}

(7) Each order interval in E is weakly compact

(To be continued...)

Proof: $(1) \Rightarrow (2)$ D directed $(\leq)$, majorized. E order complete implies $\sup D = x_0$ exists. Let $D = \{x_d : d \in D\}$, $x_d := d$. Then $\{x_d : d \in D\} \uparrow x_0$. Hence $f(\mathcal{F}(D)) \rightarrow f(x_0)$ for every order continuous linear functional on E, and hence for every $f \in E^*$. Therefore $\mathcal{F}(D) \rightarrow x_0$ weakly.

(2) $\Rightarrow$ (3). If $y_\alpha \downarrow 0$, then $-y_\alpha \uparrow 0$. The set $D = \{-y_\alpha\}$ is directed ($\leq$), bounded above, so $\mathcal{F}(D)$ converges weakly. But the cone is closed, so $\mathcal{F}(D) \to 0$ weakly. Hence $\mathcal{F}(D) \to 0$ in norm since the cone is normal, so $y_\alpha \to 0$ in norm

(3) $\Rightarrow$ (4) We will show E is order complete. Let A be a directed ($\leq$) subset of E that is majorized. Let

$$B = \text{set of all upper bounds of } A \text{ in } E$$

Then $C = B - A$ is directed ($\geq$) and $\inf C = 0$ [[Suppose $\exists\, l > 0$ such that $l \leq b - a$ $\forall a \in A, \forall b \in B$. Then

$$a \leq b - l \qquad \forall a \in A\ \forall b \in B$$

$$\Rightarrow a \leq (b - l) - l \quad \forall a \in A\ \forall b \in B \ (\text{since } b - l \in B)$$

$$\Rightarrow a \leq b - nl$$

$$\Rightarrow nl \leq b - a$$

E is Archimedean since the cone is closed so $l = 0$ $\lightning$]]

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Proof cont. (3) $\Rightarrow$ (4) Will show E is order complete. Let A be a directed ($\leq$) majorized set in E. Let B = all upper bounds of A. Let $C = B - A$. C is directed ($\geq$) and $\inf C = 0$. Let

$$\mathcal{C} = \{x_c : c \in C\} \qquad x_c = c$$

so $x_c \downarrow 0$. Then $x_c \to 0$ in norm $\Rightarrow \mathcal{F}(c) \to 0$ in norm $\Rightarrow \mathcal{F}(A)$ is Cauchy (given 0-nbhd U, choose another V s.t. $V - V \subset U$. Choose $S(a_0, b_0) = \{a - b : a \geq a_0,\ b \leq b_0\}$ Then $S(a_0,b_0) \subset V$ $\Rightarrow$ if $a_1, a_2 \geq a_0$, then $S_{a_1} - S_{a_2} \subset (b_0 - S_{a_2}) - (b_0 - S_{a_1}) \subset V - V \subset U$) Hence $\mathcal{F}(A) \to x_0$ in norm for some $x_0 \in E$. The cone in E is closed so $x_0 = \sup A \Rightarrow \sup A$ exists $\Rightarrow E$ is order complete $\Rightarrow E$ is σ-order complete.

Finally $x_n \downarrow 0 \Rightarrow \|x_n\| \to 0$ since E has order continuous norm.

(4) $\Rightarrow$ (5) Suppose $x_\alpha \downarrow 0$. We will begin by showing that there exists x_{α_k}, $k \in \mathbb{N}$, s.t. $x_{\alpha_k} \downarrow 0$. To this end, we first show

$$(*) \quad \inf_{\alpha} \sup_{\beta > \alpha} \|x_\alpha - x_\beta\| = 0$$

Proof - Assume not. Then $\exists\, \varepsilon_0 > 0$ and increasing (strictly) sequence (α_k) s.t.

$$\|x_{\alpha_k} - x_{\alpha_{k+1}}\| \geq \varepsilon$$

for all k. Let $x_0 = \inf x_{\alpha_k}$ (exists since E is σ-order complete)
$x_{\alpha_k} \downarrow x_0 \Rightarrow x_{\alpha_k} - x_0 \to 0$ in norm. But

$$\|x_{\alpha_k} - x_0\| = \|(x_\alpha - x_{\alpha_{k+1}}) + (x_{\alpha_{k+1}} - x_0)\|$$

$$\geq \|x_{\alpha_k} - x_{\alpha_{k+1}}\| \geq \varepsilon_0$$

normal cone

Hence $(*)$ holds.
Use $(*)$ to choose a strictly increasing (α_k) s.t.

$$\sup_{\beta \geq \alpha_k} \|x_{\alpha_k} - x_\beta\| \leq 1/k \quad \forall k$$

E is σ-order complete, so $\inf x_{\alpha_k} = x_0$ exists. Will show $x_0 = 0$.
For each β and each $k \in \mathbb{N}$,

$$\sup_n \{x_{\alpha_k} - x_\beta \wedge x_{\alpha_n}\} = x_{\alpha_k} - x_\beta \wedge x_0$$

If $\beta' \geq \alpha_n, \beta$, then

$$\|x_{\alpha_k} - x_\beta \wedge x_{\alpha_n}\| \leq \|x_{\alpha_k} - x_{\beta'}\| \leq 1/k$$

for all $n \geq k$. Then

$$\|x_{\alpha_k} - x_\beta \wedge x_0\| \leq 1/k$$

But $x_{\alpha_k} - x_\beta \wedge x_0 \downarrow x_0 - x_\beta \wedge x_0$, so $\|x_0 - x_\beta \wedge x_0\| = 0$, i.e.

$x_0 = x_\beta \wedge x_0$. Therefore $x_0 \le x_\beta \ \forall \beta \Rightarrow x_0 = 0$ since $x_\beta \downarrow 0$

If $0 \le f \in E^*$ and $x_\alpha \downarrow 0$, choose α_k so that $x_{\alpha_k} \downarrow 0$. Then $x_{\alpha_k} \to 0$ in norm, so $f(x_{\alpha_k}) \to 0$. Since $f(x_\alpha)$ is decreasing and $f(x_\alpha) \ge 0$, we must have $f(x_\alpha) \to 0$. Hence every continuous linear functional is order continuous

(5) $\Rightarrow$ (7) later

(7) $\Rightarrow$ (6) Suppose every order interval in E is $\sigma(E,E^*)$-compact. Let γ = all order bounded sets in E. γ has the property that it is saturated (i.e. γ contains all subsets of integer multiples of the closed convex circled hull of any finite union of its members.) Also γ covers E ($E = \bigcup_{S \in \gamma} S$). γ consists of $\sigma(E,E^*)$-relatively compact sets.

The Mackey-Arens theorem says:

$$\begin{array}{ll} E & E^* \\ \gamma & \tau_\gamma = \text{top. on } E^* \text{ of unif. convergence on sets in } \gamma \end{array}$$

$(E^*(\tau_\gamma))^* = J(E)$ because γ is a saturated class of relatively weakly compact $(\sigma(E,E^*))$ sets covering E.

Let $\gamma_0 = \{[-x,x] : x \ge 0, x \in E\}$. Clearly $\gamma_0 \subset \gamma$. Every set in γ is a subset of a set in γ_0. Therefore τ_γ is generated by the seminorms $\{p_x : x \ge 0, x \in E\}$ where

$$p_x(f) = \sup_{z \in [-x,x]} |f(z)| = |f|(x)$$

$E^*(\tau_\gamma)$ is a locally convex lattice $\left(|f| \le |g| \Rightarrow p_x(f) \le p_x(g) \ \forall x > 0 \right)$
Therefore $J(E) = (E^*(\tau_\gamma))^*$ is a lattice ideal in $(E^*)^b = E^{**}$

(6) $\Rightarrow$ (7) Suppose $J(E)$ is a lattice ideal in E^{**}. If $x \le y$ then $J[x,y] = [Jx, Jy]$ (i.e. J is interval preserving)
(If $Jx \le w \le Jy$, then

$$|w| \le |Jx| + |J(y-x)|$$

$$\Rightarrow w \in J(E)$$

$$\Rightarrow w = Jz \quad , z \in E$$

Now

$$Jz = Jz \wedge Jy = J(z \wedge y) \Rightarrow z = z \wedge y$$

Hence $z \le y$. We show $x \le z$ in a similar way. Hence $z \in [x,y]$.)
Order intervals in E^{**} are $\sigma(E^{**}, E^*)$-compact. Also,

$$\sigma(E^{**} | E^*)\Big|_{J(E)} = \sigma(E, E^*)$$

Therefore order intervals in E are weakly compact

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(Proof continued)

(7) $\Rightarrow$ (1) Suppose A is a majorized directed ($\leq$) subset of the positive cone in E with an upper bound x_0. Then $A \subset [0, x_0]$ so $\mathcal{F}(A)$ has a weak cluster point $x_1 \in [0, x_0]$. But the cone is closed, so $x_1 = \sup A$. Therefore E is order complete.

Let $0 \leq f \in E^*$, and let $x_\alpha \downarrow 0$. The net $\{x_\alpha : \alpha > \alpha_0\} = S(x_{\alpha_0}) \subset [0, x_{\alpha_0}]$ has a weak cluster point which must be zero. Hence $x_\alpha \to 0$ weakly, so $f(x_\alpha) \to 0$. Therefore f is order continuous.

$\blacksquare$

DEFINITION: A Banach lattice E is an M-space if $\forall x, y \geq 0$

$$\|x \vee y\| = \max\{\|x\|, \|y\|\}$$

An element $u \in E$ is a strong order unit if $\|x\| \leq 1 \iff -u \leq x \leq u$.

Remarks and examples (1) If X is compact, then $C(X)$ is an M-space with strong order unit 1_X

(2) If (X, Σ, μ) measure space, then $L^\infty(X, \Sigma, \mu)$ is an M-space with strong order unit 1_X

(3) c, ℓ^∞ are M-spaces with strong unit $e = (1, 1, 1, \dots, 1, \dots)$

(4) c_0 is an M-space with no strong unit

(5) If X is locally compact Hausdorff but not compact, and

$$C_0(\alpha X) = \{ f \in C(X) : f(\infty) = 0 \} \quad (\alpha X = X \cup \{\infty\})$$

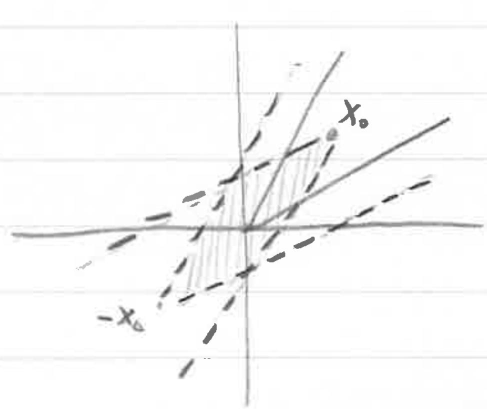
with sup norm and pointwise order, then $C_0(\alpha X)$ is an M-space without strong unit

(6) If E is a Banach lattice and if x_0 is an interior point of the cone in E, then define

$$\|x\|_0 = \inf \{ \lambda > 0 : x \in \lambda [-x_0, x_0] \}$$

(guage functional of $[-x_0, x_0]$). Then $\|\cdot\|_0$ is equivalent to the given norm and $(E, \|\cdot\|_0)$ is an M-space with strong unit.

Proof. Note that $[-x_0, x_0] = (-x_0 + K) \cap (x_0 - K)$ where K is the cone. Therefore $0 \in \operatorname{int} [-x_0, x_0]$



Since $[-x_0, x_0]$ is clearly convex and circled, its guage functional is a seminorm. Let $U = \{x \in E : \|x\| \le 1\}$. Then there is an $\varepsilon>0$ such that

$$\varepsilon_0 U \subset \underbrace{[-x_0, x_0]}_{\substack{\| \\ \text{unit ball of } \|\cdot\|_0 \\ \text{(since cone is closed)}}}$$

If $\|x\|_0 \le 1$, then $x \in [-x_0, x_0] \Rightarrow |x| \le x_0 \Rightarrow \|x\| \le \|x_0\|$

Hence

$$[-x_0, x_0] \subset \|x_0\| \, U$$

Hence $\|\cdot\|_0$ is a norm equivalent to $\|\cdot\|$.

Now

$$|x| \le |y| \Rightarrow \|x\|_0 = \inf\{\lambda>0 : x \in \lambda[-x_0, x_0]\}$$

$$\le \inf\{\lambda>0 : y \in \lambda[-x_0, x_0]\}$$

$$= \|y_0\|$$

Hence $\|\cdot\|_0$ is a Banach lattice norm.

To show an M-space, note that for $0 \le x, y$ we have

$$x, y \le x \vee y \Rightarrow \max\{\|x\|_0, \|y\|_0\} \le \|x \vee y\|_0$$

Also, since the cone in E is closed, we have

$$0 \le x \le \|x\|_0 \, x_0$$
$$0 \le y \le \|y\|_0 \, x_0$$

and so

$$0 \le x \vee y \le \max\{\|x\|_0, \|y\|_0\} x_0$$

$$\Rightarrow \|x \vee y\|_0 \le \max\{\|x\|_0, \|y\|_0\}$$

$(\|x_0\|_0 = 1)$

$\blacksquare$

(7) If E is a Banach lattice and if $0 \le u \in E$, define E_u to be the lattice ideal generated by u,

$$E_u = \bigcup_{n=1}^{\infty} n[-u, u]$$

and let $\|\cdot\|_u$ be the guage functional on E_u of $[-u, u]$. Then $(E, \|\cdot\|_u)$ is an M-space with strong unit u and the norm topology given by $\|\cdot\|_u$ is finer than the induced topology.

Proof. $[-u,u]$ is bounded in E since the cone is normal. Therefore $\|\cdot\|_u$ is finer than the induced norm topology from E. Since there is a neighborhood basis for $\|\cdot\|_u$ on E consisting of sets that are complete for the coarser topology induced by the given norm on E, $(E, \|\cdot\|_u)$ is complete.

(8) Suppose that E is an M-space and that $0 \le f, g \in E^*$. Then

$$\|f+g\| = \|f\| + \|g\|$$

Proof. Let $\varepsilon > 0$. Choose $x, y \ge 0$, $\|x\| = 1 = \|y\|$ s.t.

$$f(x) \ge \|f\| - \varepsilon$$
$$g(y) \ge \|g\| - \varepsilon$$

Now $\|x \vee y\| = \max\{\|x\|, \|y\|\} = 1$, so

$$\|f+g\| \ge (f+g)(x \vee y) \ge f(x) + g(y) \ge \|f\| + \|g\| - 2\varepsilon$$

Therefore $\|f+g\| \ge \|f\| + \|g\|$.

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(9) If E is an M space with a strong unit u, then

$$\|f\| = f(u)$$

for $0 \le f \in E^*$.

$$\left(\|x\| \le 1 \Rightarrow |x| \le u \Rightarrow |f(x)| \le f(u) \Rightarrow \|f\| \le f(u) \right)$$

Therefore the positive face P^* of the unit ball in E^*, i.e.

$$P^* = \{ f \in E^* : \|f\| = 1,\ f \ge 0 \}$$

is the intersection of the positive cone in E^* with the hyperplane

$$H_u = \{ f \in E^* : f(u) = 1 \}$$

In particular, P^* is a convex weak*-compact subset of E^*.

(10) $\operatorname{Ext} P^* =$ all lattice homomorphisms of E into $\mathbb{R}$ of norm 1 (even if E does not have a strong unit)

Proof. If f is a lattice homomorphism of E into $\mathbb{R}$ of norm 1, then $f \ge 0$ since

$$x \ge 0 \Rightarrow x = |x| \Rightarrow f(x) = f(|x|) = |f(x)| \ge 0$$

Hence $f \in P^*$. Suppose $f = \lambda g + (1-\lambda) h$ for $0 < \lambda < 1$, where $g, h \in P^*$. Then

$$0 \leq \lambda g \leq f$$
$$0 \leq (1-\lambda) h \leq f$$

If $x \in E$, then

$$|\lambda g(x)| = \lambda |g(x)| \leq \lambda g(|x|) \leq f(|x|) = |f(x)|$$

Therefore $\lambda g(x) = 0$ whenever $f(x) = 0$, and so $\lambda g = \alpha f$ for some $\alpha \geq 0$. Hence $\lambda = \alpha$ since $\|g\| = \|f\| = 1$, so $g = f$. Therefore $f = g = h$. Thus $f \in \text{Ext } P^*$.

Suppose $f \in \text{Ext}(P^*)$ and $0 \leq g \leq f$ for some $g \in E^*$. If $g \neq 0$ and $g \neq f$, then

$$f = \|g\| \left(\frac{g}{\|g\|} \right) + \|f-g\| \left(\frac{f-g}{\|f-g\|} \right)$$

and $\|g\| + \|f-g\| = \underline{1}$. Since $g/\|g\| \in P^*$ and $(f-g)/\|f-g\| \in P^*$, it follows that $f = g/\|g\|$, so $g = \|g\| f$.

Let $x \in E$.

$$f(x^+) = \sup_{g \in [0,f]} g(x) = \sup_{0 \leq \alpha \leq 1} \alpha f(x) = \sup\{f(x), 0\}$$

$$= f(x)^+$$

Therefore f is a lattice homomorphism

∎

LATTICE VERSION OF THE STONE-WEIERSTRASS THEOREM: Suppose that X is a compact Hausdorff space, that F is a sublattice of $C(X)$ that separates points of X and that contains the constants. Then F is dense in $C(X)$

Proof. WLOG X contains more than one point. Then whenever x, y are distinct points of X and $a, b \in \mathbb{R}$, there is an $f \in F$ s.t.

$$f(x) = a \qquad f(y) = b$$

In fact, choose any $g \in F$ such that $g(x) \neq g(y)$ and let

$$f(z) = \frac{g(z) - g(y)}{g(x) - g(y)} a + \frac{g(x) - g(z)}{g(x) - g(y)} b$$

Let $f \in C(X)$ and let $\varepsilon > 0$ be given. Fix $x \in X$. For each $y \neq x$, choose $g_y \in F$ so that

$$g_y(x) = f(x) \qquad g_y(y) = f(y)$$

If $V_y = \{z : g_y(z) < f(z) + \varepsilon\}$, then V_y is open and $x, y \in V_y$.

$\{V_y : y \neq x\}$ is an open cover of X, so there exists a finite subcover

$$V_{y_1}, \dots, V_{y_n}$$

Let $g_x = g_{y_1} \wedge \dots \wedge g_{y_n} \in F$. Note

$$g_x(z) < f(z) + \varepsilon \quad \forall z \in X$$

Let $V_x = \{z \in X : g_x(z) > f(x) - \varepsilon\}$. Then V_x is an open set containing x. $\{V_x : x \in X\}$ is an open cover. Choose a finite subcover $V_{x_1}, \dots, V_{x_k}$
Let

$$g = g_{x_1} \wedge \dots \wedge g_{x_k} \in F$$

Then

$$f(x) - \varepsilon \leq g(x) \leq f(x) + \varepsilon \quad \forall x \in E$$

Hence F is dense in $C(X)$.

$\blacksquare$

KAKUTANI'S REPRESENTATION THEOREM FOR M-SPACES.

If E is an M-space and if X is the weak*-closure of $\mathrm{Ext}\, P^*$ equipped with the weak* topology, and if $\Phi : E \to C(X)$ defined by

$$\Phi x(f) := f(x)$$

then Φ is isometry and a lattice isomorphism onto a closed sublattice of $C(X)$. If E contains a strong unit, then $\Phi(u) = \chi_X$ and Φ is onto.

Proof. If $f \in X$, then f is a lattice homomorphism on E.

(If $f_\alpha \in \operatorname{Ext} P^*$ and $f_\alpha \to f$ weak*, then

$$f(|x|) = \lim f_\alpha(|x|) = \lim |f_\alpha(x)| = |f(x)| \quad \forall x \ )$$

Note that $\Phi(E) \subset C(X)$ by definition of weak* topology.
Φ is a lattice homomorphism

$$(\Phi(|x|))f = f(|x|) = |f(x)| = |\Phi x(f)| = |\Phi x|(f)$$

$$\Rightarrow \Phi|x| = |\Phi x| \quad \forall x \in E$$

If we could show that Φ is an isometry on the cone of E, it would be an isometry on all of E. For if $x \in E$,

$$\|\Phi(x)\| = \| \, |\Phi(x)| \, \| = \|\Phi(x)^+ \vee \Phi(x)^-\|$$

$$= \max \{ \|\Phi(x)^+\|, \|\Phi(x)^-\| \}$$

$$= \max \{ \|\Phi(x^+)\|, \|\Phi(x^-)\| \}$$

$$= \max \{ \|x^+\|, \|x^-\| \} = \|x^+ \vee x^-\| = \|x\|$$

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Proof (continued) To complete the proof that Φ is an isometry, we only need to show that it is an isometry on the cone. If $x \geq 0$, then

$$\|x\| = \sup\{f(x) : \|f\| = 1\}$$

since $x \geq 0$ $\rightarrow$
$$= \sup\{f(x) : f \in P^*\}$$

Krein Mil'man $\rightarrow$
$$= \sup\{f(x) : f \in \overline{\operatorname{Ext} P^*}^{w^*} = X\}$$

$$= \sup\{[\Phi(x)]f : f \in X\}$$

$$= \|\Phi x\|_{C(X)}$$

Now assume E has a strong unit u. Then $P^* = \{f \in E : f(u) = 1, f \geq 0\}$ is weak* compact and convex, and so $X = \operatorname{Ext} P^*$ since X always consists of lattice homomorphisms.

$$[\Phi(u)]f = f(u) = 1 \quad \forall f \in X$$

$$\Rightarrow \Phi(u) = \chi_X$$

Hence $\Phi(E)$ is a closed sublattice in $C(X)$ that separates points and contains the constant. By Stone Weierstrass $\Phi(E) = C(X)$ ☒

Fact: If E is an M-space without a strong unit u, then $X \neq \operatorname{Ext} P^*$. If $f \in X$ and $f \notin \operatorname{Ext} P^*$, then

$$\frac{f}{\|f\|} = g_f \in \operatorname{Ext} P^*$$

and $\Phi(E)$ can be described as follows:

$$\Phi(E) = \{ \varphi \in C(X) : \varphi(f) = \|f\| \varphi(g_f) \text{ for each } f \in X \setminus \operatorname{Ext} P^* \}$$

Applications of Kakutani

PROPOSITION: Suppose that E is an order complete M-space with a strong unit u and that M is an order complete sublattice of E that contains u. Then M is the range of a positive contractive projection P on E.

Proof. $M \approx C(X) \Rightarrow C(X)$ order complete $\Rightarrow X$ Stonean. By Nachbin - Kelly

$$\begin{tikzcd} E \arrow[dr, dashed, "P"] & \\ C(X) \arrow[u, hook] \arrow[r, "I"] & C(X) \end{tikzcd}$$

(extension of I)

$$\|P\| = \|I\| = 1$$

We must show P is positive. If $z \geq 0$, $\|z\| \leq 1$, then

$$z = u - v$$

where $v \geq 0$ (since u is a strong unit). Then

$$Pz = Pu - Pv = u - Pv$$

But $\|Pv\| \leq \|v\| \leq 1$, so $Pv \leq u$. Hence $Pz = u - Pv \geq 0$.

Application: Let X be Stonean compact Hausdorff. Consider

$$C(X) \subset \ell^\infty(X)$$

These contain strong unit 1_X. Then there is a positive contractive projection P of $\ell^\infty(X)$ onto $C(X)$.

Recall Nachbin - Kelley

$$\begin{tikzcd} E \arrow[dr, "\hat{T}"] & \\ M \arrow[u, hook] \arrow[r, "T"] & C(X) \end{tikzcd} \quad X \text{ Stonean}$$

$$\|T\| = \|\hat{T}\|$$

We want to extend this to Banach lattices and positive operators

PROPOSITION: Suppose that X is Stonean (compact, T_2), that M is a closed sublattice of a Banach lattice E and that $T : M \longrightarrow C(X)$ is a positive linear map. Then T has a positive linear extension $\hat{T}$ to E of the same norm.

Proof Since T is positive and M is a Banach lattice, T is continuous. For each $x \in X$, define φ_x by

$$\varphi_x(y) = [Ty](x) \qquad \forall y \in M$$

φ_x is a positive linear functional on M for each $x \in X$. Also $\|\varphi_x\| \leq \|T\|$ for all $x \in X$. By an earlier extension theorem, φ_x has a positive linear extension $\hat{\varphi}_x$ to E such that $\|\varphi_x\| = \|\hat{\varphi}_x\|$. Define $T_1 : E \longrightarrow \ell^\infty(X)$ by

$$T_1 y = \{\hat{\varphi}_x(y) : x \in X\}$$

T_1 is a positive linear mapping and T_1 is an extension of T if we regard $C(X) \subset \ell^\infty(X)$

$$\|T_1\| = \sup_{x \in X} \{|\hat{\varphi}_x(y)|\} \leq \|T\|$$

Hence $\|T_1\| = \|T\|$. Let $\hat{T} = P \circ T_1$ where P is a positive contractive projection of $\ell^\infty(X)$ onto $C(X)$. Then $\|\hat{T}\| = \|T\|$.

∎

THE DUAL OF C(X) - NORMAL MEASURES

$C(X)^*$ = the space of Radon measures on X $(M(X))$

= the space of regular signed Borel measures on X

<u>DEFINITION</u>: $\mu \in M(X)$ is a normal measure if μ is an order continuous linear functional on $C(X)$

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Examples and Remarks

(1) If x_0 is a non-isolated point of X such that x_0 has a countable neighborhood basis (V_n), then ε_{x_0} = pt. mass at x_0 is not a normal measure.

[[Choose $g_n \in C(X)$ s.t. $0 \leq g_n \leq 1$, $g_n(x_0) = 1$, $g_n(x) = 0$ $\forall x \notin V_n$. Let $f_n = g_1 \wedge \cdots \wedge g_n$. Then $f_n \downarrow 0$, yet $\varepsilon_{x_0}(f_n) = 1$ $\forall n$]]

(2) If $\{f_\alpha\} \subset C(X)$ and $f_\alpha(x) \downarrow 0$ $\forall x \in X$, then for any $\mu \in C(X)^*$ we have $\mu(f_\alpha) \longrightarrow 0$

[[by Dini's Theorem]]

<u>PROPOSITION</u>: A Radon measure μ on X is normal iff μ vanishes on all Borel sets of first category.

Proof. Suppose μ is normal and that N is a nowhere dense closed set in X. Let

$$D = \{f \in C(X) : 0 \leq f \leq 1,\ f(x) = 1 \ \forall x \in N\}$$

D is directed $(\geq)$. 0 is a lower bound. If $x \notin N$, there is an $f_x \in C(X)$ with $f_x(x) = 0$ and $f_x(y) = 1$ for $y \in N$. Hence $f_x \in D$. If h is any lower bound of D, then $h(x) \leq 0$ for all $x \notin N$. Hence

$$N \supset \{x \in X : h(x) > 0\}$$

open

Therefore $h = 0$, so $\inf D = 0$.

Let $f_d = d \quad \forall d \in D$. Then $\{f_d\} \downarrow 0$. Since μ is normal, $|\mu|$ is normal, so $|\mu|(f_d) \to 0$, i.e.

$$\int f_d \, d|\mu| \longrightarrow 0$$

However $|\mu(N)| \le \int f_d \, d|\mu|$ since $f_d = 1$ on N and $0 \le f_d \le 1$. Therefore

$\mu(N) = 0$, so $\mu(\text{Borel set of } 1^{st} \text{ category}) = 0$.

Suppose that μ vanishes on Borel sets of first category, and that $f_\alpha \downarrow 0$. For each $n \in \mathbb{N}$, let

$$D_n = \{x : f_\alpha(x) \ge 1/n \quad \forall \alpha\}$$

Then D_n is closed and $\operatorname{int} D_n = \emptyset$ (otherwise by complete regularity there is an $f_0 \ge 0$, $f_0 \ne 0$, $f_0 \le f_\alpha \ \forall \alpha$ (↯)) Hence D_n is nowhere dense, so $\mu(D_n) = 0$ for all n.

$$\mu(\{x : \inf f_\alpha \ne 0\}) = \mu\left(\bigcup_{n=1}^{\infty} D_n\right) = 0$$

Therefore $f_\alpha(x) \downarrow 0$ a.e. (μ), so

$$\mu(f_\alpha) = \int f_\alpha \, d\mu \longrightarrow 0$$

Therefore μ is a normal measure.

$\blacksquare$

Remarks:

(1) If X does not contain any isolated points, then every normal measure μ on X vanishes on finite sets

(2) If X is separable and does not contain isolated points, there do not exist non-zero normal measures on X

Proof: Suppose μ is normal. WLOG $\mu \geq 0$. Let $\{x_n\}$ be a dense sequence in X. We know by (1) that $\mu(x_n) = 0$ for all n. Since μ is regular we can choose open sets V_{nm} such that $x_n \in V_{nm}$ and $\mu(V_{nm}) < 1/m2^n$. Define

$$W_m = \bigcup_{n=1}^{\infty} V_{nm}$$

Put $P_m = X \setminus W_m$. $P = \bigcup_{m=1}^{\infty} P_m$. W_m is open and dense

$\mu(W_m) < 1/m$, $\mu(P_m) > \mu(X) - 1/m$. P_m is closed and nowhere dense so $\mu(P) = 0$. But $\mu(X) = \mu(P)$, so $\mu(X) = 0 \Rightarrow \mu = 0$

$\blacksquare$

DEFINITION: A Banach lattice E is an L-space if

$$\|x+y\| = \|x\| + \|y\|$$

for all $x, y \geq 0$.

Remarks:

(1) l_1, $L_1(X, \Sigma, \mu)$ are L-spaces
(2) The norm dual of any M-space is an L-space
(3) The norm dual of any L-space is an M-space with a strong unit

Proof of (3): The norm is positively homogeneous and additive on the cone so it extends to a unique linear functional f_0 on E ($f_0(x) = \|x^+\| - \|x^-\|$) Clearly $f \geq 0$. Note that $\|g\| \leq 1$ for $g \in E^*$, so $g \in [-f_0, f_0]$

$$\big(\|g\| \leq 1,\ x \geq 0,\ \|x\| \leq 1 \Rightarrow |g(x)| \leq \|x\| = f_0(x)$$
$$\Rightarrow g \in [-f_0, f_0].\quad \|f_0\| = 1,\ |g| \leq f_0 \Rightarrow \|g\| \leq 1 \big)$$

Thus the dual norm on E^* is just the norm with unit ball $[-f_0, f_0]$ and we have shown that any such norm is an M-space norm with strong unit f_0.

(4) If E is an L-space, then the filter of sections of any norm bounded directed ($\leq$) set converges in norm to $\sup D$.

Proof. Suppose $\mathcal{F}(D)$ is not Cauchy. Then there exists $\varepsilon_0 > 0$ $d_n \in D$ such that $d_{n+1} \geq d_n$ and $\|d_{n+1} - d_n\| \geq \varepsilon_0$. Hence for any m

$$m\varepsilon_0 \leq \sum_{k=1}^{m} \|d_{k+1} - d_k\| = \|\sum_{k=1}^{m} (d_{k+1} - d_k)\|$$

$$= \|d_{m+1} - d_1\|$$

$$\leq 2 \sup_{d \in D} \|d\|$$

which is impossible. Therefore $\mathcal{F}(D)$ is Cauchy, so $\mathcal{F}(D) \to x_0$ and $x_0 = \sup D$ since the cone is closed

(5) Any L-space is order complete

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Correction to last time: $f_\alpha(x) \downarrow 0$ μ-a.e. (f_α continuous)
WLOG assume $0 \le f_\alpha(x) \le M$ for all $x \in X$, $\alpha \in A$. If $\varepsilon > 0$, choose N such $\mu(N) < \varepsilon/2M$ such that $f_\alpha(x) \downarrow 0$ everywhere on N^c and N open.
Dini's theorem $\Rightarrow$ $f_\alpha \to 0$ unif. on N^c, so

$$\int_{N^c} f_\alpha \, d\mu \longrightarrow 0$$

$$\Longrightarrow \int_{N^c} f_\alpha \, d\mu \le \varepsilon/2 \quad \forall \alpha \ge \alpha_\varepsilon$$

Also

$$\int_N f_\alpha \, d\mu \le M\mu(N) \le \varepsilon/2$$

DEFINITION: A compact Hausdorff space X is hyperstonean if it is Stonean and if the set $\eta(X)$ of normal measures on X separates points of $C(X)$

THEOREM (DIXMIER) Suppose X is a compact Hausdorff space. Then $C(X)$ is isomorphic as a Banach lattice to the dual of a Banach lattice if and only if X is hyperstonean.

Proof. Suppose $C(X) \approx E^*$ for some Banach lattice E. E^* is order complete, so $C(X)$ is order complete. Hence X is Stonean.

E separates points of E^*. If $x \in E$, $Q : E \longrightarrow E^{**}$ canonical, then Qx is order continuous.

$$\left(f_\alpha \downarrow 0,\ f_\alpha \in E^* \Rightarrow [\inf f_\alpha](x) = \inf f_\alpha(x), \text{ i.e. } Qx(f_\alpha) \to 0 \right)$$

Hence $Q(E) \subset \eta(X)$ if we identify $C(X)$ and E^*. Hence $\eta(X)$ separates points of $C(X)$, so X is hyperstonean.

Suppose now that X is hyperstonean. Therefore, if for each $0 \leq \nu \in \eta(X)$, we define a seminorm p_ν on $C(X)$ by

$$p_\nu(f) := \int |f| \, d\nu$$

Then the family $\{p_\nu : \nu \in \eta(X)\}$ generates a Hausdorff locally convex lattice topology τ on $C(X)$. Now

$$p_\nu(f) = \nu(|f|) = \sup_{\mu \in [-\nu,\nu]} \{\mu(f)\}$$

Therefore τ = topology on $C(X)$ of uniform convergence on order bounded sets in $\eta(X)$

Claim: $[C(X), \tau]^* = \eta(X)$.

(Proof) By Mackey-Arens, it would suffice to show that the order interval $[-\nu,\nu]$ in $\eta(X)$ is $\sigma(\eta(X), C(X))$ relatively compact for each $\nu \geq 0$ in $\eta(X)$. But $\eta(X)$ is a lattice ideal in $C(X)^*$, so $[-\nu,\nu]$ in $\eta(X)$ is the same as $[-\nu,\nu]$ in $C(X)^*$ for $0 \leq \nu \leq \eta(X)$

$[-\nu,\nu]$ in $C(X)^*$ is $\sigma(C(X)^*, C(X))$-cpt and

$$\sigma(C(X)^*, C(X))\big|_{\eta(X)} = \sigma(\eta(X), C(X))$$

Hence $[-\nu,\nu]$ in $\eta(X)$ is $\sigma(\eta(X), C(X))$-cpt.

We will show that $C(X)$ is isomorphic to $\eta(X)^*$ when $\eta(X)$ is equipped with the norm and order induced by $C(X)^*$. To show that $C(X)$ is isomorphic to $\eta(X)^*$ as a Banach space, it would suffice to show that the unit ball $[-1_X, 1_X]$ in $C(X)$ is $\sigma(C(X), \eta(X))$-compact (by Mackey-Arens)

Assume for the moment that we can show that $[-1_X, 1_X]$ is complete for the topology τ on $C(X)$.

$$C(X) \longrightarrow \overset{= F}{\prod_{0 \le \nu \in \eta(X)} L^1(\nu)} \quad \text{(product top)}$$

$$\varphi : f \longmapsto (f_\nu : 0 \le \nu \in \eta(X)) \qquad f_\nu = \dot{f}$$

(topological isomorphism if $C(X)$ has top. τ). Since $[-1_X, 1_X]$ is complete for τ, $\varphi[-1_X, 1_X]$ is complete and hence closed in F. Also $\varphi[-1_X, 1_X]$ is convex and so it is weakly closed in F. Let $[-1_X, 1_X]_\nu$ be the order interval between -1_X and 1_X in $L^1(\nu)$, so it is weakly compact in $L^1(\nu)$ because $L^1(\nu)$ has order continuous norm. Hence $I = \prod [-1_X, 1_X]_\nu$ is compact for the product of the weak topologies = weak topology on $\prod L_1(\nu)$. But

$$\varphi[-1_X, 1_X] \subset I$$

so that $\varphi[-1_X, 1_X]$ is weakly compact $\Rightarrow [-1_X, 1_X]$ is weakly compact in $C(X)$ for $\tau = \sigma(C(X), \eta(X))$. Therefore $C(X)$ is isomorphic to $\eta(X)^*$ as a Banach space. It is also isomorphic as a Banach lattice since $\eta(X) \subset C(X)^*$.

LEMMA: The unit ball $[-1_X, 1_X]$ in $C(X)$ is complete for the topology τ generated by

$$p_\nu(f) = \int |f| \, d\nu \qquad f \in C(X)$$

for $0 \leq \nu \in \eta(X)$.

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(Proof continued) τ = P-topology = Peressini topology

LEMMA: The unit ball $[-1_X, 1_X]$ is complete for τ.

Proof. Suppose $\mathcal{F}$ is a τ-Cauchy filter on $[-1_X, 1_X]$. Fix $0 \le \nu \in \eta(X)$. Let τ_ν be the locally convex topology on $C(X)$ generated by the single seminorm p_ν. Then $\tau_\nu \le \tau$, so $\mathcal{F}$ is τ_ν-Cauchy. Suppose

$$H_\nu := \{\tau_\nu\text{-limit points of } \mathcal{F}\}$$

Claim: $H_\nu \neq \phi$.
Proof. Choose $F_n \in \mathcal{F}$ s.t. $F_n \supset F_{n+1}$ and $\nu(|f-f'|) < 1/2^{n+1}$ for all $f, f' \in F_n$. For each n, choose $f_n \in F_n$. Since $\{f_n\} \subset [-1_X, 1_X]$ and X is Stonean,

$$g_n := \sup_{k \geq n} f_k$$

exists for all n. Note that

$$f_{n+p} = f_n + \sum_{k=n+1}^{n+p} (f_k - f_{k-1})$$

for all $p \in \mathbb{N}$, all n. Then

$$g_n = \sup_p f_{n+p} = f_n + \sup_p \sum_{k=n+1}^{n+p} (f_k - f_{k-1}) \quad \forall n$$

$$\nu(|g_n - f_n|) = \nu\left(\sup_p \sum_{k=n+1}^{n+p} (f_k - f_{k-1})\right)$$

$$\leq \sup_p \sum_{k=n+1}^{n+p} \nu(|f_k - f_{k-1}|) \leq \frac{1}{2^n}$$

Let $h = \inf_n g_n$. Then $g_n \downarrow h$, so $\nu(g_n) \downarrow \nu(h)$ (normality of ν)

$$\nu(|f_n - h|) \leq \nu(|f_n - g_n|) + \nu(|g_n - h|)$$

$$\leq 1/2^n + \nu(g_n) - \nu(h) \longrightarrow 0$$

Hence $f_n \to h$ for τ_ν. If $F \in \mathcal{F}$, choose $f_n' \in F \cap F_n$. Then $f_n' \to h$ for τ_ν since $f_n \to h$ for τ_ν and

$$\nu(|f_n - f_n'|) < 1/2^{n+1}$$

Therefore $h \in \overline{F}$, so $h \in H_\nu \Rightarrow H_\nu \neq \emptyset$. $\boxtimes$

Claim: H_ν is a sublattice of $[-1_X, 1_X]$
Proof. Suppose $f, g \in H_\nu$, $\varepsilon > 0$. Choose $F_\varepsilon \in \mathcal{F}$ s.t.

$$\nu(|h - h'|) < \varepsilon \quad \forall h, h' \in F_\varepsilon$$

Then $f, g \in H_\nu \Rightarrow f, g \in \overline{F_\varepsilon}^{\tau_\nu} \Rightarrow \nu(|f - h|) \leq 2\varepsilon, \nu(|g - h|) \leq 2\varepsilon$ for all $h \in F_\varepsilon$. Then

$$f \vee g - h = (f-h) \vee (g-h) \leq |f-h| + |g-h|$$

$$h - f \vee g = (h-f) \wedge (h-g) \leq |f-h| + |g-h|$$

Therefore

$$\nu(|f \vee g - h|) \leq \nu(|f-h|) + \nu(|g-h|) \leq 4\varepsilon \quad \forall h \in F_\varepsilon$$

and so $f \vee g \in H_\nu$ by same argument as in the last claim. Similarly $f \wedge g \in H_\nu$. ▨

Let $f_\nu := \sup H_\nu$. H_ν is directed $(\leq)$, so since ν is normal $f_\nu \in H_\nu$. Therefore

$$0 \leq \nu_1 \leq \nu_2 \Rightarrow p_{\nu_2}(f) \leq p_{\nu_1}(f) \Rightarrow \tau_{\nu_2} \geq \tau_{\nu_1}$$

$$\Rightarrow H_{\nu_2} \subset H_{\nu_1}$$

Hence $\{f_\nu : \nu \geq 0, \nu \in \eta(X)\}$ is directed $(\geq)$. Let $f = \inf f_\nu$. Then $f_\nu \to f$ for τ. Given $\nu_0 \geq 0$, $\nu_0 \in \eta(X)$ we know $\nu_0(f_\nu) \downarrow \nu_0(f)$ since ν_0 is normal. If $\nu \geq \nu_0$, then $H_\nu \subset H_{\nu_0}$, so $f_\nu \in H_{\nu_0}$. Therefore $f \in H_{\nu_0}$ for any ν_0, so

$$f \in \bigcap_{0 \leq \nu \in \eta(X)} H_\nu = \text{all } \tau\text{-limit points of } \mathcal{F}$$

Hence $\mathcal{F} \to f$ ▨

<u>KAKUTANI'S L-SPACE THEOREM</u>: If E is an L-space with weak order unit u, then there exists a strictly positive measure μ on a compact Stonean space X that vanishes on all Borel sets of X of first category such that E is isometric and lattice isomorphic to $L^1(\mu)$. Moreover, under this isomorphism, μ corresponds to 1_X and the ideal E_μ generated by μ corresponds to $L^\infty(\mu)$.

Proof. The lattice ideal E_u generated by μ is an M-space with a strong unit u if E_u is normed by the unit ball $[-u,u]$. Therefore there is an isometry and lattice isomorphism Φ of E_u onto $C(X)$ for a compact Hausdorff space X.

X is Stonean because E and hence E_u is order complete. Also, the norm on E is additive and positive homogeneous on the cone so it extends to a linear functional f_0 on E s.t. $f_0(x) \geq 0$ for $x \geq 0$ and $f_0(x) > 0$ for $x \geq 0, x \neq 0$. f_0 is order continuous since an L-space has order continuous norm. Then $f_0|_{E_u}$ is an order continuous strictly positive functional on E_u s.t.

$$\| f_0|_{E_u} \| = \|u\|$$

Then $f_0|_{E_u}$ corresponds to a strictly positive normal measure μ on $C(X)$ with

$$\|\mu\| = \| f_0|_{E_u} \| = \|u\|$$

Note $\mu(B) = 0$ $\forall$ Borel sets of 1st category. Let $0 \leq x \in E$. u is a weak order unit $\Rightarrow$ $B(u) = E$. Let $x_n = x \wedge nu \uparrow x$

$x_n \in E_\mu$. $\quad x_n \longleftrightarrow f_n \in C(X)$. Since $x_n \uparrow$, $f_n \uparrow$. Hence

$$f_o(x) \uparrow, \ \leq f_o(x) \Rightarrow \sup_n \int f_n d\mu \leq +\infty$$

Therefore there is an $f \in L_1(\mu)$ s.t. $f_n \uparrow f$ and $\int f_n d\mu \longrightarrow \int f d\mu$

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(Proof continued)

Let $x \geq 0$ in E and let $x_n := x \wedge nu$. Since u is a weak order unit $x_n \uparrow x$. Also, $x_n \in E_u$

$$x_n \xleftrightarrow{\Phi} f_n \in C(X)$$

We know $f_0(x_n) \uparrow f_0(x)$, i.e.

$$\sup_n \int f_n \, d\mu \leq f_0(x) < \infty$$

$\{ \int f_n \, d\mu \}$ is an increasing sequence that is bounded above. Let $f_x :=$ pointwise limit of $\{f_n\}$. Then $f_x \in L^1(\mu)$ and

$$\int f_n \, d\mu \uparrow \int f_x \, d\mu \qquad \text{(MCT)}$$

Extend Φ to the cone in E by $\Phi(x) = f_x$. Notice

$$\|x\| = \lim \|x_n\| = \lim_n f_0(x_n) = \lim \int f_n \, d\mu$$

$$= \int f_x \, d\mu = \|f_x\|_{L_1(\mu)}$$

Therefore Φ is an isometry on the cone of E into $L^1(\mu)$.

Claim: $\Phi(x \wedge y) = \Phi(x) \wedge \Phi(y) \quad \forall x, y \geq 0$ in E.

$$x_n = x \wedge nu \qquad y_n = y \wedge nu \qquad z_n = (x \wedge y) \wedge nu$$
$$= x_n \wedge y_n$$

$$x_n \uparrow x \qquad y_n \uparrow y \qquad z_n \uparrow x \wedge y$$

Hence $\Phi(x \wedge y) = \lim_n \Phi(z_n) = \lim \Phi(x_n \wedge y_n) = \lim \Phi(x_n) \wedge \Phi(y_n)$

$$= \lim \Phi(x_n) \wedge \lim \Phi(y_n)$$

$$= \Phi(x) \wedge \Phi(y)$$

By similar considerations we can see that Φ is additive and positively homogeneous and preserves sups on the cone of E. Extend Φ to E by

$$\Phi(x) = \Phi(x^+) - \Phi(x^-)$$

for $x \in E$. Then

$$x^+ \wedge x^- = 0 \Rightarrow 0 = \Phi(x^+ \wedge x^-) = \Phi(x^+) \wedge \Phi(x^-)$$

$$\Rightarrow \Phi(x^+) = \Phi(x)^+$$

Therefore Φ is a lattice homomorphism. Also

$$\|\Phi(x)\| = \|\,|\Phi(x)|\,\| = \|\Phi(|x|)\| = \|\,|x|\,\| = \|x\|$$

Hence Φ is an isometry and lattice isomorphism of E into $L^1(\mu)$.

Therefore $\Phi(E)$ is a closed sublattice of $L^1(\mu)$ containing $C(X)$. Hence $\Phi(E) = L^1(\mu)$.

$$\left[\begin{array}{c} E_u \text{ is lattice ideal} \\ \text{generated by } u \end{array}\right] \longleftrightarrow \left[\begin{array}{c} \Phi(E_u) \text{ is lattice ideal in } L^1(\mu) \\ \text{generated by } 1_X \end{array}\right]$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad L_\infty(\mu)$$

Hence $C(X) = E_u = L^\infty(\mu)$.

$\blacksquare$

Now let E be an L-space with no weak order unit.

<u>DEFINITION:</u> If E is a vector lattice and D is a collection of non-zero disjoint elements of E such that the band $B(D)$ generated by D is E, then D is called an order basis.

<u>Example:</u> In ℓ^2, $D_1 = \{(\frac{1}{n})\}$ or $D_2 = \{e^{(n)} : n \in \mathbb{N}\}$ is an order basis.

One can always Zornify to obtain an order basis for any Archimedean vector lattice $E \neq \{0\}$.

Let $D = \{u_\alpha : \alpha \in A\}$ be an order basis for the L-space E such that $u_\alpha \geq 0$ and $\|u_\alpha\| = 1$. Let

$$B_\alpha = \{x \in E : x \wedge (u_\alpha - x) = 0\}$$

(base associated with u_α)

<u>Note</u>: (1) $0 \le x \le u_\alpha$ for all $x \in B_\alpha$. Also $0, u_\alpha \in B_\alpha$, so 0 and u_α are the smallest and largest elements of B_α.

(2) If $x \in B_\alpha$ and $x' = u_\alpha - x$, then $x' \in B_\alpha$ and $x' \vee x = u_\alpha$, $x' \wedge x = 0$

(3) The supremum and infimum of arbitrary subsets $\{x_\beta : \beta \in I\}$ of B_α belong to B_α. (The sups and infs exist in E since E is order complete.)

[Let $x_0 = \sup_{\beta \in I} x_\beta$. Then $0 \le u_\alpha - x_0 \le u_\alpha - x_\beta \quad \forall \beta \in I$

Then $x_\beta \wedge (u_\alpha - x_\beta) = 0 \Rightarrow x_\beta \wedge (u_\alpha - x_0) \quad \forall \beta \in I$

$$\underbrace{\Rightarrow}_{\text{dist. law}} \quad x_0 \wedge (u_\alpha - x_0) = 0 \quad \Rightarrow x_0 \in B_\alpha$$

If $y_0 = \inf_{\beta \in I} \{x_\beta\}$, then $-y_0 = \sup_{\beta \in I} \{-x_\beta\} \Rightarrow u_\alpha - y_0 = \sup_{\beta \in I} \underbrace{(u_\alpha - x_\beta)}_{\in B_\alpha} \in B_\alpha$

$\Rightarrow y_0 \in B_\alpha$]

Hence B_α is a distributive lattice with complements, largest and smallest element, and is order complete - COMPLETE BOOLEAN ALGEBRA

Therefore B_α is isomorphic as a Boolean algebra to the Boolean algebra of clopen subsets of a Stonean space X_{u_α} and this X_{u_α} is unique up to homeomorphism.

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(Proof continued)

Let E_{u_α} = ideal generated by u_α equipped with the norm $\|\cdot\|_\alpha$ with unit ball $[-u_\alpha, u_\alpha]$. Then E_{u_α} is isometric and lattice isomorphic to $C(X_\alpha)$, where X_α is a Stonean compact Hausdorff space.

$$E_{u_\alpha} \qquad\qquad C(X_\alpha)$$

$$u_\alpha \longleftrightarrow 1_{X_\alpha}$$

$B_\alpha \longleftrightarrow$ characteristic functions of clopen sets of X_α

By the uniqueness, X_α is homeomorphic to X_{u_α}.

We know that E = band $B(D)$ generated by D. The lattice ideal $I(D)$ generated by D is

$$I(D) = \bigoplus_{\alpha \in A} E_{u_\alpha}$$

↑ finite sums

Let X = topological disjoint union of X_α ($g_\alpha : X_\alpha \to X$ equip X (disjoint union of the X_α) with the finest topology that makes each injection g_α continuous.) Then X is a locally compact Hausdorff space. A Radon measure μ on X is a continuous linear functional on the locally convex space $K(X)$ of all continuous functions on X with compact support in X equipped with the topology of uniform convergence on compact sets in X.

Now

$$(*) \qquad I(D) = \bigoplus_{\alpha \in A} E_{u_\alpha} \longleftrightarrow \bigoplus_{\alpha \in A} C(X_\alpha) = K(X)$$

On E, $\|x\|$ is additive and positive homogeneous on the cone, so it extends to a strictly positive linear functional f_0 on E. Then

$$f_0\big|_{I(D)} \longleftrightarrow \mu$$

where μ is a strictly positive Radon measure.

Let $0 \le x \in E$. Since $E = B(I(D))$, it follows that

$$x = \sup\{x \wedge n \sum_{i=1}^{m} u_{\alpha_i} : \alpha_i \in A \ \ n, m \text{ arb}\}$$

directed upward set in $\oplus E_{u_\alpha}$

We get $x_\beta \uparrow x$, where $x_\beta \longleftrightarrow f_\beta \in K(X)$ with (f_β) increasing.

Now $f_0(x_\beta) \uparrow f_0(x)$, so

$$\sup_\beta \int f_\beta \, d\mu < \infty$$

Hence (f_β) is an increasing norm bounded family in $L_1(\mu)$, so $f_\beta \longrightarrow f_x$ in $L^1(\mu)$.

$$\|f_x\|_1 = \lim_\beta \|f_\beta\|_1 = \lim_\beta f_0(x_\beta) = \lim_\beta \|x_\beta\| = \|x\|$$

Therefore $x \longleftrightarrow f_x$ is an isometry as before.
Proceed as before.

$\blacksquare$

Remark: Every separable Banach lattice has a weak order unit.

Proof. Suppose D is a countable dense subset of the Banach lattice E consisting of non-zero elements.
If

$$|x| \wedge |d| = 0 \quad \forall d \in D$$

Then $x = 0$ (for $\exists\, d_n \longrightarrow x$ in norm $\Rightarrow |d_n| \longrightarrow |x|$ in norm

$$\Rightarrow |x| \wedge |x| = \lim |x| \wedge |d| = 0$$

$$\Rightarrow x = 0)$$

Let $u = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n|}{\|x_n\|}$ where $D = (x_n)$. Then

$$|x| \wedge u = 0 \Rightarrow |x| \wedge x_n = 0 \quad \forall x_n \in D$$

since $|u| \geq \frac{1}{2^n \|x_n\|} |x_n|$. Therefore $x = 0$, so u is a weak order unit.

$\blacksquare$

PROPOSITION: If E is an L-space and if F is a Banach lattice such that every norm bounded directed ($\le$) set has a sup, then for a linear map $T: E \to F$

order continuous $\Leftrightarrow$ order seq. cont. $\Leftrightarrow$ order bdd

$\Leftrightarrow$ continuous

The space $\mathcal{L}(E,F)$ of continuous linear maps of E into F is an order complete vector lattice.

Proof. Note that F is order complete since every directed ($\le$) set of positive elements is norm bounded. Hence

$$L^b(E,F) = \text{order bounded linear maps}$$

is an order complete vector lattice. To complete the proof we will show

(1) If $T \in \mathcal{L}(E,F)$, then $|T|$ exists in $L^b(E,F)$.

(2) If $0 \le S \in \mathcal{L}(E,F)$, then S is order continuous.

Recall that if $x \ge 0$,

$$|T|x = \sup\left\{ \sum_{i=1}^{n} |Tx_i| : x = \sum_{i=1}^{n} x_i,\ x_i \ge 0 \right\}$$

But

$$\left\| \sum_{i=1}^{n} |Tx_i| \right\| \leq \sum_{i=1}^{n} \|Tx_i\| \leq \|T\| \sum_{i=1}^{n} \|x_i\| = \|T\| \, \|x\|$$

L-space

Hence sup exists, so $|T|$ exists

(2) $x_\alpha \downarrow 0 \Rightarrow x_\alpha \xrightarrow{\text{norm}} 0 \Rightarrow Sx_\alpha \longrightarrow 0$ in norm

$\{Sx_\alpha\}$ is decreasing and order bounded since $S \geq 0$. Therefore $Sx_\alpha \downarrow 0$

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LEMMA: If μ_1, μ_2 are positive normal measures on a σ-Stonean compact Hausdorff space X such that μ_2 is strictly positive ($0 \le f \in C(X)$, $f \neq 0 \Rightarrow \mu(f) > 0$), then μ_1 and μ_2 are not disjoint.

Proof. WLOG $\|\mu_1\| = 1 = \|\mu_2\|$. Suppose $\mu_1 \wedge \mu_2 = 0$. Then

$$(*) \qquad 0 = (\mu_1 \wedge \mu_2)(1_X) = \inf\{\mu_1(f) + \mu_2(g) : 0 \le f, g,\ f+g = 1_X\}$$

Hence $\exists\, f_n, g_n \ge 0$ in $C(X)$ with $f_n + g_n = 1_X$ and

$$\mu_1(f_n) + \mu_2(g_n) \le 1/2^n$$

for all n.

Since $g_n \le 1_X$ and X is σ-Stonean, it follows that

$$h_n := \sup\{g_k : k \ge n\}$$

exists for all n. Also, if

$$h_{np} = \sup\{g_k : n \le k \le n+p\}$$

then $h_{np} \overset{p}{\nearrow} h_n$ for all n. Since μ_2 is normal,

$$\mu_2(h_{np}) \overset{p}{\nearrow} \mu_2(h_n) \quad \forall n$$

Now

$$h_{np} \leq g_n + g_{n+1} + \dots + g_{n+p}$$

$$\Rightarrow \mu_2(h_{np}) \leq \mu_2(g_n) + \dots + \mu_2(g_{n+p})$$

$$\leq 1/2^n + \dots + 1/2^{n+p} \leq 1/2^{n-1}$$

for all p and all n. Hence $\mu_2(h_n) \leq 1/2^{n-1}$. On the other hand, if

$$h := \inf h_n$$

(which exists by σ-Stonean property) then

$$\mu_2(h_n) \downarrow \mu_2(h)$$

so $\mu_2(h) = 0$. Observe that

$$1 = \mu_1(1_X) = \mu_1(f_n) + \mu_1(g_n) \leq 1/2^n + \mu_1(g_n)$$

Since $g_n \leq h_n \leq 1_X$, we get

$$1 - \frac{1}{2^n} \leq \mu_1(g_n) \leq 1$$

$$\Rightarrow \mu_1(g_n) \longrightarrow 1$$

$$\Rightarrow \mu_1(h_n) \longrightarrow 1 = \mu_1(h)$$

Therefore $h \neq 0$, but $\mu_2(h) = 0$ ↯

$\boxtimes$

<u>PROPOSITION</u>: If E is a Banach lattice, then the band in E^{**} generated by $Q(E)$ is exactly the band of order continuous linear functionals on E^*.

Proof. If $x \geq 0$ in E and if $f_\alpha \downarrow 0$ in E^* then

$$Qx(f_\alpha) = f_\alpha(x) \downarrow 0$$

Hence Qx is order cont. on E^*. Hence $B(QE) \subset B((E^*)^0) = E^{*0}$ = order cont. linear functionals.

Suppose φ_0 is a positive order cont. functional on E^* such that $\varphi_0 \perp B(Q(E))$. Then $\varphi_0 \perp QE$. We will show $\varphi_0 = 0$. Let

$$B_{\varphi_0} = \{f \in E^* : \varphi_0(|f|) = 0\}$$

Note: Clearly B_{φ_0} is a lattice ideal in E^*, in fact it is a band ($D \subset B_{\varphi_0}$; $D(\leq)$ D positive elements and $x_0 = \sup D$ exists in E^*. $\{f_\alpha\} \in D$ Now $0 \leq f_\alpha \uparrow f_0$ so $\underset{=0}{\varphi(f_\alpha)} \uparrow \underset{=0}{\varphi(f_0)}$

$\Rightarrow f_0 \in B_{\varphi_0}$) To prove $\varphi_0 = 0$ we will show $B_{\varphi_0}^{\perp} = \{0\}$. Suppose not. Choose $f_0 \geq 0$, $f_0 \neq 0$, $f_0 \in B_{\varphi_0}^{\perp}$. Choose $x_0 \geq 0$ $x_0 \neq 0$ so that $f_0(x_0) > 0$. Let

$$B_{x_0} = \{ f \in E^* : |f|(x_0) = 0 \}$$

B_{x_0} is also a band.

φ_0 is strictly positive on $B_{\varphi_0}^{\perp}$ and x_0 is strictly positive on $B_{\varphi_0}^{\perp}$. Choose $g_0 \in B_{x_0}$, $h \in B_{x_0}^{\perp}$ so that $f_0 = g_0 + h_0$. Then $g_0(x_0) = 0$, so $h_0(x_0) > 0$. Also, $0 \leq h_0 \leq f_0 \in B_{\varphi_0}^{\perp}$. Hence $h_0 \in B_{\varphi_0}^{\perp}$, so

$$h_0 \in B_{\varphi_0}^{\perp} \cap B_{x_0}^{\perp}$$

Consider $(E^*_{h_0}, \|\cdot\|_{h_0}) \approx C(X)$. Then

$$\varphi_0 \big|_{E^*_{h_0}}, \ Qx_0 \big|_{E^*_{h_0}} \longleftrightarrow \mu_1, \mu_2$$

where μ_1, μ_2 are both strictly positive normal measures. Therefore μ_1 and μ_2 are not disjoint by the lemma. But μ_1 and μ_2 are disjoint by construction. Hence $\varphi_0 = 0$.

$\blacksquare$

<u>THEOREM</u>: Suppose E is a Banach lattice. TFAE

(1) E is weakly sequentially complete

(2) Each monotone increasing norm bounded sequence converges in norm

(3) $Q(E)$ is a band in E^{**}

(4) $Q(E) = E^{*\circ}$

(5) No closed sublattice of E is topologically isomorphic and lattice isomorphic to c_0.

Proof (1) $\Rightarrow$ (2) $x_n \le x_{n+1} \le \cdots$ If $0 \le f \in E^*$, then

$$\sup f(x_n) < +\infty$$

since (x_n) is bounded, so (x_n) is weakly Cauchy. Therefore $x_n \to x_0$ weakly, so $x_n \to 0$ in norm (normality of cone and monotone convergence)

(3) $\Rightarrow$ (4) from last proposition

11/2 BANACH LATTICES

We need the following proposition to prove the last theorem.

<u>PROPOSITION</u>: If E is a Banach lattice and if M is a closed lattice ideal in E such that $M \neq M^{\perp\perp}$, then c_0 is isomorphic as a Banach lattice to a closed sublattice of M. If E is order complete, we can actually get that ℓ_∞ is isomorphic to a closed sublattice of $M^{\perp\perp}$.

Proof. Choose $0 \le u \in M^{\perp\perp} \setminus M$. Then $(E_u, \|\cdot\|_u)$ is an M-space with strong unit u which is isomorphic to some $C(X)$, X compact Hausdorff

$$u \longleftrightarrow 1_X$$

Let $M \cap E_u = M_u$. Then M_u is a closed lattice ideal in E_u. Since closed lattice ideals in $C(X)$ = closed algebraic ideals, we can find a closed set A in X s.t.

$$M_u \approx \{f \in C(X) : f(t) = 0 \ \ \forall t \in A\}$$

$A \neq \emptyset$ since $u \notin M$, so $1_X \notin M_u$. Also, $u = \sup\{[0,u] \cap M\}$, i.e. $1_X = \sup\{[0, 1_X] \cap M_u\}$. Therefore A has empty interior. Therefore A is a closed nowhere dense set.

Let

$$\mathcal{B} = \{B \subset X : B \text{ is clopen and } B \cap A = \emptyset\}$$

Then $\mathcal{B}$ is directed $(\le)$

<u>Case 1</u>: Suppose B does not have a largest element. Consider the set $\{\chi_B : B \in \mathcal{B}\} =: D \in C(X)$. The filter $\mathcal{F}(D)$ of sections of D does not converge for the norm on $E_M = C(X)$ induced by E, for if $\mathcal{F}(D) \longrightarrow f_0 \in C(X) = E_M$, then $f_0 = \sup D$. But

$$\chi_B \wedge (1_X - \chi_B) = 0$$

and so

$$f_0 \wedge (1_X - f_0) = 0$$

$$\Rightarrow f_0 = \chi_{B_0}$$

where B_0 is clopen. Also, $B_0 \cap A = \phi$, since $f_0 \in M$ and M is closed $\Rightarrow f_0 \in M_M \Rightarrow f_0(t) = 0 \ \forall t \in A \Rightarrow B_0 \cap A = \phi$. Therefore $\mathcal{B}$ would have B_0 as largest element $\lightning$.

Since $\mathcal{F}(D)$ does not converge, $\exists\, \varepsilon > 0$ and increasing sequence $B_n \subset \mathcal{B}$ s.t.

$$\| \chi_{B_{n+1}} - \chi_{B_n} \| \ge \varepsilon_0$$

Now

$$x_n \in E_M \longleftrightarrow \chi_{B_{n+1}} - \chi_{B_n} \in C(X)$$

where $x_n \in [0,u]$ since $\chi_{B_{n+1}} - \chi_{B_n} \in [0, 1_X]$. The x_n's are disjoint in $[0,u]$ and $\|x_n\| \geq \varepsilon_0$ for all n. By the homework problem

$$(\lambda) \longmapsto \sum_{n=1}^{\infty} \lambda_n x_n$$

is a lattice and top isomorphism of c_0 into E. But

$$\sum_{n=1}^{k} \lambda_n x_n \in M$$

so $\sum_{n=1}^{\infty} \lambda_n x_n \in M$. Hence $c_0 \hookrightarrow M$

Case 2: Suppose $\mathcal{B}$ has a largest element B_0. Then $\chi_{B_0} \in M_u$. Since $B_0 \cap A = \phi$, we can't have $B_0 \cup A = X$, so there is a $t_0 \in X$ s.t. $t_0 \notin B_0 \cup A$. Let $C = B_0 \cup \{t_0\}$. Then C is closed and $C \subset A^c$. We can choose open sets G_n s.t.

$$C \subset G_n \subset \overline{G_n} \subset G_{n+1} \subset A^c$$

for all n. By Urysohn $\exists\ (f_n) \in C(X)$ s.t. $0 \leq f_n \leq 1$ and

$$f_n(\overline{G_n}) = 0 \quad f(G_{n+1}^c) = 0$$

Then $f_n(t) = 0 \ \forall t \in A$, so $f_n \in M_u \ \forall n$. If f_n converges in the norm of E to $f_0 \in M_u$, then $f = \sup f_n$ since (f_n) is increasing. Hence f is the characteristic function of a clopen set C_1 disjoint from A. Then $t_0 \in C_1$ but $t_0 \notin B_0$, so B_0 is not the largest element of $\mathcal{B}$ is.

Choose $\varepsilon > 0$ and subsequence (f_{n_k}) s.t

$$\| f_{n_{k+1}} - f_{n_k} \| \geq \varepsilon$$

for all k. Set $g_k = f_{n_k}$. Notice that if $k < l < m < n$, then

$$(g_l - g_k) \wedge (g_n - g_m) = 0$$

Let $x_n = g_{2n} - g_{2n-1}$ and $\|x_n\| \geq \varepsilon_0$. HW $\Rightarrow c_0 \hookrightarrow M$.

Now suppose E is order complete. Take $(\lambda_n) \in \ell^\infty$. Then

$$\sum_{n=1}^{k} \lambda_n x_n \leq \|(\lambda_n)\|_\infty u$$

(since x_n's are disjoint) Also $\sum_{n=1}^{k} \lambda_n x_n \in M$. For $(\lambda_n) \geq 0$ in ℓ_∞, define

$$T((\lambda_n)) = \sup_k \sum_{n=1}^{k} \lambda_n x_n \in M^{\perp\perp}$$

This defines a top and lattice isomorphism of ℓ^∞ into $M^{\perp\perp}$.

$\blacksquare$

__COROLLARY:__ If E is an order complete Banach lattice, then E has an order continuous norm iff $\ell^\infty \not\hookrightarrow E$

11/5 BANACH LATTICES

(Proof continued) Suppose ℓ^∞ is not isomorphic to a closed sublattice of E, then if M is a closed lattice ideal in E, then $M = M^{\perp\perp}$. Hence every closed lattice ideal in E is a band.

Suppose $x_\alpha \downarrow 0$ where $x_\alpha \in [0, x_0]$ for all α. Given $\varepsilon > 0$ let I_α = closed lattice ideal generated by $(x_\alpha - \varepsilon x_0)^+$. Let P_α be the corresponding band projection onto I_α. Then

$$(I - P)(x_\alpha - \varepsilon x_0) = -(x_\alpha - \varepsilon x_0)^- \leq 0$$

Therefore

$$\|x_\alpha\| \leq \|P_\alpha x_\alpha\| + \|(I-P)x_\alpha\| \leq \|P_\alpha x_0\| + \|(I - P_\alpha)\varepsilon x_0\|$$

(*)

$$\leq \|P_\alpha x_0\| + \varepsilon \|x_0\|$$

We want to show $\|P_\alpha x_0\|$ is small for "large" α. Let $y_\alpha = (I - P_\alpha)x_0$. Note that

$$x_\alpha \geq P_\alpha x_\alpha \geq \varepsilon P_\alpha x_0$$

so that $0 = \inf x_\alpha \geq \inf \varepsilon P_\alpha x_0 \geq 0$. Hence $\inf P_\alpha x_0 = 0$. Therefore

$$\sup y_\alpha = x_0 - \inf P_\alpha x_0 = x_0$$

The band $B(\{y_\alpha : \alpha \in A\}) = \{y_\alpha : \alpha \in A\}^{\perp\perp}$ contains $y_{\alpha'}$ for all α'. Therefore $x_0 \in B(\{y_\alpha\})$ = closed lattice ideal generated by the family $\{y_\alpha\}$. Hence there is an $y_0 \in I(\{y_\alpha\})$ s.t.

$$\|x_0 - y_0\| < \varepsilon$$

The family $\{y_\alpha\}$ is directed $(\leq)$ so there exist α_0 and a number β_0 such that

$$0 \leq y_0 \leq \beta_0 y_{\alpha_0}$$

Now $x_0 - x_0 \wedge y_0 = (x_0 - y_0)^+$, so

$$|x_0 - x_0 \wedge y_0| \leq |x_0 - y_0|$$

Then

$$x_0 \wedge y_0 \leq x_0 \wedge \beta_0 y_{\alpha_0} \leq \text{projection of } x_0 \text{ onto the band generated by } y_{\alpha_0}$$

$$\leq y_{\alpha_0} = (I - P_{\alpha_0})x_0 \leq x_0$$

$$\Longrightarrow \|P_{\alpha_0} x_0\| = \|x_0 - y_{\alpha_0}\| \leq \|x_0 - x_0 \wedge y_0\|$$

$$\leq \|x_0 - y_0\| < \varepsilon$$

Hence $\|x_\alpha\| \to 0$ in $(*)$, so E has order continuous norm. $\boxtimes$

<u>**COROLLARY:**</u> An order complete separable Banach lattice has order continuous norm.

Remainder of an earlier proof: If every continuous linear functional on a Banach lattice is order continuous, then every order interval in E is weakly compact.

Proof. If every continuous linear functional on E is order continuous, then E is order complete. (c.f. $(3) \Rightarrow (4)$). To see this, let S be a directed ($\leq$) set in the positive cone of E with an upper bound. Let $\mathcal{U}$ = all upper bounds of S. Then $\mathcal{U}$ is directed ($\geq$). Then $\mathcal{U} - S$ is directed ($\geq$) and $\inf(\mathcal{U} - S) = 0$ (since E is Archimedean). Let $D = \mathcal{U} - S$. Then since every continuous linear functional is order continuous, it follows that $\mathcal{F}(D) \to 0$ weakly, so $\mathcal{F}(D) \longrightarrow 0$ in norm since D is directed ($\geq$) and the cone is normal. Therefore $\mathcal{F}(S)$ is norm Cauchy (c.f. $(3) \Rightarrow (4)$). But then $\mathcal{F}(S) \longrightarrow x_0$ in norm and $x_0 = \sup S$.

Now suppose $0 \leq x \in E$. Then $(E_x, \|\cdot\|_x)$ is an order complete M-space, so $(E_x, \|\cdot\|_x) \approx C(X)$ where X is Stonean compact. Since E^* separates points of E and hence of E_x and every continuous linear functional is order continuous, it follows that the space $N(X)$ of normal measures on X separates points of $C(X)$. Therefore X is hyperstonean. By Dixmier's theorem, $C(X) = N(X)^*$. Let $T: C(X) \longrightarrow E$ be the canonical injection. $T \geq 0$, so $T^*: E^* \longrightarrow C(X)^*$ is positive and order continuous

($\mu_\alpha \downarrow 0 \Rightarrow (\inf T^* \mu_\alpha)(f) = \inf \{ T^* \mu_\alpha (f) \} = \inf \mu_\alpha (Tf) = 0$
$\forall f \geq 0$ in $C(X)$, so $\inf T^* \mu_\alpha = 0 \Rightarrow T^* \mu_\alpha \downarrow 0$)

T^* maps order continuous functionals on E into normal measures, so $T^*(E^*) \subset \eta(X)$. Therefore T is continuous for $\sigma(C(X), \eta(X))$ and $\sigma(E, E^*)$, so T is weakly compact.

$$[-x, x] \approx T[-1_X, 1_X] = \text{weakly compact}$$

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PROPOSITION: If E is a Banach lattice with an order continuous norm, then every band projection on E^* is $\sigma(E^*,E)$ - continuous and every band in E^* is $\sigma(E^*,E)$ - closed

Proof. Let τ be the topology on E^* of uniform convergence on order bounded in E. τ is given by the family of seminorms

$$\mathcal{P} = \{p_x : 0 \le x \in E\}$$

where

$$p_x(f) = |f|(x) = \sup_{z \in [-x,x]} f(z)$$

Since E has order continuous norm, each order bounded set in E is relatively weakly compact. Therefore, by the Mackey-Arens Theorem

$$[E^*(\tau)]^* = E$$

Therefore the weak topology for τ is just $\sigma(E^*,E)$
Let P be a band projection on E^*. Then

$$|Pf| \le |f| \implies p_x(Pf) \le p_x(f) \quad \forall f \in E^*$$

for all $0 \le x \in E$. Therefore P is τ continuous, so P is $\sigma(E^*,E)$ continuous.

If B is a band in E^*, then B is a projection band since E^* is order complete. Let P be the band projection onto $B^{\perp}$. Then

$$B = P^{-1}(0)$$

so B is $\sigma(E^*, E)$-closed since P is $\sigma(E^*, E)$ continuous.

$\boxed{/\!/}$

Example: (1) c_0 has order continuous norm, so every band in ℓ_1 is w^* closed ($\sigma(\ell_1, c_0)$)

(2) ℓ_∞ does not have an order continuous norm. ℓ_1 is a band in $(\ell_\infty)^*$. But ℓ_1 is not w^* closed in ℓ_∞^*

<u>PROPOSITION</u>: If A is a relatively weakly compact subset of an L-space E, then the solid hull

$$S_A = \{y \in E : |y| \le |x|, x \in A\}$$

is relatively weakly compact.

Proof. By the Eberlein theorem it suffices to show that every sequence (y_n) in S_A has a weakly convergent subsequence. For each n, choose $x_n \in A$ s.t. $|y_n| \le |x_n|$. Define

$$x = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n|}{\|x_n\|}$$

Let E_x be the lattice ideal generated by x. Then $\{x_n\}$, $\{(y_n)\}$ are subsets of E_x. Also, x is a weak unit in E_x. The norm closure $\overline{E_x}$ in E is an L-space and x is a weak unit in $\overline{E_x}$
For suppose $u \wedge x = 0$ for $u \in \overline{E_x}$. E has order continuous norm, so $\overline{E_x}$ is the band generated by E_x (since $l_\infty \not\hookrightarrow E$) There exists $u_\alpha \uparrow u$ where $u_\alpha \in E_x$ and $u_\alpha \geq 0$. Then $u_\alpha \wedge x = 0 \Rightarrow u_\alpha = 0$ $\Rightarrow u = 0$).
Hence

$$\overline{E_x} \approx L^1(\mu, X)$$

μ strictly positive Radon measure on a compact X

$$x_n \leftrightarrow f_n \in L^1$$

$$y_n \leftrightarrow g_n \in L^1$$

where $|f_n| \geq |g_n|$. Since $\{f_n\}$ is relatively weakly compact,

$$\lim_{\mu(E) \to 0} \sup_n \int_E |f_n| \, d\mu = 0$$

$$\Rightarrow \lim_{\mu(E) \to 0} \sup_n \int_E |g_n| \, d\mu = 0$$

So (g_n) is relatively weakly compact in $L_1(\mu) \Rightarrow y_n$ has a weakly convergent subsequence. ▨

<u>PROPOSITION</u>: (Grothendieck) If X is a compact Hausdorff space and if A is a norm bounded subset of $C(X)^* = M(X)$, TFAE

(1) A is relatively weakly compact

(2) If $(f_n) \subset C(X)$ converges weakly to 0, then $\lim \mu(f_n) = 0$ unif. in $\mu \in A$;

(3) If (f_n) is a norm bounded disjoint sequence in $C(X)$, then $\lim \mu(f_n) = 0$ unif in A

(4) If $(\mathcal{U}_n)$ is a disjoint sequence of open sets in X, then $\lim \mu(\mathcal{U}_n) = 0$ unif. for $\mu \in A$.

Proof. (1) $\Rightarrow$ (2) Let $(f_n) \longrightarrow 0$ weakly. Then $(|f_n|) \longrightarrow 0$ weakly. For each $m \in N$ let

$$g_m(x) := \sup_{n \geq m} |f_n(x)|$$

Then g_m is a Baire-$\underline{1}$ function on X, (g_m) is decreasing, and $g_m(x) \longrightarrow 0$ pointwise. By MCT, $\mu(g_m) \longrightarrow 0$ $\forall \mu \in C(X)^*$

Since A is relatively weakly compact, so is $|A| = \{|\mu| : \mu \in A\}$ Let Y = weak closure of $|A|$ in $C(X)^*$.

$$\mu(g_m) \xrightarrow{m} 0 \quad \forall \mu \in Y$$

We can regard each g_m as a continuous function on Y for the topology induced on Y by $\sigma(C(X)^*, C(X)^{**})$. $g_n(x) \downarrow 0$, so by Dini's theorem, $g_n \longrightarrow 0$ unif on Y, i.e. $|\mu|(g_m) \longrightarrow 0$ unif. for $\mu \in A$ But

$$|\mu(f_n)| \le |\mu|(|f_n|) \le |\mu|(|g_n|)$$

Therefore $\mu(f_n) \to 0$ unif for $\mu \in A$.

(2) $\Rightarrow$ (3) (f_n) norm bounded & disjoint $\Rightarrow (f_n(x)) \to 0$ pointwise and is unif. bdd $\Rightarrow (f_n) \to 0$ weakly

(3) $\Rightarrow$ (4) Suppose $\mu(U_n)$ does not approach 0 unif in $\mu \in A$. $\exists$ compact $C_n \subset U_n$ s.t. $\mu(C_n)$ does not approach 0 unif for $\mu \in A$
Take

$$f_n = \begin{cases} 1 & \text{on } C_n \\ 0 & \text{on } U_n^c \end{cases}$$

where $0 \le f_n \le 1$. This will give a contradiction to (3).

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(Proof continued)

(4) $\Rightarrow$ (1) We use the following lemma

<u>LEMMA:</u> Suppose $A = \{\mu_j\}$ satisfies (4). Then for each compact $B \subset X$ and each $\eta > 0$, there is an open set $\mathcal{U} \supset B$ s.t.

$$|\mu_j|(\overline{\mathcal{U}} \setminus B) < \eta \quad \forall j$$

Proof. Use the regularity of the μ_j's to choose a sequence $(\mathcal{U}_n)$ of open sets such that

$$\mathcal{U}_n \supset \overline{\mathcal{U}_{n+1}} \supset B$$

and $|\mu_j|(\overline{\mathcal{U}}_n \setminus B) < 1/n$ for $j = 1, \dots, n$. Then

$$|\mu_j|(B) \le |\mu_j|(\mathcal{U}_n) \le |\mu_j|(\overline{\mathcal{U}}_n) \le |\mu_j|(\overline{\mathcal{U}}_n \setminus B) + |\mu_j|(\overline{\mathcal{U}}_n \cap B)$$

$$\le \underbrace{|\mu_j|(\overline{\mathcal{U}}_n \setminus B)}_{\to 0 \text{ as } n \to 0} + |\mu_j|(B)$$

Hence $\overset{(*)}{\lim_n} |\mu_j|(\mathcal{U}_n) = |\mu_j|(B)$ for all j. If this limit $(*)$ is not uniform in j, then for some $\varepsilon_0 > 0$ there would exist $\mu_{j_p}, \mathcal{U}_{k_p}$ such that

$$|\mu_{j_p}|(\mathcal{U}_{k_p}) - |\mu_{j_p}|(B) > 2\varepsilon_0$$

$$|\mu_{j_p}|(\overline{U}_{k_{p+1}}) - |\mu_{j_p}|(B) < \varepsilon_0$$

Therefore

$$|\mu_{j_p}|(U_{k_p}) - |\mu_{j_p}|(\overline{U}_{k_{p+1}})$$

$$= \left[|\mu_{j_p}|(U_{k_p}) - |\mu_{j_p}|(B)\right] - \left[|\mu_{j_p}|(\overline{U_{k_{p+1}}}) - |\mu_{j_p}|(B)\right]$$

$$> 2\varepsilon_0 - \varepsilon_0 = \varepsilon_0$$

$$\Longrightarrow \quad |\mu_{j_p}|(U_{k_p} \setminus \overline{U}_{k_{p+1}}) > \varepsilon_0$$

Since $|\mu|(E) \leq 4 \sup\{|\mu(F)| : F \subset E\}$, it follows that there is an open set $W_p \subset U_{k_p} \setminus \overline{U}_{k_{p+1}}$ s.t.

regularity

$$|\mu_{j_p}(W_p)| > \varepsilon_0/4$$

Then $\{W_p : p \in \mathbb{N}\}$ are disjoint open sets which do not satisfy (4)

Now to show (4) $\Rightarrow$ (1), i.e. to show A is relatively weakly compact, it would suffice to show that each sequence (μ_n) in A has a weakly convergent subsequence. Define

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} |\mu_n|$$

Then (μ_n) is in the band $B(\mu)$ generated by μ. Note that $I(\mu)$

$$\left(I(\mu) = \{ \nu \in C(X)^* : |\nu| \le \alpha\mu, \text{ some } \alpha \} \right)$$

is contained in the set of measures absolutely continuous w.r.t. μ. Since $C(X)^*$ has order continuous norm,

$$B(\mu) = \overline{I(\mu)}$$

and so $B(\mu)$ is contained in the set of measures absolutely continuous w.r.t. μ. By RNP $\Rightarrow \exists f_j \in L^1(\mu, X)$ s.t.

$$\mu_j(E) = \int_E f_j \, d\mu$$

To show that μ_j has a weakly convergent subsequence, it would suffice to show that the measures

$$E \longrightarrow \mu_j(E)$$

are uniformly countably additive (Dunford-Pettis). Suppose not. Then there is a disjoint sequence (C_i) of Borel sets and an $\varepsilon_0 > 0$ s.t. for each n there exist j_n and $k_n > n$ s.t.

$$\left| \mu_{j_n} \left(\bigcup_{i=k_n}^{\infty} C_i \right) \right| > \varepsilon_0$$

Choose k_n s.t.

$$|\mu_{i_n}(\underbrace{\bigcup_{i=k_n}^{k_{n+1}-1} C_i}_{B_n})| > \varepsilon_0$$

Thus we can find a disjoint sequence (B_j) of Borel sets and a subsequence (ν_j) of (μ_n) s.t.

$$|\nu_j(B_j)| > \varepsilon_0$$

Because of the regularity of ν_j we can assume B_j is compact. Since $(\nu_j) \subset A$, (4) holds for (ν_j) so lemma implies that there exists open sets $U_j \supset B_j$ s.t.

$$|\nu_j|(\overline{U_k} \setminus B_k) < 2^{-(k+1)}\varepsilon_0$$

for all j,k. Let $V_k = U_k \setminus \bigcup_{j=1}^{k-1} \overline{U_j}$. Then

$$V_k \triangle B_k \subset (U_k \triangle B_k) \cup \left(\bigcup_{j=1}^{k-1} (\overline{U_j} \triangle B_j)\right)$$

since the B_j's are disjoint. Hence

$$|\nu_k|(V_k \triangle B_k) \le \frac{\varepsilon_0}{2^{k+1}} + \sum_{j=1}^{k} \frac{\varepsilon_0}{2^j} \le \frac{\varepsilon_0}{2}$$

$$\Rightarrow |\nu_k(V_k)| > \varepsilon_0/2 \quad \lightning$$

$\blacksquare$

11/12 BANACH LATTICES

<u>PHILLIP'S LEMMA</u>: If (λ_n) is a sequence in $ba(\mathbb{N})$ that converges to 0 for $\sigma(ba(\mathbb{N}), \ell^\infty)$ and if

$$\nu_n = \lambda_n \Big|_{c_0}$$

then $\|\nu_n\|_1 \to 0$

Proof: See Day's book

<u>THEOREM</u>: Suppose that X is a σ-Stonean compact Hausdorff space. Then if $(\mu_n) \subset C(X)^*$ converges weak*, then (μ_n) converges weakly.

Proof. WLOG we can assume $\mu_n \to 0$ w^*. Hence (μ_n) is norm bounded.

To prove $\mu_n \to 0$ weakly, it would suffice to show that μ_n is weakly relatively compact. For then every subsequence has a convergent subsequence, so the original sequence must converge.

Suppose (μ_n) is not relatively weakly compact. By Grothendieck's theorem there exists a disjoint norm bounded sequence (f_k) in $C(X)$ s.t.

$$\lim_k \mu_n(f_k)$$

is not uniform in n. Hence there is an $\varepsilon_0 > 0$ and

Subsequences (ν_n) of (μ_n), (g_k) of (f_k) s.t.

$$|\nu_n(g_n)| > \varepsilon_0 \quad \forall n$$

For each subset $J \subset \mathbb{N}$ define

$$g_J := \sup_{n \in J} g_n$$

(σ-Stonean used here). Define λ_n on the power set $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$ by

$$\lambda_n(J) = \nu_n(g_J)$$
$$\lambda_n(\emptyset) = 0$$

Then $(\lambda_n) \subset ba(\mathbb{N})$ [$J_1 \cap J_2 = \emptyset \Rightarrow g_{J_1} \wedge g_{J_2} = 0$ by disjointness

$$\Rightarrow g_{J_1 \cup J_2} = g_{J_1} \vee g_{J_2} = g_{J_1} + g_{J_2}$$

$$\Rightarrow \lambda_n(J_1 \cup J_2) = \lambda_n(J_1) + \lambda_n(J_2)$$

Also $|\lambda_n(J)| \leq \|\nu_n\| \sup_k \|g_k\|$]. Observe

$$\lim_n \lambda_n(J) = \lim_n \nu_n(g_J) = 0$$

$\nu_n \to 0 \ w^*$

Hence $\lambda_n \to 0$ weak* (Given $\varepsilon > 0$ and $u \in \ell^\infty$ choose a simple functions v on $\mathbb{N}$ s.t. $\|u - v\| < \varepsilon / 2\sup\|\lambda_n\|$. Then $\lambda_n(v) \to 0$, so $\lambda_n(u) \to 0$.)

By Phillip's Lemma, if $\rho_n = \lambda_n|_{c_0}$, then $\|\rho_n\|_1 \longrightarrow 0$. Hence

$$\sum_{j=1}^{\infty} |\lambda_n(\{j\})| \xrightarrow{n} 0$$

$$\Longrightarrow |\lambda_n(\{n\})| = |\nu_n(g_n)| \longrightarrow 0$$

But $|\nu_n(g_n)| > \varepsilon_0$ ↯

$\boxed{/}$

THEOREM: If E is a Banach lattice, TFAE

① E is weakly sequentially complete

② Each monotone increasing norm bounded sequence in E converges in norm.

③ $Q(E)$ is a band in E^{**}

④ $c_0 \not\hookrightarrow E$

Proof. (1) $\Rightarrow$ (2) (x_n) monotone increasing norm bounded $\Rightarrow \sup f(x_n) < +\infty$ for all $0 \le f \in E^*$ $\Rightarrow$ (x_n) is weak Cauchy. Hence $x_n \longrightarrow x_0$ weakly. But x_n increasing $\Rightarrow x_n \longrightarrow x_0$ in norm

(2) $\Rightarrow$ (3) E is σ-order complete. (Let $(x_n) \geq 0$ be majorized in E by x_0. Let $y_n = \sup_{1\le k\le n} x_k$. Then $y_n \uparrow$ and is majorized by x_0.

(y_n) is norm bounded because since $y_n \in [0, x_0]$ $\forall n$. Therefore y_n converges in norm to y_0, and $y_0 = \sup y_n = \sup x_k$.)

If $(y_n) \downarrow 0$ in E, then $-y_n \uparrow 0$ and $(-y_n)$ is norm bounded so $-y_n$ converges to some $y_0 \Rightarrow y_0 = 0 \Rightarrow y_n \to 0$ in norm. By an earlier result, $Q(E)$ is a lattice ideal in E^{**}

Suppose $A \subset Q(E)$ consists of positive elements and that $u = \sup A$ exists in E^{**}. A is norm bounded since $A \subset [0, u]$ in E^{**}

directed $(\leq)$

Let

$$M = \sup\{\|x\| : x \in A\}$$

Suppose the filter $\mathcal{F}(A)$ of sections of A is not Cauchy. Then there is an $\varepsilon_0 > 0$ such that for all $x \in A$, there exist $y_x \in A$ $y_x \geq x$ with $\|y_x - x\| \geq \varepsilon_0$. Chose $x \in A$ s.t. $\|x_1\| \geq M - 1$. After x_k has been selected chose $x_{k+1} \in A$ s.t. $x_{k+1} \geq y_{x_k}$ and

$$\|x_{k+1}\| \geq M - \frac{1}{k+1}$$

Then $(x_n) \uparrow$ is norm bounded, so must converge. However

$$\|x_{n+1} - x_n\| \geq \|y_{x_n} - x_n\| \geq \varepsilon_0$$

Hence $\mathcal{F}(A)$ converges in norm. The limit must be the supremum u of $A \Rightarrow u \in Q(E)$ since $Q(E)$ is norm closed in E^{**}.
Hence $Q(E)$ is a band in E^{**}

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(Proof continued)

(3) $\Rightarrow$ (4) Suppose $T: c_0 \to E$ is a topological and lattice isomorphism. If $e^{(n)}$ is the n^{th} unit vector in c_0, let

$$z_n := \sum_{k=1}^{n} Te^{(k)}$$

Since $T \geq 0$, $\{z_n\}$ is monotone increasing. Also

$$\|z_n\| \leq \|T\| \, \|\sum_{k=1}^{n} e^{(k)}\|_{c_0} = \|T\|$$

since (z_n) is norm bounded. Therefore (Qz_n) is norm bounded and increasing. (Qz_n) has a w^*-cluster point u, and $u = \sup Qz_n$ since Qz_n increases and the cone in E^{**} is w^*-closed. Hence $u \in Q(E)$ since $Q(E)$ is a band in E^{**}, so $\exists z \in E$ s.t. $Qz = u$. Note $z = \sup z_n$ because Q is a lattice isomorphism

$$\left[\!\left[Q(z \wedge z_n) = Q(z) \wedge Q(z_n) = u \wedge Q(z_n) = Q(z_n) \Rightarrow z \wedge z_n = z_n \right.\right.$$
$$\left.\left. \Rightarrow z \geq z_n \ \forall n, \text{ etc} \right]\!\right]$$

The unit ball in c_0 is mapped into $[-z, z]$. For if $x \in c_0$, then

$$x = \sum_{n=1}^{\infty} x_k e^{(k)}$$

since $\|x - \sum_{n=1}^{k} x_n e^{(n)}\|_{c_0} = \sup_{n \geq k+1} |x_n| \to 0$. Therefore

$$|Tx| = \left|\sum_{k=1}^{\infty} x_k Te^{(k)}\right| \leq \sum_{k=1}^{\infty} Te^{(k)} = z$$

if $\|x\|_{c_0} \leq 1$, then $|x_k| \leq 1 \ \forall k$

E has order continuous norm by (3), so $[-z,z]$ is weakly compact. Hence T is weakly compact $\Rightarrow c_0$ is reflexive $\lightning$.

(4) $\Rightarrow$ (3) If no closed sublattice of E is isomorphic to c_0, then $M = M^{\perp\perp}$ for every closed lattice ideal M in E. Also, ℓ_∞ is not isomorphic to a closed lattice ideal of E, so $Q(E)$ is a lattice ideal in E^{**} which is closed. If $Q(E) \neq Q(E)^{\perp\perp}$ the same earlier result would imply that $Q(E)$ contains a sublattice isomorphic to c_0. But then $c_0 \hookrightarrow E$ $\lightning$. Therefore $Q(E) = Q(E)^{\perp\perp}$ so $Q(E)$ is a band in E^{**}.

(3) $\Rightarrow$ (1) Suppose (x_n) is weakly Cauchy sequence. Then (Qx_n) is norm bounded in E^{**} and a weak* Cauchy sequence. Therefore $Qx_n \rightarrow \varphi \in E^{**}$ weak*

Want to show $\varphi \in Q(E)$. For each $f \geq 0$ in E^*, define

E^*_f = lattice ideal generated by f equipped with norm $\|\cdot\|_f$ with unit ball $[-f,f]$

Then E^*_f is an order complete M space with strong unit f, so

$E^*_f \approx C(X_f)$ for compact Stonean X_f. Let τ be the topology on E^{**} of uniform convergence on the order intervals in E^*

Claim: $E^{**}(\tau) \hookrightarrow \prod \{ C(X_f)^* : 0 \le f \in E^* \} =: F$

$$\varphi \longmapsto \left(\varphi|_{E^*_f} : 0 \le f \in E^* \right)$$

Claim: $[E^{**}(\tau)]^* = I(E^*)$ = lattice ideal generated by E^* in E^{***}

Why? τ is a locally convex lattice topology finer than $\sigma(E^{**}, E^*)$ so $[E^{**}(\tau)]^*$ will be a lattice ideal in E^{***} containing E^*. Hence $[E^{**}(\tau)]^* \supset I(E^*)$. The topology τ_2 on E^{**} of uniform convergence on order intervals in $I(E^*)$ is finer than τ and by the Mackey Arens theorem, the dual $[E^{**}(\tau_2)]^* = I(E^*)$ [[order intervals in $I(E^*)$ are also order intervals in E^{***} and order intervals in E^{***} are w^*-compact and therefore $\sigma(I(E^*), E^{**})$ compact.]]

$$\tau_2 \ge \tau \ge \sigma(E^{**}, I(E^*))$$

Hence $E^{**}(\tau)$ still embeds in F for weak topology $\sigma(E^{**}, I(E^*))$ on $E^{**}(\tau)$ and product of weak topologies $\sigma(C(X_f)^*, C(X_f)^{**})$.

$$Qx_n \longrightarrow \varphi \quad \text{for } \sigma(E^{**}, E^*)$$

$$\left(Qx_n|_{E^*_f} \right) \longrightarrow \varphi_{E^*_f} \quad \text{for } \sigma((E^*_f)^*, E^*_f) \quad \text{since } E^*_f \subset E^*$$

By Grothendieck $Qx_n|_{E_\xi^*} \longrightarrow \varphi|_{E_\xi^*}$ for $\sigma\big((E_\xi^*)^*, (E_\xi^*)^{**}\big)$

$$\begin{array}{cc} \uparrow & \uparrow \\ C(x_\xi)^* & C(x_\xi)^{**} \end{array}$$

$$\Longrightarrow \quad Qx_n \longrightarrow \varphi \quad \text{for } \sigma(E^{**}, I(E^*))$$

Since τ is a locally convex lattice topology on E^{**}, every band in E^{**} is closed for τ $(|P\xi| \le |\xi|)$. Hence $Q(E)$ is closed for τ and hence for $\sigma(E^{**}, I(E^*))$, so

$$Qx_n \longrightarrow \varphi$$

$$\Longrightarrow \quad \varphi \in Q(E)$$

$$\Longrightarrow \quad \varphi = Qx_0 \;, \; x_0 \in E$$

$$\Longrightarrow \quad x_n \longrightarrow x_0 \text{ weakly}$$

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THEOREM: If E is a Banach lattice TFAE

(1) E is reflexive
(2) no closed sublattice of E is isomorphic to c_0 or ℓ_1
(3) no closed sublattice of E or E^* is isomorphic to c_0
(4) E, E^* have monotone convergence property

Proof (1) $\Rightarrow$ (2) Obvious

(2) $\Rightarrow$ (3) Suppose no sublattice is isomorphic to c_0 but there is a sublattice of E^* that is isomorphic to c_0. Now E is order complete and E^* = all order continuous linear functionals on E. Let $T: c_0 \longrightarrow E^*$ be a lattice isomorphism into E^*. Let $f_n = Te^{(n)}$ where $(e^{(n)})$ is unit vector basis. Then $f_n \wedge f_m = 0$ for $n \neq m$ since T is a lattice isomorphism. For each $\lambda = (\lambda_n) \in c_0$, we have

$$\lambda = \sum_{n=1}^{\infty} \lambda_n e^{(n)}$$

so $T(\lambda) = \sum \lambda_n f_n$. Since T is a topological isomorphism, $\exists$ constants m, M s.t.

$$\forall (\lambda_n) \in c_0 \qquad m \|(\lambda_n)\|_{c_0} \leq \|\sum \lambda_n f_n\| \leq M \|(\lambda_n)\|_{c_0}$$

Let $B(f_n)$ be the band in E^* generated by f_n. $E^* = B(f_n) \oplus B(f_n)^{\perp}$ since E^* is order complete. It can be shown that $E = B(f_n)^{\circ} \oplus (B(f_n)^{\perp})^{\circ}$

where $A^\circ = \{x \in E : f(x) = 0 \ \forall f \in A\}$, $A \subset E^*$. Moreover, $B(f_n)^\circ$ and $(B(f_n)^\perp)^\circ$ are bands in E. (Penultimate statement is a consequence that E^* = all order cont. functionals on E and E is order complete.)

For each $n \in \mathbb{N}$,

$$m = m\,\|e^{(n)}\|_{c_0} \le \|f_n\| \le M\,\|e^{(n)}\|_{c_0} = M$$

Now

$$\|f_n\| = \sup_{\substack{\|x\|=1 \\ x \ge 0}} f_n(x) = \sup_{\substack{\|x\|=1 \\ x \ge 0 \\ x = y+z,\ y \in B(f_n)^\circ,\ z \in (B(f_n)^\perp)^\circ}} \big(\underbrace{f_n(y)}_{0} + f_n(z)\big)$$

Therefore, we can choose $x_n \in (B(f)^\perp)^\circ$ s.t. $\|x_n\| = 1$ and

$$m \le f_n(x_n) \le M$$

Now $f_k \in B(f_\ell)^\perp$ if $\ell = k$, and so $f_k(x_\ell) = 0$ if $\ell \ne k$. Also

$$E^* = B(f_\ell)^\perp + B(f_k)^\perp$$

(since $B(f_k) \subset B(f_\ell)^\perp$.) Therefore

$$(B(f_k)^\perp)^\circ \cap (B(f_\ell)^\perp)^\circ = \{0\} \qquad k \ne \ell$$

for if $x \in$ this intersection, then given $f \in E^*$, $f = g + h$ where $g \in B(f_\ell)^\perp$, $h \in B(f_k)^\perp$, then $f(x) = g(x) + h(x) = 0 \quad \forall f \in E^* \Rightarrow x = 0$). Note that

$$x_n \wedge x_m \in (B(f_n)^\perp)^\circ \cap (B(f_m)^\perp)^\circ$$

lattice ideals

$$\Longrightarrow \quad x_n \wedge x_m = 0 \quad \text{if } m \neq n$$

Define $S : \ell_1 \longrightarrow E$ by

$$S(\lambda) = \sum_{n=1}^{\infty} \lambda_n x_n$$

$\left(\left\| \sum_{n=k}^{k+p} \lambda_n x_n \right\| \le \sum_{n=k}^{k+p} |\lambda_n| \Rightarrow S(\lambda)\right.$ makes sense for $\lambda = (\lambda_n) \in \ell_1$ and

$\left. \|S\| \le 1 \right)$

If $0 \le \lambda = (\lambda_n)$, then

for any k

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\| \ge \left\| \sum_{n=1}^{k} \lambda_n x_n \right\|$$

$$\ge \frac{1}{M} \left(\sum_{n=1}^{k} f_n \right) \left(\sum_{m=1}^{k} \lambda_m x_m \right)$$

$$\left\| \sum_{n=1}^{k} f_n \right\| < M \, \| \underbrace{(1, 1, \dots, 1}_{k}, 0, 0, \dots) \|_{c_0} = M$$

$$= \frac{1}{M} \sum_{n=1}^{k} \lambda_n f_n(x_n) \geq \frac{m}{M} \sum_{n=1}^{k} \lambda_n$$

Since $x_n \wedge x_m = 0$ for $n \neq m$, we have

$$|S(f_n)| = \Big| \sum_{n=1}^{\infty} \lambda_n x_n \Big| = \sum_{n=1}^{\infty} |\lambda_n| x_n = S(|\lambda_n|)$$

so S is a lattice homomorphism and

$$\| S(\lambda_n) \| = \| S(|\lambda_n|) \| \geq \frac{m}{M} \| (|\lambda_n|) \|_{\ell_1}$$

$$= \frac{m}{M} \| (\lambda_n) \|_{\ell_1}$$

Hence S is a topological and lattice isomorphism of ℓ_1 into E
Hence E^* does not have a sublattice isomorphic to c_0.

(3) $\Rightarrow$ (4) Last proposition applied to E and E^*

(4) $\Rightarrow$ (1) If E has the monotone convergence property, then $Q(E)$ is a band in E^{**}. If E^* has MCP then E^{**} = order continuous linear functionals on E^*. By an earlier result, the band generated by $Q(E)$ in E^{**} is equal to all order cont. linear functionals on E^*. Hence $Q(E) = E^{**}$.

$\boxed{/\!/}$

Spaces of Linear Operators on Banach Lattices

Let E, F be Banach lattices

Facts that we have already proved:

(1) The space $L^+(E,F)$ of all differences of positive linear maps is contained in the space $\mathcal{L}(E,F)$ of all continuous linear maps and in the space $L^b(E,F)$ of all order bounded linear maps

(2) If F is order complete, then $L^+(E,F) = L^b(E,F)$ and $L^b(E,F)$ is an order complete vector lattice containing $L^0(E,F) =$ order cont. maps and $L^{s0}(E,F) =$ seq. order cont. maps as bands.

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3) If $E = F = c$ = space of convergent sequences and $T_i : E \longrightarrow F$ are given by

$$T_1 x = (x_1, \lim x_n, x_3, \lim x_n, \dots)$$

$$T_2 x = (x_2, \lim x_n, x_4, \lim x_n, \dots)$$

then $T_1, T_2 \geq 0$ and $T = T_1 - T_2 \in L^+(E,F)$. However T^+ does not exist. The proof shows that even though T has range in c_0, there does not exist a positive operator with range in c_0 that dominates T. Therefore

$$L^b(c, c_0) = L^+(c, c_0) \subsetneq \mathcal{L}(c, c_0)$$

c_0 is order complete

Fact: This example was modified by S. Kaplan to provide an example of two compact Hausdorff spaces X, Y s.t.

$$L^+(C(X), C(Y)) \subsetneq L^b(C(X), C(Y)) = \mathcal{L}(C(X), C(Y))$$

<u>PROPOSITION</u>: If $F = C(Y)$ for a compact Stonean Y, then $\mathcal{L}(E,F) = L^+(E,F) = L^b(E,F)$ and $\mathcal{L}(E,F)$ is a Banach lattice for the operator norm.

Proof. Y compact Stonean $\Rightarrow F$ is order complete, so $L^+(E,F)=L^b(E,F)$ is an order complete vector lattice. Also $L^b(E,F)=\mathcal{L}(E,F)$

To see that the operator norm is a Banach lattice norm, note that if B is a bounded set in $C(Y)$ s.t. $B=-B$, then $\sup B$ exists and

$$\|\sup B\| = \sup\{\|b\| : b\in B\}$$

Why? Since $B=-B$, $\sup B \geq b\vee(-b)$ for any $b\in B$. Then

$$\sup B \geq |b| \ \forall\, b\in B \Rightarrow \|b\| = \|\,|b|\,\| \leq \|\sup B\|$$

If $\sup\{\|b\| : b\in B\}\leq 1$, then $B\subset[-1_Y, 1_Y]$. Hence $\sup B\in[-1_Y,1_Y]$, so $\|\sup B\|\leq 1$.

If $T\in\mathcal{L}(E, C(Y))$, then

$$\|\,|T|\,\| = \sup_{\substack{\|x\|\leq 1\\ x\geq 0}} \|\,|T|x\,\| = \sup_{\substack{\|x\|\leq 1\\ x\geq 0}} \Big\| \sup_{|z|\leq x} Tz \Big\|$$

$$= \sup_{\substack{\|x\|\leq 1\\ x\geq 0}} \sup_{|z|\leq x} \|Tz\| = \sup_{\|z\|\leq 1}\|Tz\| = \|T\|$$

Also, if $0\leq T_1\leq T_2$, then $\|T_1\|\leq\|T_2\|$

$$\|T_1\| = \sup_{\substack{\|x\|\leq 1\\ x\geq 0}}\|T_1 x\| \leq \sup_{\substack{\|x\|\leq 1\\ x\geq 0}}\|T_2 x\| = \|T_2\|$$

Therefore the operator norm is a lattice norm

$\boxtimes$

PROPOSITION: If E is an L-space and if F has the MCP, then $\mathcal{L}(E,F) = L^+(E,F) = L^b(E,F) = L^o(E,F) = L^{so}(E,F)$ is a Banach lattice for the operator norm.

Proof We have shown all the spaces are equal if E is an L-space and F is a Banach lattice in which every directed $(\leq)$ subset of the cone which is norm bounded converges. This statement about F is equivalent to F having MCP [[Let D be directed $(\leq)$ norm bounded in the cone and let F have MCP. $Q(D)$ is directed $(\leq)$ and norm bounded in F^{**}. Therefore $Q(D)$ has a w^*-cluster point v and $v = \sup D$. But $Q(F) \supset Q(D)$ and $Q(F)$ is a band in F^{**} so $v = Q(F) \Rightarrow v = Qx_0$ for $x_0 \in F \Rightarrow x_0 = \sup D$. Now $D \subset [0, x_0] \leftarrow$ weakly compact, so $\mathcal{F}(D)$ has a weak cluster point It must be x_0. Hence $\mathcal{F}(D) \longrightarrow x_0$ weakly, so $\mathcal{F}(D) \longrightarrow x_0$ in norm (directed upward).]]

Recall, if $x \geq 0$

$$|T|x = \sup \underbrace{\left\{ \sum_{n=1}^{k} |Tx_n| : x_n \geq 0, \sum_{n=1}^{k} x_n = x \right\}}_{\text{directed set}}$$

Observe

$$\left\| \sum_{n=1}^{k} |Tx_n| \right\| \leq \sum_{n=1}^{k} \|Tx_n\| \leq \|T\| \sum_{n=1}^{k} \|x_n\| \underset{\uparrow\ L\text{-space}}{=} \|T\| \|x\|$$

Hence $|T|x = \sup D$ exist and $\mathcal{F}(D) \to |T|x$. Also

$$\| \, |T|x \, \| \leq \|T\| \, \|x\|$$

$$\Rightarrow \| \, |T| \, \| \leq \|T\| \Rightarrow \| \, |T| \, \| = \|T\|$$

We also have $0 \leq T_1 \leq T_2 \Rightarrow \|T_1\| \leq \|T_2\|$ as before. Hence the operator norm is a Banach lattice norm

$\boxed{}$

Example: $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$T_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$T_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} T_1 & T_1 \\ -T_1 & T_1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}$$

In general $T_n : \mathbb{R}^{2^n} \to \mathbb{R}^{2^n}$ is given by

$$T_n = \frac{1}{\sqrt{2}} \begin{pmatrix} T_{n-1} & T_{n-1} \\ -T_{n-1} & T_{n-1} \end{pmatrix}$$

$\|T_n\| = 1 \quad \forall n$. Also

$$|T_n| = \frac{1}{(\sqrt{2})^n} \left(\mathbb{1} \otimes \mathbb{1} \right)$$

$$\| \, |T_n| \, \| \geq \frac{1}{(\sqrt{2})^n} \left\| \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\| \cdot \frac{1}{\sqrt{2^n}}$$

$$= \frac{1}{\sqrt{2^{2n}}} \left\| \begin{pmatrix} 2^n \\ 2^n \\ \vdots \\ 2^n \end{pmatrix} \right\| = \frac{1}{\sqrt{2^{2n}}} \sqrt{2^{3n}} = \sqrt{2^n}$$

$\longrightarrow \infty$ as $n \to \infty$

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PROPOSITION: Suppose E and F are Banach lattices and that M is a closed subspace of $\mathcal{L}(E,F)$ s.t. $|T|\in M$ when $T\in M$. Then

$$\|T\|_r := \| \, |T| \, \|$$

defines a Banach lattice norm on M equivalent to the usual operator norm.

Proof. Note

$$\|T\| = \sup_{\|x\|\le 1} \|Tx\| \le \sup_{\|x\|\le 1} \| \, |T| \, |x| \, \| \le \sup_{\substack{\|x\|\le 1 \\ x\ge 0}} \| \, |T| x \, \| = \| \, |T| \, \|$$

$$\Longrightarrow \|T\| \le \|T\|_r$$

Also, $T \longrightarrow \|T\|_r$ is a seminorm since $|S+T| \le |S| + |T|$. Then it is actually a norm and

$$id : (M, \|\cdot\|_r) \longrightarrow (M, \|\cdot\|)$$

is cont., 1-1, and onto. To complete the proof, it would suffice to show that $(M, \|\cdot\|_r)$ is complete (by Open Mapping theorem)

Given a $\|\cdot\|_r$ - Cauchy sequence in M, choose a subsequence (T_n) s.t.

$$\|T_{n+1} - T_n\|_r < \frac{1}{2^{n+1}} \quad \forall n$$

(T_n) is also $\|\cdot\|$-Cauchy, so $\exists\, T_0 \in M$ s.t. $T_n \longrightarrow T_0$ in the norm $\|\cdot\|$.

For any $x \geq 0$ and $|z| \leq x$, we have

$$T_0 z - T_k z = \sum_{n=k}^{\infty} (T_{n+1} - T_n) z \leq \sum_{n=k}^{\infty} |T_{n+1} - T_n| x$$

$$\Rightarrow \underbrace{\sup_{|z| \leq x} (T_0 - T_k) z}_{|T_0 - T_k| x} \leq \sum_{n=k}^{\infty} |T_{n+1} - T_n| x$$

$$\Rightarrow \|(T_0 - T_k)\|_r = \sup_{\substack{\|x\| \leq 1 \\ x \geq 0}} \|\, |T_0 - T_k| x \,\|$$

$$\leq \sup_{\substack{\|x\| \leq 1 \\ x \geq 0}} \left\| \sum_{n=k}^{\infty} |T_{n+1} - T_n| x \right\|$$

$$\leq \sup_{\substack{\|x\| \leq 1 \\ x \geq 0}} \sum_{n=k}^{\infty} \|\, |T_{n+1} - T_n| x \,\|$$

$$\leq \sum_{n=k}^{\infty} \|\, |T_{n+1} - T_n| \,\| \leq 1/2^k \longrightarrow 0$$

Hence $T_n \longrightarrow T_0$ in $\|\cdot\|_r$, so $(M, \|\cdot\|_r)$ is complete. $\boxtimes$

PROPOSITION: Suppose E, F are Banach lattices and that $T : E \longrightarrow F$ is a bounded linear operator.

(1) If $T \geq 0$, then $T^* \geq 0$

(2) If $T \in L^+(E,F)$, then $T^* \in L^\circ(F^*, E^*)$. Also $|T|$ exists and is in $L^\circ(F^*, E^*)$

(3) If $|T|$ exists, then $|T|^* \geq |T^*|$ and $\|T^*\|_r \leq \|T\|_r$

(4) If $|T|$ exists, then $|T|^* = |T^*|$ if and only if $|T^*|$ is weak* continuous

Proof (1) If $0 \leq x \in E$, $0 \leq g \in F^*$, then

$$T^*g\,(x) = g(Tx) \geq 0$$

Hence $T^*g \geq 0$ $\forall g \geq 0$, so T^* is positive

(2) If $g_\alpha \downarrow 0$ in F^* and $0 \leq x \in E$, then if $T \geq 0$ it follows from (1) that $(T^* g_\alpha)$ is decreasing. Also,

$$\begin{aligned}(\inf T^* g_\alpha)(x) &= \inf(T^* g_\alpha(x)) \\ &= \inf(g_\alpha(Tx)) \\ &= (\inf g_\alpha)(Tx) = 0\end{aligned}$$

Hence $(T^* g_\alpha) \downarrow 0$, so T^* is order continuous. Therefore any $T \in L^+(E,F)$ has an order continuous adjoint. Rest follows from $L^0(F^*, E^*)$ is band.

(3) If $|T|$ exists, then $|T| \geq T, -T$, so

$$|T|^* \geq T^*, -T^*$$

$$\Longrightarrow |T|^* \geq \underbrace{|T^*|}_{\text{exists by (2)}}$$

Hence

$$\| T^* \|_r = \| \, |T^*| \, \| \leq \| \, |T|^* \| = \| \, |T| \, \| = \|T\|_r$$

(4) If $|T|$ exists and $|T^*|$ is w^*-continuous, then $|T^*| = S^*$ for some $S \in \mathcal{L}(E,F)$. Then $S \geq T, -T$ (calculation), so $S \geq |T|$

$$\Longrightarrow \quad |T^*| = S^* \geq |T|^* \geq |T^*|$$

$$\Longrightarrow \quad |T^*| = |T|^*$$

▨

Remarks: (1) We know that $\mathcal{L}(L^1[0,1], \ell^2) = L^b(L^1[0,1], \ell^2)$ and that these spaces are a Banach lattice for the operator norm. By a homework problem, $\mathcal{L}(L^p[0,1], \ell^2)$ properly contains $L^b(L^p[0,1], \ell^2)$ for any $p > 1$, and so $\mathcal{L}(L^p, \ell^2)$ is not a lattice.

(2) $\mathcal{L}(\ell_2, \ell_2)$ is not a lattice (contradicts claim in D-S)

Proof (Peressini - Hilbert)

$$T = (t_{ij}) \qquad t_{ij} = \begin{cases} 0 & i \neq j \\ \dfrac{1}{i-j} & i \neq j \end{cases}$$

$$T : \ell^2 \longrightarrow \ell^2 \qquad \|T\| = \pi$$

Define $x_j = y_j = \dfrac{1}{\sqrt{j}\,\log j}$ $\quad j > 1$, and $y_1 = x_1 = y_2$

Then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |t_{ij}|\, x_j y_j = +\infty$$

(See Hardy - Littlewood - Polya Inequalities - Hilbert matrix)

These facts will imply that T is not an order bounded operator. Let

$$z_i = \sum_{j=1}^{\infty} |t_{ij}|\, y_j$$

$$u_j^{(i)} = (\operatorname{sign} t_{ij})\, y_j$$

Then $u^{(i)} = (u_j^{(i)})_{j \in \mathbb{N}} \in [-y, y]$ in ℓ_2. Note

$$z_i = \sum_{j=1}^{\infty} t_{ij} u_j^{(i)} = (Tu^{(i)})_i$$

If T is order bounded, there would exist a $u \in l_2$ s.t.

$$T[-y, y] \subset [-u, u]$$

In particular, $Tu^{(i)} \in [-u, u]$, so that if $0 \leq v \in l_2$, then

$$-\langle u, v \rangle \leq \langle Tu^{(i)}, v \rangle \leq \langle u, v \rangle$$

In particular, if $v = e^{(i)}$

$$-u_i \leq (Tu^{(i)})_i \leq u_i \quad \forall i$$

$$\Rightarrow |z_i| = |(Tu^{(i)})_i| \leq u_i$$

Hence

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |t_{ij}| y_j x_i = \sum_{i=1}^{\infty} z_i x_i \leq \sum_{i=1}^{\infty} u_i x_i$$

$$= \langle u, x \rangle < +\infty$$

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COMPACT OPERATORS

LEMMA: If A is a relatively compact set in an M-space E and if $C \subset A$, then $\sup C$ exists in E and is equal to $\sup C$ in E^{**}. Moreover, the set $\{\sup C : C \subset A\}$ is relatively compact in E.

Proof. Since E is an M-space, E^{**} is an M-space with strong unit, so E^{**} is isometric and lattice isomorphic to $C(X)$ for a compact Hausdorff space

Suppose we identify E canonically with a sublattice of E^{**} and E^{**} with $C(X)$ as above. Recall that the Arzela-Ascoli theorem asserts that a subset of $C(X)$ is relatively compact iff it is uniformly bounded and equicontinuous

equicontinuous - given $\varepsilon > 0$, $s \in X$ $\exists$ nbhd V_s of s s.t.

$$\sup_{f \in K} \sup_{t \in V_s} |f(s) - f(t)| < \varepsilon$$

Suppose $C \subset A$. Then C is norm bounded and so the directed ($\leq$) set $D(C)$ = all suprema of finite sets in C is still norm bounded. C is equicontinuous, so $\exists \varepsilon > 0$ and $s \in S$ $\exists$ nbhd V_s of s s.t.

$$\sup_{f \in C} \sup_{t \in V_s} |f(s) - f(t)| < \varepsilon$$

Suppose $g \in D(C)$. Then

$$g = f_1 \vee f_2 \vee \dots \vee f_n$$

where $f_i \in C_i$. Then

$$|g(s) - g(t)| \leq \max_{1 \leq i \leq n} |f_i(s) - f_i(t)|$$

Hence $D(C)$ is equicontinuous; so $D(C)$ is relatively compact and directed ($\leq$). $\mathcal{F}(D(C))$ has a cluster point $u \in \overline{D(A)}$. Therefore $u = \sup D(C)$ and $\mathcal{F}(D(C)) \rightarrow u$ in the norm of E and of E^{**}. But $\sup D(C) = \sup C$ so $\sup C$ exists and is the same in E and E^{**} $\forall C \subset A$.

$$\{\sup C : C \subset A\} \subset \overline{D(A)} \leftarrow \text{compact}$$

$\blacksquare$

__PROPOSITION__: If E is a Banach lattice and F is an M-space, then the space $C(E,F)$ of all compact maps from E into F is a Banach lattice for the operator norm.

Proof. If $T \in C(E,F)$ and $U = B_E$, then $T(U)$ is relatively compact in the M-space F. If $0 \leq x \in U$, then the supremum of the set $\{Tz : |z| \leq x\}$ exists in F and F^{**} and these sups are equal. Hence $|T|$ exists and

$$|T|x = \sup\{Tz : |z| \leq x\}$$

Also $\{|T|x : 0 \le x \in U\}$ is relatively compact by the lemma so $|T|$ is compact since

$$|T|x = |T|x^+ - |T|x^-$$

and $x^+, x^- \in U$ if $x \in U$. For $0 \le x \in U$, the set $B_x := \{Tz : |z| \le x\}$ is a subset of F s.t. $B_x = -B_x$ and $\sup B_x$ exists and is the same in F as in F^{**}. By an earlier argument

$$\| \sup_{|z| \le x} Tz \| = \sup_{|z| \le x} \|Tz\|$$

Therefore

$$\| |T| \| = \sup_{\substack{\|x\| \le 1 \\ x \ge 0}} \| |T|x \| = \sup_{\substack{\|x\| \le 1 \\ x \ge 0}} \| \sup_{|z| \le x} Tz \|$$

$$= \sup_{\|z\| \le 1} \|Tz\| = \|T\|.$$

For any Banach lattices E, F, it is true that $0 \le S \le T$, $S, T \in \mathcal{L}(E,F)$ then $\|S\| \le \|T\|$. Hence $T \mapsto \|T\|$ is a lattice norm.

$\boxtimes$

Example: $\mathcal{L}(c, c_0)$ is not a lattice, but $\mathcal{C}(c_0, c)$ is a Banach lattice

Recall (metric) approximation property means that the identity operator on the space can be approximated uniformly on compact sets by finite rank operators (of norm ≤ 1)

$C(X)$ has the metric approximation property. (Actually satisfies property that given a compact set K in $C(X)$ and an $\varepsilon > 0$ there exists $0 \leq \mu_i \in C(X)^*$, $0 \leq f_i \in C(X)$ s.t.

$$\| f - \sum_{i=1}^{n} \mu_i(f) f_i \| < \varepsilon$$

positive!

$\forall f \in K$)

T is of finite rank iff $T \in E^* \otimes F$

<u>PROPOSITION:</u> If E is an L-space and F is a Banach lattice, the space $\mathcal{C}(E, F)$ of compact operators from E into F is a Banach lattice for the operator norm.

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PROPOSITION: If E, F are Banach lattices and if $T: E \to F$ is a bounded operator of finite rank, then $|T|$ exists and is the limit in the operator norm of positive finite rank operators.

Proof Since T has finite rank, there exists $x_i^* \in E^*$, $y_i \in F$ $1 \le i \le n$, s.t

$$Tx = \sum_{i=1}^{n} x_i^*(x)\, y_i$$

$\forall x \in E$. Then

$$|Tx| = \left| \sum_{i=1}^{n} x_i^*(x) y_i \right| \le \sum_{i=1}^{n} |x_i^*(x)|\,|y_i|$$

$$\le \|x\| \underbrace{\left(\sum_{i=1}^{n} \|x_i^*\|\, y_i \right)}_{=:y}$$

Hence the range of T is contained in the lattice ideal F_y generated by y. Also, this shows that $T: E \to F_y$ is continuous, so since it has finite rank, it is compact. But $F_y = C(X)$ for some compact Hausdorff X, so $|T|: E \to F_y$ exists and is compact by an earlier result.

Let U be the unit ball in E and let $K = |T|(U)$. Then K is relatively compact. Since $C(X)$ has the positive (metric) approximation property, $\exists\ 0 \le y_i^* \in F_y^* = C(X)^*$, $0 \le z_i \in F_y = C(X)$ $1 \le i \le p$ s.t.

$$\| y - \sum_{i=1}^{p} y_i^*(y) z_i \|_y < \varepsilon$$

for all $y \in K$ (where ε has been preassigned). Hence

$$\| |T|x - \sum_{i=1}^{p} y_i^* (|T|x) z_i \|_y < \varepsilon$$

for all $x \in U$. Let $z_i^* = |T|^* y_i^*$ $i = 1, \dots, p$. Then

$$\| |T|x - \sum_{i=1}^{p} z_i^*(x) z_i \|_y < \varepsilon$$

for all $x \in U$, so

$$\| |T|x - \sum_{i=1}^{p} z_i^*(x) z_i \| < \varepsilon \|y\|$$

for all $x \in U$

$$\Rightarrow \| |T| - \underbrace{\sum_{i=1}^{p} z_i^*(\cdot) z_i}_{\text{positive}} \| < \varepsilon \|y\|$$

$\blacksquare$

Remark: We have shown that if E is an L-space and F has the monotone convergence property, then $\mathcal{L}(E,F)$ is a Banach lattice for the operator norm.

Note that if F has the MCP and if F is identified with a sublattice of F^{**} then $\exists$ a positive contractive projection of F^{**} onto F, namely the band projection onto F.

We will show that if E is an L-space and F is the range of a positive contractive projection $P : F^{**} \to F$, then $\mathcal{L}(E,F)$ is a Banach lattice for the operator norm.

Proof. Let $T \in \mathcal{L}(E,F)$. Then $T^* \in \mathcal{L}(F^*, E^*)$ and $E^* = C(X)$ for X stonean. Hence T^* is the difference of positive operators from F^* into E^*, so T^{**} is also the difference of positive operators from E^{**} into F^{**}. Now F^{**} is order complete, so $|T^{**}|$ exists and is continuous.

$$T_1 - T \leq P \circ |T^{**}| \circ Q_E$$

$$\Big(x \geq 0 \Rightarrow Q_E x \geq 0 \Rightarrow |T^{**}| Q_E x \geq T^{**} Q_E x$$

$$\Rightarrow P \circ |T^{**}| Q_E x \geq P T^{**} Q_E x$$

$$= P Q_F T x = Tx \Big)$$

Hence T is the difference of positive operators.

Since F is the range of a positive projection P on F^{**}, it follows that F is order complete.

(Let D be a majorized subset of F. Then $Q_F(D)$ is majorized in F^{**} so $\sup Q_F(D) = u$ exists in F^{**} $\Rightarrow$ $Pu = \sup D$ in F)

Therefore $|T|$ exists and is in $\mathcal{L}(E,F)$. Also $|T| \le P \circ |T^{**}| \circ Q_F$
Hence

$$\|T\|_r \le \|T^{**}\|_r$$

Since P is contractive and Q_F is an isometry,

$$\Rightarrow \|T\|_r = \|T^{**}\|_r$$

Since $T \longrightarrow T^*$ is $\|\cdot\|_r$-decreasing. We know that $\mathcal{L}(F^*, E^*)$ is a Banach lattice for the operator norm, so $\|T^*\|_r = \|T^*\| = \|T\|$

$$\Rightarrow \|T\|_r = \|T^{**}\|_r \le \|T^*\|_r$$

$$= \|T\|$$

Since in general $\|T\| \le \|T\|_r$, it follows that $\|T\| = \|T\|_r$.

$\blacksquare$

Suppose E is an L-space and F is a Banach lattice. If $T: E \longrightarrow F$ is a bounded operator of norm ≤ 1, then $|Q_F T|$ exists and has norm ≤ 1. [Take $P = Q^*_{F^*} : F^{****} \rightarrow F^{**}$ and use last result]

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PROPOSITION: If E is an L-space and F is a Banach lattice then the space $K(E,F)$ of compact operators from E into F is a Banach lattice for the operator norm.

Proof. Since F^{**} is the range of a positive contractive projection from F^{****}, it follows that the space $\mathcal{L}(E,F^{**})$ is a Banach lattice for the operator norm. We can identify $K(E,F)$ with a subspace of $\mathcal{L}(E,F^{**})$ and the operator norm on $\mathcal{L}(E,F^{**})$ induces the operator norm on $K(E,F)$.

Since E is an L-space, then $E^* = C(X)$ for some compact Hausdorff X, so E^* has the approximation (metric) property, which is equivalent to saying that $K(E,F)$ = closure of the bounded operators of finite rank from E into F. Since $\mathcal{L}(E,F^*)$ is a Banach lattice, it follows that if $T_n \to T$, then $|T_n| \to |T|$ in the operator norm. We have shown that if $T: E \to F$ is a bounded operator of finite rank, then $|T|$ is the limit in the operator norm of a sequence of of positive operators of finite rank. Therefore, if $T \in K(E,F)$, there is a sequence $\{S_k\}$ of operators of finite rank s.t. $S_k \to T$. $|S_k| = \lim_m T_{mk}$ where T_{mk} are bounded and have finite rank with range in F.

$$|S_k| \longrightarrow |T| \quad \text{in } \mathcal{L}(E,F^{**})$$

$$T_{mk} \longrightarrow |S_k|$$

Hence $|T|$ is the operator norm limit of a sequence of operators of finite rank from E into F. Hence $|T| \in K(E,F)$.

$\blacksquare$

Remark: It can be shown that if E is a Banach lattice and if F is a Banach lattice that does not contain $\ell^\infty(n)$ unif in n, then if the closure of the space of finite rank operators from E to F is a Banach lattice for the operator norm, then E is an L-space.

<u>DEFINITION</u>: Suppose that E is a Banach lattice and that F is a Banach space.

(1) If $T \in \mathcal{L}(E,F)$, then T is order summable if $\exists$ L-space L and $0 \le T_1 \in \mathcal{L}(E,L)$, $T_2 \in \mathcal{L}(L,F)$ s.t.

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "0 \le T_1"'] & & F \\ & L \arrow[ur, "T_2"'] & \end{tikzcd}$$

(2) If $S \in \mathcal{L}(F,E)$, then S is majorizing if $\exists$ M-space M and $S_1 \in \mathcal{L}(F,M)$, $0 \le S_2 \in \mathcal{L}(M,E)$ s.t.

$$\begin{tikzcd} F \arrow[rr, "S"] \arrow[dr, "S_1"'] & & E \\ & M \arrow[ur, "S_2 \ge 0"'] & \end{tikzcd}$$

Background:

Let G be a B-space. If $\{x_n\}$ is a sequence in G, then (x_n) is summable to x if the net

$$\left\{ \sum_{n \in H} x_n : H \text{ finite}, H \subset \mathbb{N} \right\}$$

converges to x. If $x^* \in G^*$ and (x_n) is summable, then

$x = \sum_{n \in \mathbb{N}} x_n$

$$\sum_{n \in \mathbb{N}} x^*(x_n) = x^*(x)$$

$$\Updownarrow$$

$$\sum_{n \in \mathbb{N}} |x^*(x_n)| < \infty$$

$$\Updownarrow$$

$$B = \left\{ \sum_{n \in \mathbb{N}} \alpha_n x_n : |\alpha_n| \leq 1, H \text{ finite} \right\} \text{ is weakly bounded}$$

But then $\sup_{\|x^*\| \leq 1} \sum_{n=1}^{\infty} |x^*(x_n)| < \infty$ [[choose α_n s.t. $\alpha_n x^*(x_n) = |x^*(x_n)|$]]

Define $\|\cdot\|_\varepsilon$ on the space $\ell^1(G)$ of all summable sequences in G by

$$\|(x_n)\|_\varepsilon := \sup_{\|x^*\| \leq 1} \sum_{n=1}^{\infty} |x^*(x_n)|$$

Then $\ell^1(G)$ is a Banach space.

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If G is a Banach lattice and if $x_n \geq 0$ $\forall n$ for $(x_n) \in \ell^1[G]$ then

$$\|(x_n)\|_{\varepsilon} = \sup_{\|x^*\| \leq 1} \sum |\langle x_n, x^* \rangle| = \sup_{\substack{\|x^*\| \leq 1 \\ x^* \geq 0}} \sum \langle x_n, x^* \rangle$$

$$= \left\| \sum_{n=1}^{\infty} x_n \right\|$$

DEFINITION: If G is a B-space, then $(x_n) \subset G$ is absolutely summable if

$$\sum_{n=1}^{\infty} \|x_n\| < +\infty$$

If (x_n) is absolutely summable, then it is summable. We denote by $\ell^1(G)$ all absolutely summable sequences in G with norm

$$\|(x_n)\|_{\pi} := \sum_{n=1}^{\infty} \|x_n\|$$

Then $\ell^1(G)$ is a B-space, and if G is a B-lattice then $\ell^1(G)$ is a Banach lattice.

LEMMA: If E is a Banach lattice and F is a Banach space, then for any $T \in \mathcal{L}(E,F)$ that maps positive summable sequences into abs. summable sequences, then

$$(*) \quad \sup \left\{ \sum_{n=1}^{\infty} \|Tx_n\| : 0 \leq (x_n) \in \ell^1[E],\ \|(x_n)\|_\varepsilon \leq 1 \right\}$$

is finite and

$$p(x) := \sup \left\{ \sum_{n=1}^{\infty} \|Tx_n\| : 0 \leq (x_n) \in \ell^1[E],\ x = \sum x_n \right\}$$

defines an additive and positively homogeneous functional on the cone of E.

Proof. Suppose $(*)$ is not finite. Then $\exists\ 0 \leq (x_n^p) \in \ell^1[E]$ s.t.

$$\|(x_n^p)\|_\varepsilon \leq 1/2^p$$

$$\sum_{n=1}^{\infty} \|Tx_n^p\| > 1$$

Choose finite $H_p \subset \mathbb{N}$ so that $\sum_{n \in H_p} \|Tx_n^p\| > 1$. List $\{x_n^{(p)} : n \in H_p\}$ as

$$H_1\ H_2\ H_3 \cdots H_n \cdots$$

i.e. seq $\{z_k\}$, which is summable and $\|(z_k)\|_\varepsilon \leq 1$. Then

$$\sum_{k=1}^{\infty} \|Tz_k\| = +\infty \quad ↯$$

since (z_k) is a positive summable seq. which is not mapped into an absolutely summable seq.

$(*)$ finite $\Rightarrow p(x) < \infty \ \forall x \geq 0$ since $\|\sum x_n\| = \|(x_n)\|_\varepsilon$ for $x_n \geq 0$

$p(\alpha x) = \alpha p(x)$ for $x \geq 0, \alpha \geq 0$ trivial. If

$$x = \sum x_n \qquad y = \sum y_n$$

for $x, y, x_n, y_n \geq 0$, then $\sum (x_n + y_n) = x + y$, so

$$p(x+y) \geq \sum_{n=1}^{\infty} \left(\|Tx_n\| + \|Ty_n\| \right)$$

$$= \sum_{n=1}^{\infty} \|Tx_n\| + \sum_{n=1}^{\infty} \|Ty_n\|$$

$$\Rightarrow p(x+y) \geq p(x) + p(y)$$

Now suppose $0 \leq x, y$ and that $\sum z_n = x + y$, for $0 \leq (z_n) \in \ell^1[E]$. Let

$$r_m = \sum_{n=1}^{m} z_n$$

$$s_m = r_m \wedge x \qquad x_n = s_n - s_{n-1}$$

$$t_m = (r_m - x) \vee 0 \qquad y_n = t_n - t_{n-1}$$

Then $z_n = r_n - r_{n-1} = (r_n \wedge x + r_n \vee x - x) - (r_{n-1} \wedge x + r_{n-1} \vee x - x)$

$$= (r_n \wedge x + (r_n - x) \vee 0) - (r_{n-1} \wedge x + (r_{n-1} - x) \vee 0)$$

$$= s_n + t_n - s_{n-1} - t_{n-1}$$

$$= x_n + y_n$$

Also $\sum_{n=1}^{k} x_n = s_k - s_0 = s_k = r_k \wedge x \longrightarrow x$ since $r_k \longrightarrow x + y$

Hence

$$x = \sum_{n=1}^{\infty} x_n$$

Similarly,

$$y = \sum_{n=1}^{\infty} y_n$$

Thus

$$\sum_{n=1}^{\infty} \|Tz_n\| \leq \sum_{n=1}^{\infty} \|Tx_n\| + \sum_{n=1}^{\infty} \|Ty_n\| \leq p(x) + p(y)$$

$$\Rightarrow p(x+y) \leq p(x) + p(y)$$

$\blacksquare$

THEOREM: A Banach lattice E can be renormed to be an L-space if and only if every positive summable sequence is absolutely summable.

Proof. If E is an L-space and if $0 \leq (x_n) \in \ell^1[E]$, then for any finite set H

$$\left\| \sum_{n \in H} x_n \right\| = \sum_{n \in H} \|x_n\|$$

In particular, for any k,

$$\sum_{n=1}^{k} \|x_n\| = \left\| \sum_{n=1}^{k} x_n \right\| \leq \left\| \sum_{n=1}^{\infty} x_n \right\| < M < \infty$$

and so (x_n) is abs. summable.

Since summability and abs. summability are invariant under a change to an equivalent norm, this proves $\Rightarrow$

If every positive summable seq. is absolutely summable on E, we can take $T = I$ in the lemma. For each $x \geq 0$

$$p(x) = \sup \left\{ \sum_{n=1}^{\infty} \|x_n\| : 0 \leq (x_n) \in \ell^1[E], x = \sum x_n \right\}$$

is finite, additive and positively homogeneous on the cone of E. Let

$$\|x\|_1 = p(|x|)$$

Then $\|x\| \leq \|x\|_1 \quad \forall x \in E$

Claim: $0 \leq z_1 \leq z_2 \Rightarrow p(z_1) \leq p(z_2)$. For let

$$z_2 = \sum_{n=1}^{\infty} s_n \qquad s_n \geq 0$$

Then

$$z_1 = \sum_{n=1}^{\infty} s_n \wedge z_1$$

for

$$\sum_{n=1}^{k} s_n \to z_2 \Rightarrow \left(\sum_{n=1}^{k} s_n\right) \wedge z_1 \to z_2 \wedge z_1 = z_1$$

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(Proof continued) We have shown that (6) implies $\exists x_0^* \in E^*$ s.t.

$$\|Tx\| \le \langle |x|, x_0^* \rangle$$

for all $x \in E$.

Define $\Phi : E \longrightarrow C(X)$ by

$$[\Phi(x)](x^*) = \langle x, x^* \rangle \qquad x^* \in X, x \in E$$

Φ is a lattice isomorphism. Φ is also an isometry since

$$\|\Phi(x)\| = \| \, |\Phi(x)| \, \| = \|\Phi(|x|)\|$$

$$= \sup_{\substack{\|x^*\| \le 1 \\ x^* \ge 0}} \langle |x|, x^* \rangle = \| \, |x| \, \| = \|x\|$$

for all $x \in E$.

Define a functional F on the sublattice $\Phi(E)$ of $C(X)$ by

$$F(\Phi(x)) := x_0^*(x)$$

Then F is a positive linear functional on $\Phi(E)$ of norm $\|x_0^*\|$. Therefore F can be extended to a positive Radon measure μ on X of norm $\|x_0^*\|$

$$\|Tx\| \le \langle |x|, x_0^* \rangle = F(\Phi(|x|)) = \mu(|x|) = \int \langle |x|, x^* \rangle d\mu(x^*)$$

(7) $\Rightarrow$ (1) Let $I = \{ x \in E : \int_X \langle x, x^* \rangle \, d\mu(x^*) = 0 \}$. Then I is a closed lattice ideal in E.

$\prod$ Lemma: If E is a Banach lattice is a closed lattice ideal in E, then $E/I \overset{=\dot{E}}{}$ is a Banach lattice for the cone

$$\dot{K} = \{ \dot{x} \in \dot{E} : x \in \text{cone in } E \}$$

Also the canonical map $\varphi : E \to E/I$ is a lattice homomorphism.

Example: $C[0,1]/\text{Constants} \sim C_0[0,1] = \text{all } g \in C[0,1] \text{ s.t. } g(0) = 0$
Then $\dot{K} = C_0[0,1]$. So we need an ideal and not just a subspace

Proof of lemma: Suppose $\dot{x} \in \dot{K} \cap (-\dot{K})$. Then $\exists\, x_1, x_2 \in \dot{x}$ s.t. $x_1 \geq 0$, $x_2 \leq 0$. Then

$$0 \leq x_1 \leq x_1 - x_2 \in I$$

$$\Rightarrow x_1 \in I$$

$$\Rightarrow \dot{x} = 0$$

Now easy to see that $\dot{K}$ is a cone in $\dot{E}$. Next we will show that $\varphi(x^+) = \varphi(x)^+$.

$$x^+ \geq x, 0 \Rightarrow \varphi(x^+) \geq \varphi(x) = \dot{x}, \; \varphi(0) = \dot{0}$$

Suppose $\dot{z} \geq \dot{x}, \dot{0}$. Then $\exists\, z_1, z_2 \in \dot{z}$, $x_1 \in \dot{x}$ s.t.

$$z_1 \geq x_1 \qquad z_2 \geq 0$$

Then $z_1 \vee z_2 \geq x_1, 0$ and $z_1 \vee z_2 - z_1 = 0 \vee (z_2 - z_1) = (z_2 - z_1)^+ \in I$

Hence $z_1 \vee z_2 \in \dot{z}$. $z_1 \vee z_2 \geq x_1, 0 \Rightarrow z_1 \vee z_2 \geq x_1^+$

$$\dot{z} \geq \varphi(z_1 \vee z_2) \geq \varphi(x_1^+) = \varphi(x^+)$$

$$|x_1^+ - x^+| \leq |x_1 - x| \in I$$

Hence $\varphi(x^+) = \varphi(x)^+$.

Finally, given $|\dot{x}| \leq |\dot{y}|$, if $y \in \dot{y}$ choose $x \in \dot{x}$ with $|x| \leq |y|$. Then

$$\|\dot{x}\| \leq \|y\| \Rightarrow \|\dot{x}\| \leq \|\dot{y}\|$$

$\rfloor\!\rfloor$

Back to original proof, E/I is a Banach lattice.
Suppose $\dot{x} \in E/I$ and $x_1, x_2 \in \dot{x}$

$$\left| \int_X \langle |x_1|, x^* \rangle \, d\mu(x^*) - \int_X \langle |x_2|, x^* \rangle \, d\mu(x^*) \right|$$

$$\leq \int_X \left| \langle |x_1| - |x_2|, x^* \rangle \right| d\mu(x^*)$$

$$\leq \int \langle |x_1 - x_2|, x^* \rangle \, d\mu(x^*) = 0$$

Hence we can define

$$\|\dot{x}\|_1 = \int \langle |x|, x^* \rangle \, d\mu(x^*)$$

on E/I. $\dot{x} \longmapsto \|\dot{x}\|_1$ is a lattice norm on E/I. Also $\dot{x} \longmapsto \|\dot{x}\|$ is additive on the cone of E/I. Let $L = E/I$ equipped with $\|\cdot\|_1$ and completed. Then L is an L-space

Now

$$(*) \qquad \|Tx\| \leq \int \langle |x|, x^* \rangle \, d\mu(x^*) \quad \forall x$$

and so $\ker T \supset I$. Therefore T induces a linear map S on E/I into F

$$S\dot{x} = Tx \qquad x \in \dot{x}$$

S has norm ≤ 1 on $(E/I, \|\cdot\|_1)$. Therefore S extends to L into F. Call this T_2. Let $T_1 = \varphi$.

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "0 \leq T"'] & & F \\ & L \arrow[ur, "T_2"'] & \end{tikzcd}$$

lattice homomorphism

$\blacksquare$

12/12 BANACH LATTICES

Tensor Products

E, F vector spaces. $B(E,F) :=$ all bilinear functionals on $E \times F$.

Define

$$\chi : E \times F \longrightarrow B(E,F)' = \text{all linear functionals on } B(E,F)$$

$$\chi(x,y) := u_{xy} = x \otimes y$$

where $u_{xy}(b) = b(x,y)$. Define

$$E \otimes F = \text{linear hull of } \chi(E \times F) \text{ in } B(E,F)'$$

Basic property of $\otimes$

$$\begin{tikzcd}
E \times F \arrow[d, "\chi"'] \arrow[dr, "b"] & \\
E \otimes F \arrow[r, dashed, "T"'] & G = \text{vector space}
\end{tikzcd}
\qquad T \circ \chi = b$$

$$T \longleftrightarrow b$$

$$L(E \otimes F, G) \longleftrightarrow B(E,F;G)$$

Algebraic Properties

$$(x_1+x_2)\otimes y = x_1\otimes y + x_2\otimes y \quad \forall x_1,x_2\in E \ \forall y\in F$$

$$\lambda x\otimes y = \lambda(x\otimes y) = x\otimes \lambda y$$

Every element of $E\otimes F$ can be written as $\sum_{n=1}^{k} x_n\otimes y_n$ (non-unique representation) If $(x_n),(y_n)$ are linearly independent, then k is the <u>rank</u> of u

Suppose E,F are Banach spaces. If $u\in E\otimes F$, then

$$(*) \qquad \|u\|_\pi := \inf\left\{\sum_{n=1}^{k}\|x_n\|\|y_n\| : u=\sum_{n=1}^{k}x_n\otimes y_n\right\}$$

greatest cross norm

Note $\|\cdot\|_\pi$ is the Minkowski functional of the convex circled hull of $\{x\otimes y : \|x\|\le 1, \|y\|\le 1\}$

We have the correspondence

$$\mathcal{L}(E\otimes_\pi F, G) \leftrightarrow \mathcal{B}(E,F;G)$$

In particular

$$(E\otimes_\pi F)^* = \mathcal{B}(E,F)$$

But $\underset{b}{\mathcal{B}(E,F)} \sim \underset{T}{\mathcal{L}(E,F^*)}$ $\qquad Tx(y) = b(x,y)$

$b \longleftrightarrow T$

$$\ell \circ \chi = b$$

$$\|T\| = \sup_{\substack{\|x\|\le 1 \\ \|y\|\le 1}} |Tx(y)| = \sup_{\substack{\|x\|\le 1 \\ \|y\|\le 1}} |b(x,y)| = \sup_{\substack{\|x\|\le 1 \\ \|y\|\le 1}} |\ell(x\otimes y)|$$

$$= \sup_{\substack{u \in \text{convex} \\ \text{circled hull} \\ \text{of } \{x\otimes y : \|x\|\le 1, \|y\|\le 1\}}} |\ell(u)| = \sup_{\|u\|_\pi \le 1} |\ell(u)| = \|\ell\|$$

$$\begin{array}{ccccc} \ell & \longleftrightarrow & b & \longleftrightarrow & T \\ (E\otimes_\pi F)^* & \approx & B(E,F) & \approx & \mathcal{L}(E,F^*) \end{array}$$

$$\|T\| = \|\ell\|$$

Cross norm property : $\|x \otimes y\|_\pi = \|x\|\,\|y\|$

$\le$ clear from definition.

Conversely, $\exists\ \|x^*\| \le 1$, $\|y^*\| \le 1$ s.t. $x^*(x) = \|x\|$ and $y^*(y) = \|y\|$. If

$$x \otimes y = \sum_{n=1}^{k} x_n \otimes y_n$$

then

$$\|x\|\,\|y\| = \langle x\otimes y, x^* \otimes y^* \rangle = \sum_{n=1}^{k} \langle x_n, x^* \rangle \langle y_n, y^* \rangle$$

$$\le \sum_{n=1}^{k} \|x_n\|\,\|y_n\|$$

$E \otimes F$ can also be regarded as a subspace of $\mathcal{B}(E^*, F^*)$

$$\sum_{n=1}^{k} x_n \otimes y_n \longleftrightarrow \left[(x^*, y^*) \to \sum_{n=1}^{k} x^*(x_n)\, y^*(y_n) \right]$$

Define new norm by

$$\|b\|_\varepsilon = \sup_{\substack{\|x^*\| \le 1 \\ \|y^*\| \le 1}} |b(x^*, y^*)|$$

$$\left\| \sum_{n=1}^{k} x_n \otimes y_n \right\|_\varepsilon = \sup_{\substack{\|x^*\| \le 1 \\ \|y^*\| \le 1}} \left| \sum_{n=1}^{k} x^*(x_n)\, y^*(y_n) \right|$$

least cross norm

Notice that $\| x \otimes y \|_\varepsilon = \|x\| \, \|y\|$

Examples:

completion

$\ell^1[E]$ = all summable sequences in E = $\ell^1 \widetilde{\otimes}_\varepsilon E$

$\ell^1(E)$ = all absolutely summable sequences in E = $\ell^1 \widetilde{\otimes}_\pi E$

Suppose E is a B-lattice and F is a B-space. Every element of $E \otimes F$ can be regarded as an order summable linear map from E^* into F.

$$u = \sum_{n=1}^{k} x_n \otimes y_n \longleftrightarrow T_u$$

where

$$T_u(x^*) = \sum_{n=1}^{k} x^*(x_n)\, y_n$$

T_u finite rank $\Rightarrow$ order summable.

T order summable from E^* to F. Norm given by

$$(*) \qquad \|T\|_\ell := \sup\left\{ \sum_{m=1}^{p} \|T x_m^*\| : x_m^* \geq 0, \left\|\sum_{m=1}^{p} x_m^*\right\| = 1 \right\}$$

Then new tensor norm

$$E \otimes_\ell F : \left\| \sum_{k=1}^{n} x_n \otimes y_n \right\|_\ell = \sup\left\{ \sum_{m=1}^{p} \left\| \sum_{k=1}^{n} x_m^*(x_n)\, y_n \right\| : x_m^* \geq 0, \left\| \sum_{m=1}^{p} x_m^* \right\| = 1 \right\}$$

$E \tilde{\otimes}_\ell F$ is always a Banach lattice.

$(E \tilde{\otimes}_\ell F)^*$ = all order summing operators from E into F^*

END OF COURSE

12/7 BANACH LATTICES

(Proof continued)

Want to show $0 \leq x \leq y \Rightarrow p(x) \leq p(y)$. Now

$$y = y - x + x$$

and so $p(y) = p(y-x) + p(x) \geq p(x)$. Then if $x_1, x_2 \in E$,

$$\|x_1 + x_2\|_1 = p(|x_1 + x_2|) \leq p(|x_1| + |x_2|) = p(|x_1|) + p(|x_2|)$$

$$= \|x\|_1 + \|x\|_2$$

By the lemma

$$\sup\left\{ \sum_{n=1}^{\infty} \|x_n\| : 0 \leq (x_n) \in \ell'[E], \|\Sigma x_n\| = \|(x_n)\|_2 \leq 1 \right\} = M < \infty$$

$$\Rightarrow \|x\|_1 \leq M \|x\|.$$

∎

<u>PROPOSITION</u>: If E is a Banach lattice, F is a B-space and $T \in \mathcal{L}(E,F)$, then TFAE

(1) T is order summable $\left(\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "0\le T_1"'] & & F \\ & L \arrow[ur, "T_2"'] & \end{tikzcd}\right.$

(2) $\exists\ C>0$ s.t. $\sum_{n=1}^{k} \|Tx_n\| \le c \|\sum_{n=1}^{k} x_n\| \quad \forall\, x_n \ge 0,\ 1\le n\le k$

(3) T maps positive summable sequences in E into absolutely summable sequences in F

(4) $\exists\ x_T^* \in E^*,\ x_T^* \ge 0$, s.t. $\|Tx\| \le x_T^*(|x|)\ \forall x\in E$

(5) T is continuous for the topology of unif. convergence on order bounded sets in E and the norm top of F

(6) T^* maps B_{F^*} into an order interval in E^*

(7) $\exists$ positive Radon measure μ on the compact set $X = \{x^* \in E^* : \|x^*\| \le 1,\ x^* \ge 0\}$ equipped with the w^* topology s.t.

$$\|Tx\| \le \int_X x^*(|x|)\, d\mu(x^*)$$

$\forall x \in E$.

Proof. (1) $\Rightarrow$ (2) If $x_n \ge 0$ for $n=1,\dots,k$. We can write

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "0 \le T_1"'] & & F \\ & L \arrow[ur, "T_2"'] & \end{tikzcd}$$

Then

$$\sum_{n=1}^{k} \|Tx_n\| \le \|T_2\| \sum_{n=1}^{k} \|T_1 x_n\| = \|T_2\| \, \|T_1(\sum_{n=1}^{k} x_n)\|$$

$$\le \underbrace{\|T_2\| \, \|T_1\|}_{C} \, \|\sum_{n=1}^{k} x_n\|$$

(2) $\Rightarrow$ (3) Suppose $0 \le (x_n) \in \ell^1[E]$. For any k,

$$\|(x_n)\|_\varepsilon = \|\sum_{n=1}^{\infty} x_n\| \ge \|\sum_{n=1}^{k} x_n\| \ge \frac{1}{C} \sum_{n=1}^{k} \|Tx_n\|$$

$$\Rightarrow \sum_{n=1}^{\infty} \|Tx_n\| \le C \, \|(x_n)\|_\varepsilon < \infty$$

(3) $\Rightarrow$ (4) By the lemma,

$$p(x) = \sup \left\{ \sum_{n=1}^{\infty} \|Tx_n\| : \begin{matrix} 0 \le (x_n) \in l_1[E] \\ \sum x_n = x \end{matrix} \right\}$$

defines an additive positively homogeneous functional on the cone of E which can be extended to a linear functional x_T^* in E^*. Notice $x_T^* \ge 0$. Also

$$\|Tx\| \le \|Tx^+\| + \|Tx^-\| \le p(x^+) + p(x^-)$$

$$= p(|x|) = x_T^*(|x|)$$

(4) $\Rightarrow$ (5)

$$T : E \longrightarrow F$$

E: typical seminorm; F: norm

$0 \le x^* \in E \quad p_{x^*}(x) = x^*(|x|)$

Now $x_T^*(|x|) = p_{x_T^*}(x)$ is one of these seminorms

(5) $\Rightarrow$ (6)

$$T : E \longrightarrow F$$

E: unf conv on order int; F: unf conv. on B_{F^*}

$$\Rightarrow T^* : F^* \longrightarrow E^*$$

takes B_{F^*} into order interval

(6) $\Rightarrow$ (7) Suppose $T(B_{F^*}) \subset [-x_0^*, x_0^*]$

If $0 \le z \in E$ and $\|y^*\| \le 1$, $y^* \in F^*$, then

$$|y^*(Tz)| = |T^* y^*(z)| \le x_0^*(z)$$

$$\Rightarrow \|Tz\| = \sup_{\|y^*\|\le 1} |y^*(Tz)| \le x_0^*(z)$$

Therefore, for any $x \in E$,

$$\|Tx\| \le \|Tx^+\| + \|Tx^-\| \le x_0^*(x^+) + x_0^*(x^-) = x_0^*(|x|)$$

If $X = \{x^* \in E^* : \|x^*\| \le 1, x^* \ge 0\}$ with w^* top., we can define

$$\Phi : E \longrightarrow C(X)$$

$$\Phi(x)(x^*) = x^*(x)$$

Then Φ is an isometry and lattice isomorphism into $C(X)$

$$\Phi(x^+)(x^*) = x^*(x^+) = \sup_{0 \le y^* \le x^*} y^*(x)$$

$$= \sup_{0 \le y^* \le x^*} \Phi(x)\, y^* = \Phi(x)^+(x^*)$$

Problem Set #1

(Three quickies just to get your motor running!)

Due Friday, Sept. 7

1. Suppose that E is an ordered vector space in which every element has a minimal decomposition into positive elements in the following sense: Given $x \in E$, there exist $y \geq 0$, $z \geq 0$ in E such that:

   (1) $x = y - z$

   (2) If $x = y' - z'$ where $y' \geq 0$, $z' \geq 0$, then $y \leq y'$ and $z \leq z'$

Prove that E is a vector lattice

2. Suppose that $P[0,1]$ is the vector space of all functions defined on the unit interval $[0,1]$ that are polynomials with real coefficients. Equip $P[0,1]$ with the pointwise ordering:

$$p_1 \leq p_2 \quad \text{if} \quad p_1(t) \leq p_2(t) \quad \text{for all } t \in [0,1]$$

Show that if $p \in P[0,1]$ is not in the cone and if $q \in P[0,1]$ satisfies $q \geq p$, $q \geq 0$, th[en] it is always possible to construct a $q_1 \in P[0,1]$ such that $q_1 \geq 0$, $q_1 \geq p$, $q \geq q_1$, $q \neq q_1$. (I[t] follows that $P[0,1]$ is not a vector lattice.)

3 Suppose that $M[0,1]$ is the vector space of Lebesgue measurable functions (not equivalence classes) on $[0,1]$ equipped with the order

$$f \leq g \quad \text{if} \quad f(t) \leq g(t) \quad \text{for all } t \in [0,1]$$

$M[0,1]$ is a σ-order complete vector lattice by Math 441
Show that $M[0,1]$ is not order complete.

$\frac{3}{3}$

Fine job!

1) **Lemma:** If $x^+ := \sup\{0,x\}$ exists for every $x \in E$, then E is a vector lattice.

Proof. Let $x, y \in E$ and define

$$m := (x-y)^+ + y$$

Then

$$x - y \le (x-y)^+ \Rightarrow x = (x-y) + y \le (x-y)^+ + y = m$$

$$0 \le (x-y)^+ \Rightarrow y \le y + (x-y)^+ = m$$

Hence m is an upper bound for $\{x,y\}$. If b is any upper bound, then

$$x - y = (b-y) - (b-x) \le (b-y)$$
$$0 \le b-y$$

and so $(x-y)^+ \le b-y$. Therefore

$$m = (x-y)^+ + y \le b - y + y = b$$

Hence $m = \sup\{x,y\}$. Therefore the supremum of every two-element set exists

It is clear that the infimum of every two-element set exist by taking

$$\inf\{x,y\} := -\sup\{-x,-y\}$$

Therefore E is a vector lattice. OK

<u>THEOREM</u>: Suppose that E is an ordered vector space in which every element has a minimal decomposition into positive elements in the following sense: Given $x \in E$, there exists $y \geq 0$, $z \geq 0$ in E such that:

(1) $x = y - z$

(2) If $x = y' - z'$, where $y' \geq 0$, $z' \geq 0$, then $y \leq y'$ and $z \leq z'$.

Then E is a vector lattice.

Proof. By the lemma it suffices to show that $\sup\{0,x\}$ exists for every $x \in E$.

Let $x \in E$ and define $x^+ := y$, where $x = y - z$ is the minimal decomposition (such a decomposition is unique by condition (2).) Clearly $0 \leq x^+$. Also, since $z \geq 0$,

$$x = y - z \leq y = x^+$$

Therefore x^+ is an upper bound for $\{0,x\}$. Suppose $0 \leq b$ and $x \leq b$. Then

$$X = b - (b - x)$$

By condition (2), $x^+ = y \le b$, and so $\sup\{x, 0\} = x^+$ exists.

OK

$\frac{1}{1}$

$\blacksquare$

2) Suppose that $P[0,1]$ is the vector space of all functions defined on the unit interval $[0,1]$ that are polynomials with real coefficients. Equip $P[0,1]$ with the pointwise ordering

$$p_1 \leq p_2 \iff p_1(t) \leq p_2(t) \quad \forall t \in [0,1]$$

Suppose $p, q \in P[0,1]$ satisfy

$$\begin{aligned} &(i) \quad q \geq p \\ &(ii) \quad q \geq 0 \\ &(iii) \quad q \neq p,\ q \neq 0 \end{aligned}$$

Then there exist $q_1 \in P[0,1]$ such that $q_1 \geq 0$, $q_1 \geq p$, and $q_1 < q$.

Proof. Since q and $q-p$ are continuous, they are bounded on $[0,1]$. Let

$$M_1 := \max_{0 \leq t \leq 1} q(t) \qquad M_2 := \max_{0 \leq t \leq 1} (q(t) - p(t))$$

Note that $M_1 > 0$ and $M_2 > 0$ by (i), (ii) and (iii). Choose $\alpha > 0$ such that

$$\alpha < \min\left\{ \frac{1}{M_1}, \frac{1}{M_2} \right\}$$

and define

$$q_1 := q - \alpha q (q-p)$$

Then $q_1(t) \le q(t)$ $\forall t \in [0,1]$. Moreover

$$\alpha M_1 < 1 \Rightarrow \alpha q(t) < 1 \ \forall t \Rightarrow \alpha q(t)(q(t)-p(t)) \le q(t) - p(t)$$

$$\Rightarrow p(t) \le q(t) - \alpha q(t)(q(t)-p(t)) \quad \forall t$$

$$\Rightarrow p \le q_1$$

$$\alpha M_2 < 1 \Rightarrow \alpha(q(t)-p(t)) < 1 \ \forall t \Rightarrow \alpha q(t)(q(t)-p(t)) \le q(t) \ \forall t$$

$$\Rightarrow 0 \le q(t) - \alpha q(t)(q(t)-p(t)) \ \forall t$$

$$\Rightarrow 0 \le q_1$$

Since $q \neq 0$, there exists some interval (a,b) such that $q(t) > 0$ for all $t \in (a,b)$. Now $q-p$ is a polynomial, so it is impossible for $q(t)-p(t)$ to be zero for all $t \in (a,b)$. Therefore there exists some $t_0 \in [0,1]$ with

This is where this argument fails for functions assumed only to be continuous

$$q(t_0)(q(t_0) - p(t_0)) > 0$$

Then $q_1(t_0) < q(t_0)$. Hence $q_1 \neq q$, so $q_1 < q$.
Finally, it is clear that q_1 is a polynomial, so $q_1 \in P[0,1]$.

$\frac{1}{1}$ $\boxtimes$

Therefore $P[0,1]$ is not a vector lattice.

3) Suppose that $M[0,1]$ is the vector space of Lebesgue measurable functions on $[0,1]$ equipped with the order

$$f \leq g \iff f(t) \leq g(t) \quad \forall t \in [0,1]$$

Then $M[0,1]$ is not order complete.

Proof. Let E be a non-measurable subset of $[0,1]$. Let

$$\mathcal{F} := \{ \chi_{\{x\}} : x \in E \}$$

Then $\mathcal{F}$ is a family of measurable functions on $[0,1]$. Suppose $\sup \mathcal{F}$ did exist, and let $f := \sup \mathcal{F}$. Then

$$\chi_{\{x\}} \leq f \Rightarrow 1 = \chi_{\{x\}}(x) \leq f(x) \quad \forall x \in E$$

Suppose there were some $x_0 \in E$ with $1 < f(x_0)$. If

$$g(t) := \begin{cases} f(t) & t \neq x_0 \\ \frac{1}{2}(1 + f(x_0)) & t = x_0 \end{cases}$$

Then $\chi_{\{x\}} \leq g$ for every $x \in E$ and $g < f$ [since $g(x_0) < f(x_0)$], which contradicts the choice of f. Therefore $f(x) = 1$ for every $x \in E$. If $x \notin E$, then $0 \leq f(x)$. Suppose therefore were an $x_0 \notin E$ with $0 < f(x_0)$. If

$$h(t) := \begin{cases} f(t) & t \neq x_0 \\ \frac{1}{2} f(x_0) & t = x_0 \end{cases}$$

Then $\chi_{\{x\}} \leq h$ for every $x \in E$ and $h < f$, again a contradiction. Therefore $f(x) = 0$ for every $x \notin E$. Hence

$$f = \chi_E$$

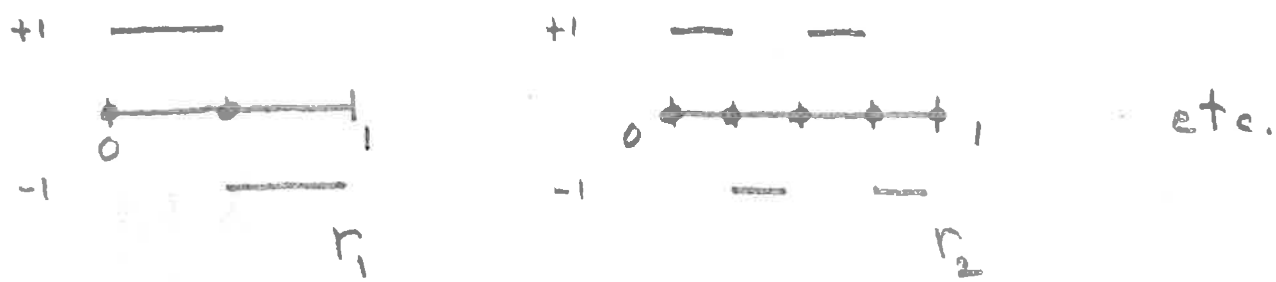
But χ_E is not measurable! Therefore $\sup \mathcal{F}$ does not exist in $M[0,1]$. $\square$

OK

$\frac{1}{1}$

Math 450 Problem Set #2
(Due Fri. Sept. 21)

The <u>Rademacher functions</u> $\{r_n(t)\}$ are defined on $[0,1]$ as follows: Divide the interval $[0,1]$ into 2^n subintervals of equal length that are disjoint. The graph of f_n is then obtained by alternating the values $+1$ and -1 on the interior of these subintervals with the value 0 assigned to endpoints



The Rademacher functions form an orthonormal sequence in $L^2[0,1]$ in the sense that $\int_0^1 r_n(t)\, r_m(t)\, dt = \delta_{mn}$

If $p>1$, the <u>Rademacher</u> <u>coefficients</u> (λ_n) of $f \in L^p[0,1]$ are defined by $\lambda_n = \int_0^1 f(t)\, r_n(t)\, dt$. It can be shown that there is a constant C_p such that $\|(\lambda_n)\|_2 \le C_p \|f\|_p$ for all $f \in L^p[0,1]$. Consequently, $Tf = (\lambda_n)$ defines a bounded linear mapping of $L^p[0,1]$ into ℓ^2.

4. Prove that T does not have an absolute value $|T| : L^p[0,1] \longrightarrow \ell^2$

5. Define $S : L^1[0,1] \longrightarrow L^\infty[0,1]$ by $Sf = \sum_{n=1}^{\infty} \lambda_n \chi_{\left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]}$
Prove S is not weakly compact, that $|S|$ exists, and that $|S|$ is compact.

Larry Riddle

$\frac{3}{3}$ Fine work!

④ Define $T : L^p[0,1] \longrightarrow \ell^2$ $(1<p<\infty)$ by

$$(Tf)_n := \int_0^1 f r_n \, d\mu$$

where (r_n) are the Rademacher functions.

<u>CLAIM</u>: T does not have an absolute value $|T| : L^p[0,1] \to \ell^2$.

Proof. Suppose $|T|$ does exist. For each n there exists a $g_n \in L^q[0,1]$ $(1/p + 1/q = 1)$ such that

$$(|T|f)_n = \int f g_n \, d\mu$$

We claim that for each n, $g_n \geq 1$ a.e.. To see this, first let

$$E_n = \{\omega : g_n(\omega) < r_n(\omega)\}.$$

Now $\chi_{E_n} \geq 0$, so $|T|\chi_{E_n} \geq T\chi_{E_n}$. In particular

$$\int_{E_n} g_n \, d\mu \geq \int_{E_n} r_n \, d\mu$$

$$\Rightarrow \quad 0 \geq \int_{E_n} (r_n - g_n) \, d\mu$$

But on E_n, $r_n - g_n > 0$, so we must have $\mu(E_n) = 0$. Hence

$$g_n \geq r_n \text{ a.e.}$$

Now let

$$F_n = \{\omega : g_n(\omega) < -r_n(\omega)\}$$

As before, $|T|\chi_{F_n} \geq (-T)\chi_{F_n}$, so

$$\int_{F_n} g_n \, d\mu \geq -\int_{F_n} r_n \, d\mu$$

$$\Rightarrow \quad 0 \leq \int_{F_n} (g_n + r_n) \, d\mu$$

But on F_n, $g_n + r_n < 0$, so $\mu(F_n) = 0$. Therefore

$$g_n \geq -r_n \quad \text{a.e.}$$

Combining this with the other inequality $g_n \geq r_n$ a.e., we see that

$$g_n \geq |r_n| = 1 \quad \text{a.e.}$$

But if $f(\omega) = 1 \;\; \forall \omega \in [0,1]$, then $f \in L^p[0,1]$, and

$$(|T|f)_n = \int g_n \, d\mu \geq 1$$

Hence the coordinates of $|T|f$ do not converge to 0! Therefore $|T|f \notin \ell^2$, a contradiction which shows that $|T|$ does not exist.

OK

$\frac{1}{1}$

(5) Define $S : L^1[0,1] \longrightarrow L^\infty[0,1]$ by

$$Sf := \sum_{n=1}^{\infty} \left(\int_0^1 f r_n \, d\mu \right) \chi_{(1/2^n, 1/2^{n-1}]}$$

a) S is not weakly compact.

Weakly compact operators from L_1 to $\mathfrak{X}$ take bdd unif. int. sets into norm compact set. Since $\|S(r_n) - S(r_m)\| \geq 1 \ \forall n,m$, (Sr_n) can't belong to a norm compact set.

For each m, let

$$f_m := 2^m \chi_{[0, 1/2^m]}$$

Then $\|f_m\|_1 = 1$, so the sequence (f_m) is bounded. Now

$$\int_0^1 f_m r_n \, d\mu = \begin{cases} 2^m & \text{if } 1 \leq n \leq m \\ 0 & \text{if } n > m \end{cases}$$

Since

$$\int_{[0, 1/2^m]} r_n \, d\mu = \begin{cases} 1 & \text{if } 1 \leq n \leq m \\ 0 & \text{if } n > m \end{cases}$$

This is wrong

Therefore

$$Sf_m = \sum_{n=1}^{m} 2^m \chi_{(1/2^n, 1/2^{n-1}]} = 2^m \chi_{(1/2^m, 1]}$$

and so $\|Sf_m\|_\infty = 2^m$. But weakly compact sets are norm bounded. Therefore S(unit ball of L_1) is not relatively weakly compact since $\|Sf_m\| \to \infty$.

OK

(b) $|S|$ exists and $|S|(f) = \left(\int f\,d\mu\right)\chi_{[0,1]}$

Let $A : L^1[0,1] \to L^\infty[0,1]$ be given by

$$Af := \left(\int f\,d\mu\right)\chi_{[0,1]}$$

Suppose $f \in L^1[0,1]$ and $f \geq 0$. If $|g| \leq f$ in $L^1[0,1]$, then

$$r_n g \leq f \quad \forall n$$

$$\Rightarrow \int r_n g\,d\mu \leq \int f\,d\mu \quad \forall n$$

$$\Rightarrow Sg \leq Af$$

Hence Af is an upper bound for the set $\{Sg : |g| \leq f\}$ in $L^\infty[0,1]$. Suppose h is any upper bound for this set. Since $|r_n f| \leq f$ for each n

$$S(r_n f)(\omega) \leq h(\omega) \quad \forall \omega \notin E_n$$

where $\mu(E_n) = 0$. Let $E = \bigcup_{n=1}^{\infty} E_n$, so $\mu(E) = 0$. Suppose $\omega \notin E$ and $\omega \in (1/2^n, 1/2^{n-1}]$ for some n. Then $\omega \notin E_n$, so

$$S(r_n f)(\omega) \leq h(\omega)$$

But

$$S(r_n f)\omega = \int_0^1 r_n r_n f \, d\mu = \int_0^1 f \, d\mu$$

Hence

$$\int_0^1 f \, d\mu \le h(\omega) \quad \forall \omega \notin E$$

Therefore $Af \le h$ in $L^\infty[0,1]$. We therefore conclude that

$$\sup\{Sg : |g| \le f\} = Af$$

for $f \ge 0$. By our proposition, $|S|$ exists and $|S|(f) = Af$ for each $f \ge 0$. But then for any $f \in L^1[0,1]$

$$|S|(f) = |S|(f^+ - f^-) = |S| f^+ - |S| f^-$$

$$= \left(\int f^+ d\mu\right) \chi_{[0,1]} - \left(\int f^- d\mu\right) \chi_{[0,1]} \qquad \text{OK}$$

$$= \left(\int f \, d\mu\right) \chi_{[0,1]}$$

(c) $|S|$ is compact

2/2

This is immediate since $|S|$ has a finite dimensional range (in fact its range has dimension 1 : $\mathrm{Rng}(|S|) = \mathrm{sp}\{\chi_{[0,1]}\}$. OK

Larry Riddle

3/3

Fine work!

(6) **THEOREM:** The lattice operations in $C[a,b]$ are weakly sequentially continuous, i.e. if $f_n \to f$ weakly, then $|f_n| \to |f|$ weakly.

Proof. Suppose $f_n \to f$ weakly in $C[a,b]$. Let $\omega \in [a,b]$ and define

$$\mu_\omega(E) := \begin{cases} 1 & \text{if } \omega \in E \\ 0 & \text{if } \omega \notin E \end{cases}$$

Then $\mu_\omega \in C[a,b]^*$. Then

$$\int f_n \, d\mu_\omega \longrightarrow \int f \, d\mu_\omega$$

$$\Rightarrow \quad f_n(\omega) \to f(\omega)$$

Therefore f is the pointwise limit of the f_n's. Now the set $\{f_n : n \in \mathbb{N}\}$ is uniformly bounded since (f_n) is weakly Cauchy. Since

$$\lim_n |f_n|(\omega) = |f|(\omega)$$

for each $\omega \in [a,b]$, the dominated convergence theorem implies that

$$\int |f_n| \, d\mu \longrightarrow \int |f| \, d\mu$$

for each $\mu \in C[a,b]^*$. Therefore $|f_n| \to |f|$ weakly. ▨

1/1

⑦ <u>Example</u>: The lattice operations in $L^1[0,1]$ are not weakly sequentially continuous.

Proof. Let $f_k(t) := \sin k\pi t$. Then each f_k belongs to $L^1[0,1]$. The Riemann-Lebesgue lemma says that

$$\int_0^1 g(t) f_k(t)\, dt \longrightarrow 0$$

for every $g \in L^1[0,1]$. Hence in particular

$$\int_0^1 g(t) f_k(t)\, dt \longrightarrow 0$$

for every $g \in L^\infty[0,1]$. Therefore $f_k \to 0$ weakly in $L^1[0,1]$. But for each k

$$\int_0^1 |f_k(t)|\, dt = k \int_0^{1/k} \sin k\pi t\, dt = k\left[-\frac{\cos k\pi t}{k\pi} \Big|_0^{1/k} \right]$$

$$= \frac{1}{\pi}\left(-\cos\pi + \cos 0 \right) = \frac{2}{\pi}$$

Since the constant function $\mathbb{1}$ belongs to $L^\infty[0,1]$, this shows that $|f_k|$ does not converge weakly to 0. ☒

$\frac{1}{1}$

⑧ <u>THEOREM</u>: Suppose that (X,Σ,μ) is a finite measure space. If C is a relatively weakly compact subset of $L^1(X,\Sigma,\mu)$, then the solid hull

$$|C| := \{ g \in L^1(\mu) : |g| \le |f| \text{ for some } f \in C \}$$

of C is also relatively weakly compact.

Proof. Since (X,Σ,μ) is a finite measure space, a subset of $L^1(\mu)$ is relatively weakly compact if and only if it is bounded and uniformly integrable. Therefore there exists an $M>0$ such that

$$\|f\|_1 \le M \quad \forall f \in C$$

If $g \in |C|$, then $|g| \le |f|$ for some $f \in C$, so

$$\|g\|_1 = \int |g| \, d\mu \le \int |f| \, d\mu = \|f\|_1 \le M$$

Therefore $|C|$ is bounded. Now let $\varepsilon>0$. There exist a $\delta>0$ such that

$$\mu(E)<\delta \implies \sup_{f\in C} \int_E |f| \, d\mu < \varepsilon$$

Let $g \in |C|$, so $|g| \le |f|$ for some $f \in C$. Then if $\mu(E)<\delta$ we have

$$\int_E |g| \le \int_E |f| \, d\mu < \varepsilon$$

Therefore

$$\mu(E)<\delta \Rightarrow \sup_{g\in|C|} \int_E |g|\,d\mu < \varepsilon$$

and so $|C|$ is uniformly integrable. Thus $|C|$ is relatively weakly compact.

$$\frac{1}{1}$$

Math 450 Problem Set #4
(Due Monday, Oct 8)

Consider the vector lattice $C[0,1]$ of continuous real-valued functions on the unit interval. For each band M in $C[0,1]$, define

$$O_M = \{x \in [0,1] : f(x) \neq 0 \text{ for some } f \in M\} \qquad G_M = [0,1] \setminus \overline{O}_M$$

9. Prove that O_M is an open subset of $[0,1]$ and that if O_M is dense in $[0,1]$, then $M = C[0,1]$.

10. If O_M is not dense in $[0,1]$, then show that

$$M = \{h \in C[0,1] : h(x) = 0 \text{ for all } x \in G_M$$

11. Prove that

$$N = \{f \in C[0,1] : f(x) = 0 \text{ for } 0 \leq x \leq \tfrac{1}{2}\}$$

is a band in $C[0,1]$ and determine $N^\perp$.

12. Prove that the only projection bands in $C[0,1]$ are the trivial bands: $\{0\}$ and $C[0,1]$. (Hint: For a given projection band P in $C[0,1]$, compute the components in P and $P^\perp$ of the function that is identically equal to one on $[0,1]$.)

Larry Riddle

4/4

Fine work!

Consider the vector lattice $C[0,1]$ of continuous real-valued functions on the unit interval. For each band M in $C[0,1]$ define

$$O_M := \{x \in [0,1] : f(x) \neq 0 \text{ for some } f \in M\}$$

$$G_M := [0,1] \setminus \overline{O_M}$$

9 **PROPOSITION:** O_M is an open subset of $[0,1]$ and if O_M is dense in $[0,1]$, then $M = C[0,1]$.

Proof. Let $x \in O_M$. Then there exists $f \in M$ such that $f(x) \neq 0$. Since f is continuous there exists an open interval I containing x such that $f(y) \neq 0$ for each $y \in I$. Therefore $I \subset O_M$, so O_M is open.

Now suppose O_M is dense. Let $g \in M^\perp$. If there exist an $x \in [0,1]$ with $g(x) \neq 0$, then there exists a $y \in O_M$ and an $f \in M$ such that $f(y) \neq 0$ and $g(y) \neq 0$. But then

$$|f| \wedge |g| \, (y) \neq 0$$

which contradicts $g \in M^\perp$. Hence $g(x) = 0$ for all $x \in [0,1]$, so $M^\perp = \{0\}$. Since M is a band in the Archimedean vector lattice $C[0,1]$ OK

$$C[0,1] = \{0\}^\perp = (M^\perp)^\perp = M$$

(10) **PROPOSITION**: $M = \{h \in C[0,1] : h(x) = 0$ for all $x \in G_M\}$.

Proof. Let E be the set on the right. Suppose $h \in M$. If $x \in G_M$, then $x \notin O_M$, so $f(x) = 0$ for every $f \in M$. In particular, $h(x) = 0$. Therefore $h \in E$, so $M \subset E$.

Now suppose $f \in M^\perp$ and g is an arbitrary element of E. If $x \in G_M$ then $g(x) = 0$, so

$$|f| \wedge |g| \, (x) = 0.$$

If $x \notin G_M$, then $x \in \overline{O_M}$. Suppose $f(x) \neq 0$. Then there exists a $y \in O_M$ and an $h \in M$ such that $f(y) \neq 0$ and $h(y) \neq 0$. But this contradicts the fact that f belongs to $M^\perp$. Hence $f(x) = 0$. Therefore

$$|f| \wedge |g| (x) = 0.$$

Therefore $|f| \wedge |g| = 0$, and so $f \in E^\perp$. Hence $M^\perp \subset E^\perp$, from which we obtain

$$E \subset E^{\perp\perp} \subset M^{\perp\perp} = M.$$

Therefore $M = E$. OK

$\blacksquare$

Note: We did not need to assume O_M was not dense. Indeed, if O_M is dense, then $G_M = \emptyset$, so every $h \in C[0,1]$ satisfies $h(x) = 0 \ \forall x \in G_M$ vacuously. We would then get $C[0,1] = E = M$, which proves (9) again.

(11) <u>Example</u>: The set

$$N = \{f \in C[0,1] : f(x) = 0 \text{ for } 0 \le x \le 1/2\}$$

is a band in $[0,1]$.

Proof. It is obvious that N is a lattice ideal of $C[0,1]$. Now suppose $A \subset N$ and $f := \sup A$ exists in $C[0,1]$. Suppose $f(x_0) > 0$ for some $x_0 \in [0,1/2]$. We may clearly assume that $x_0 \ne 1/2$. Now choose a continuous function g on $[0,1]$ such that

$$0 \le g \le 1$$
$$g(x_0) = 0$$
$$g([1/2, 1]) = 1$$

and define $h := fg$. For $x \in [0,1/2]$,

$$0 \le f(x)\, g(x) = h(x)$$

since $f(x) \ge 0$ and $0 \le g(x) \le 1$. For $x \in [1/2, 1]$

$$h(x) = f(x) g(x) = f(x) \ge k(x) \quad \forall k \in A$$

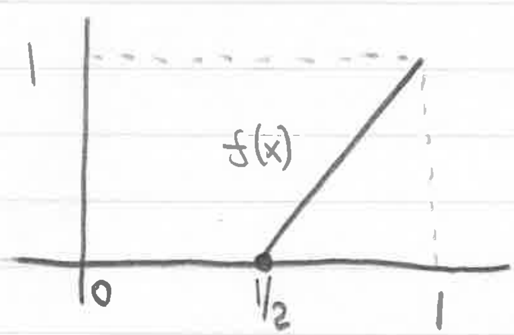
Therefore h is an upper bound for A. Also, $h \le f$, since $h = f$ on $[1/2, 1]$, and on $[0,1/2]$, f is positive and $g \le 1$. However $h(x_0) = 0 < f(x_0)$, which contradicts the fact that f is the supremum of A. Therefore we must have $f(x) = 0$ for all $x \in [0,1/2]$, and so $f \in N$. Hence N is a band.

Claim: $N^{\perp} = \{ h \in C[0,1] : f(x) = 0 \text{ for } 1/2 \le x \le 1 \}$

Proof. Let E denote the set on the right. If $h \in E$, then

$$|h| \wedge |f| = 0 \quad \forall f \in N$$

since $h(x) = 0$ on $[1/2, 1]$ and $f(x) = 0$ on $[0, 1/2]$. Therefore $E \subset N^{\perp}$.
Suppose $h \in N^{\perp}$. The function f whose graph is



belongs to N, and so $|h| \wedge |f| = 0$. Therefore $h(x) = 0$ for $1/2 < x \le 1$.
But h is continuous, so $h(1/2) = 0$ also. Hence $h \in E$, and thus
$N^{\perp} \subset E$. Therefore $N^{\perp} = E$.

OK

$\boxtimes$

(12) **PROPOSITION**: The only projection bands in $C[0,1]$ are the trivial bands.

Proof. Let P be a projection band. Then $C[0,1] = P + P^\perp$. Let $f \in C[0,1]$ be the function

$$f(x) = 1 \quad \forall x \in [0,1]$$

Then $f = f_1 + f_2$, where $f_1 \in P$ and $f_2 \in P^\perp$. Suppose $f_1(x) \neq 0$. Then $f_2(x) = 0$ since $|f_1| \wedge |f_2| = 0$. But $f_1(x) + f_2(x) = f(x) = 1$, so $f_1(x) = 1$. Therefore f_1 takes only the values 0 and 1, i.e.

$$f_1 = \chi_A$$

for some $A \subset C[0,1]$. But the only continuous characteristic functions are 0 and 1. Hence either $f_1 = 0$ or $f_1 = 1$. Therefore either $1 \in P$ or $1 \in P^\perp$. OK

[[Claim: If M is lattice ideal in $C[0,1]$ and $1 \in M$, then $M = C[0,1]$. For suppose $f \in C[0,1]$. Then $\exists \alpha > 0$ s.t. $|f(x)| < \alpha$ for all x. If $1 \in M$, then $\alpha 1 \in M$. But $|f| \leq |\alpha| 1 = |\alpha 1|$, so $f \in M$ since M is a lattice ideal.]]

Therefore either $P = C[0,1]$ or $P^\perp = C[0,1]$. In the latter case $P = \{0\}$.

OK

$\boxtimes$

Math 450 Problem Set #5
(Due Friday Oct 19)

13. Suppose that E is a Banach lattice and that $\{x_n\}$ is a sequence of disjoint positive elements of E that is bounded above and such that for some $\varepsilon > 0$, $\|x_n\| \geq \varepsilon$ for all n. Prove that some sublattice of E is topologically and lattice isomorphic to c_0 in its usual order and norm (Note: Vector lattices E, F are said to be lattice isomorphic if there is a 1-1, onto, linear $T: E \to F$ such that $T(x \vee y) = Tx \vee Ty$ and $T(x \wedge y) = Tx \wedge Ty$ for all $x, y \in E$)

14. Suppose that E, F are Banach lattices and that $T: E \to F$ is a positive linear map. Then:

(a) T is a <u>lattice homomorphism</u> if T preserves lattice operations (eg. $T(x \vee y) = Tx \vee Ty$ for all x, y in E)

(b) T is <u>almost interval preserving</u> if $T[0,x]$ is dense in $[0,Tx]$ for all $x \geq 0$

(c) T is <u>interval preserving</u> if $T[0,x] = [0,Tx]$ for all $x \geq 0$

Prove that if T is almost interval preserving, then its adjoint T^* is a lattice homomorphism

15 Prove that if T^* is a lattice homomorphism, then T is almost interval preserving. (Hint: A separation argument can be used.)

16. Prove that T is a lattice homomorphism if and only if T^* is interval preserving (or almost interval preserving) (Hint. Again a separation argument can be used – this time for the weak*-topology)

Larry Riddle

4/4 Excellent!

(13) THEOREM: Suppose that E is a Banach lattice and that (x_n) is a sequence of disjoint positive elements of E that is bounded above and such that for some $\varepsilon > 0$, $\|x_n\| \geq \varepsilon$ for all n. Then some sublattice of E is topologically and lattice isomorphic to c_0.

Proof. Let $\hat{c}_0$ be the subspace of c_0 consisting of all sequences with only finitely many non-zero components. Define $T: \hat{c}_0 \to E$ by

$$T((\alpha_n)) = \sum_{n=1}^{\infty} \alpha_n x_n$$

Then if $\alpha = (\alpha_n) \in \hat{c}_0$, with $\alpha_n = 0 \ \forall n \geq m$

$$|T(\alpha)| = \left|\sum_{n=1}^{\infty} \alpha_n x_n\right| \leq \sum_{n=1}^{m} |\alpha_n x_n| = \sum_{n=1}^{m} |\alpha_n| x_n$$

$$\leq (\sup|\alpha_n|) \sum_{n=1}^{m} x_n$$

since the x_n's are disjoint (rule 14) $$= \|\alpha\| \sup_{n \leq m} x_n \leq \|\alpha\| u$$

(where u is an upper bound for the set $\{x_n : n \in \mathbb{N}\}$). Hence

$$\|T\alpha\| \leq \|\alpha\| \|u\|,$$

so T is continuous. Since $\hat{c}_0$ is dense in c_0, T has a continuous linear extension to c, also denoted by T.

Moreover, we also have

$$(T\alpha)^+ = \left(\sum_{n=1}^{m} \alpha_n x_n\right)^+ = \sum_{n=1}^{m} (\alpha_n x_n)^+ \quad [\![\text{since } \alpha_n x_n\text{'s are disjoint}]\!]$$

$$= \sum_{n=1}^{m} \alpha_n^+ x_n \qquad [\![\text{since } x_n \geq 0]\!]$$

$$= \sum_{\alpha_n \geq 0} \alpha_n x_n$$

$$= T(\alpha^+)$$

Therefore $(T\alpha)^+ = T(\alpha^+)$ for every $\alpha \in \hat{c}_0$. If $\alpha \in c_0$, choose $\alpha_n \in \hat{c}_0$ such that $\alpha_n \longrightarrow \alpha$. Then $\alpha_n^+ \longrightarrow \alpha^+$, so

$$T(\alpha^+) = \lim (T\alpha_n^+) = \lim (T\alpha_n)^+ = (T\alpha)^+$$

(since $+$ is continuous). Therefore T is a lattice homomorphism on c_0.
Finally, if $\alpha = (\alpha_n) \in \hat{c}_0$ and $\alpha_n = 0$ $\forall n \geq m$, then

$$|T(\alpha)| = T(|\alpha|) = \sum_{n=1}^{m} |\alpha_n| x_n \geq |\alpha_k| x_k \quad \forall k$$

$$\Rightarrow \|T\alpha\| \geq |\alpha_k| \|x_k\| \geq |\alpha_k| \varepsilon \quad \forall k$$

$$\Rightarrow \|T\alpha\| \geq \sup_k |\alpha_k| \varepsilon = \|\alpha\| \varepsilon$$

$\frac{1}{1}$

Therefore T is 1-1 and $T^{-1}|_{R(T)}$ is continuous.
Hence c_0 is topologically and lattice isomorphic to $T(c_0)$, which is by consequence a sublattice of E.

$\boxtimes$ OK

(14) **PROPOSITION**: Suppose E and F are Banach lattices and $T: E \to F$ is a positive linear map. If T is almost interval preserving, then T^* is a lattice homomorphism.

Proof. Let $f \in F^*$. Then if $\alpha \geq 0$ in E,

$$\begin{aligned}
(T^* f)^+ \alpha &= \sup \{ T^* f(z) : z \in [0, \alpha] \} \\
&= \sup \{ f(Tz) : z \in [0, \alpha] \} \\
&= \sup \{ f(w) : w \in T[0, \alpha] \} \\
&= \sup \{ f(w) : w \in \overline{T[0, \alpha]} \} \\
&= \sup \{ f(w) : w \in [0, T\alpha] \} \\
&= f^+(T\alpha) \\
&= (T^* f^+) \alpha
\end{aligned}$$

Since f is continuous

Hence $(T^* f)^+ = T^*(f^+)$ for all $f \in F^*$, so T^* is a lattice homomorphism.

OK

$\blacksquare$

$\frac{1}{1}$

(15) PROPOSITION: Let T be as in problem 14. If T^* is a lattice homomorphism, then T is almost interval preserving.

Proof. Suppose there exists $\alpha \in [0, Tx] \setminus \overline{T[0,x]}$. Then there exists an $f \in F^*$ such that

$$f(\alpha) > \sup f(\overline{T[0,x]})$$

On the other hand,

$$f(\alpha) \le f^+(\alpha) \le f^+(Tx) = T^*(f^+)x = (T^*f)^+x$$

$$= \sup\{T^*f(z) : z \in [0,x]\}$$

$$= \sup\{f(Tz) : z \in [0,x]\}$$

$$= \sup\{f(w) : w \in T[0,x]\}$$

$$= \sup\{f(w) : w \in \overline{T[0,x]}\} < f(\alpha)$$

which is a contradiction. Hence $[0, Tx] = \overline{T[0,x]}$, so T is almost interval preserving.

OK

$\frac{1}{1}$

(16) **PROPOSITION**: Each of the following statements about a positive linear map $T: E \to F$ implies all the others:

(a) T is a lattice homomorphism;
(b) T^* is interval preserving;
(c) T^* is almost interval preserving.

Proof. (a) $\Rightarrow$ (b) : Let $f \geq 0$. Since $[0,f]$ is weak* compact in F^*, $T^*[0,f]$ is weak*-compact in E^*. Suppose $T^*[0,f] \neq [0, T^*f]$. Then chose $g \in [0,T^*f] \setminus T^*[0,f]$ and a weak*-continuous linear functional α on E^* for which

$$\alpha(g) > \sup \alpha (T^*[0,f])$$

But the weak*-continuous linear functionals on E^* are exactly the elements of E, so $\alpha \in E$ and

$$g(\alpha) > \sup \{ h(\alpha) : h \in T^*[0,f] \}$$

On the other hand

$$g(\alpha) \leq g(\alpha^+) \leq T^*f(\alpha^+) = f(T\alpha^+) = f((T\alpha)^+)$$

$$= \sup \{ k(T\alpha) : k \in [0,f] \}$$

$$= \sup \{ T^*k(\alpha) : k \in [0,f] \}$$

$$= \sup\{h(\alpha) : h \in T^*[0,f]\} < g(\alpha).$$

This contradiction shows that T^* is interval preserving. OK

(b) $\Rightarrow$ (c) is obvious

(c) $\Rightarrow$ (a). Let $\alpha \in E$ and let $f \geq 0$. Then

$$f(T\alpha^+) = T^*f(\alpha^+) = \sup\{g(\alpha) : g \in [0, T^*f]\}$$

$$= \sup\{g(\alpha) : g \in \overline{T^*[0,f]}\}$$

$$= \sup\{g(\alpha) : g \in T^*[0,f]\}$$

$$= \sup\{T^*h(\alpha) : h \in [0,f]\}$$

$$= \sup\{h(T\alpha) : h \in [0,f]\}$$

$$= f((T\alpha)^+)$$

Hence $T\alpha^+ = (T\alpha)^+$, so T is a lattice homomorphism. OK

Note: (c) $\Rightarrow$ (a) also follows from Problem 14 applied to T^{**} since E is a sublattice of E^{**}.

$\blacksquare$

[Note: It is obvious that T^* is positive so that it makes sense to ask if T^* is (almost) interval preserving]

Math 450 Problem Set #6
(Due Monday Oct 29)

17. Suppose that X is a compact Hausdorff space

(a) If the set of isolated points of X is dense in X prove that every normal measure on X is atomic (A measure μ on X is <u>atomic</u> if $\mu = \sum_{n=1}^{\infty} \lambda_n \varepsilon_{x_n}$ for $(\lambda_n) \in \ell^1$ (where ε_{x_n} is the point mass at x_n.))

(b) If every Radon measure on X is normal, show that X is finite.

18. The Banach lattice ℓ^∞ can be identified with $C(\beta N)$ where βN is the Stone-Čech compactification of the discrete space N of natural numbers. Find a concrete description of the normal measures on βN

19. Suppose that E is a Banach lattice and that the set E° of order continuous linear functionals on E separates points of E. For each $0 \le f \in E^\circ$, define a seminorm p_f on E by

$$p_f(x) = f(|x|) \qquad x \in E$$

Show that the topology $\mathcal{T}$ determined by the family $\mathcal{P} = \{p_f : 0 \le f \in E^\circ\}$ of seminorms is a locally convex lattice topology on E such that the dual of E for $\mathcal{T}$ is E°. (Hint: Use the Mackey-Arens Theorem)

Larry Riddle

4/4

Fine work (as usual!)

(17) **THEOREM:** Let X be a compact Hausdorff space. If the set of isolated points of X is dense in X, then every normal measure on X is atomic

Proof. Let I denote the set of isolated points of X. Then for any finite subset F of I,

$$\sum_{x\in F} |\mu|(\{x\}) = |\mu|(F) \le |\mu|(X).$$

Therefore

$$\sum_{x\in I} |\mu|(\{x\}) = \lim_F \sum_{x\in F} |\mu|(\{x\}) \le |\mu|(X) < \infty$$

and so only countably many of the elements in I have non-zero measure. Let I_0 denote this countable set.

Let K be a compact subset of $I \setminus I_0$. Then

$$K = \bigcup_{x\in K} \{x\}$$

and each set $\{x\}$ is open for $x \in K$ since x is isolated. Therefore K is finite, and so

$$\mu(K) = \sum_{x\in K} \mu(\{x\}) = 0$$

since $K \cap I_0 = \phi$. Since μ is regular and $\mu(K) = 0$ for each compact subset of $I\setminus I_0$, we must have $\mu(I\setminus I_0) = 0$.

Now let A be any Borel set and let K be any compact subset of $A \setminus I$. Then K is nowhere dense since I is dense in X. Since μ is normal, we have $\mu(K) = 0$. Once again we use the regularity to conclude that $\mu(A \setminus I) = 0$.

Hence for any Borel set A

$$\mu(A) = \mu(A \setminus I) + \mu(A \cap I \setminus I_0) + \mu(A \cap I_0)$$

$$= 0 + 0 + \mu(A \cap I_0) = \mu(A \cap I_0)$$

$$= \sum_{x \in A \cap I_0} \mu(\{x\}) \qquad (\text{since } A \cap I_0 \text{ is countable})$$

$$= \sum_{x \in I_0} \mu(\{x\})\, \varepsilon_x(A)$$

Therefore

$$\mu = \sum_{x \in I_0} \mu(\{x\})\, \varepsilon_x$$

OK

Note that

$$\sum_{x \in I_0} |\mu(\{x\})| \leq \sum_{x \in I_0} |\mu|(\{x\}) \leq |\mu|(X) < \infty$$

so that the if $(x_n : n \in \mathbb{N})$ is any enumeration of I_0 and $\lambda_n := \mu(\{x_n\})$, the sequence (λ_n) belongs to l_1 and

$$\mu = \sum_{n=1}^{\infty} \lambda_n \varepsilon_{x_n}.$$

$\blacksquare$

__PROPOSITION:__ Let X be a compact Hausdorff space. If every Radon measure on X is normal, then X is finite.

Proof. Let $x \in X$. Then by hypothesis the Radon measure ε_x is normal, and so ε_x vanishes on all Borel sets of first category. But $\varepsilon_x(\{x\}) = 1$, so that the point $\{x\}$ is not of first category, i.e. $\{x\}$ has non-empty interior. Therefore x is an isolated point. But then

$$X = \bigcup_{x \in X} \{x\}$$

is an open covering of the compact set X, and so X must be finite

OK

$\boxtimes$

(18) **LEMMA**: Let X be a compact Hausdorff space and (μ_n) a sequence of uniformly bounded regular signed Borel measures. If $(\alpha_n) \in \ell_1$, then

$$\mu := \sum_{n=1}^{\infty} \alpha_n \mu_n$$

is a regular Borel measure.

Proof. First note that if $|\mu_n| < M$ for each n, then for any Borel set E

$$(*) \qquad \sum_{n=1}^{\infty} |\alpha_n \mu_n(E)| \le \sum_{n=1}^{\infty} |\alpha_n| \, |\mu_n|(E) \le \|(\alpha_n)\| M < \infty$$

and so μ is a well-defined set function. Now for each k

$$\lambda_k := \sum_{n=1}^{k} \alpha_n \mu_n$$

is clearly a countably additive measure, and

$$\mu(E) = \lim_k \lambda_k(E)$$

for each Borel set E. Hence by Nikodym's theorem μ is countably additive. So it just remains to show the regularity. Let E be a Borel set and let $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that

$$\sum_{k=n_0+1}^{\infty} |\alpha_k| < \varepsilon/2M$$

For each $i \le n_0$ we can choose a compact set $K_i \subset E$ and an open set $V_i \supset E$ such that

$$|\mu_i(C)| < \varepsilon / 2|\alpha_i| n_0$$

for every Borel set $C \subset V_i \setminus K_i$. Let

$$K = \bigcup_{i=1}^{n_0} K_i$$

$$V = \bigcap_{i=1}^{n_0} V_i$$

Then $K \subset E \subset V$, K is compact, V is open, and if $C \subset V \setminus K$, then $C \subset V_i \setminus K_i$ for each $i \le n_0$, so

$$|\mu(C)| = \left|\sum_{i=1}^{\infty} \alpha_i \mu_i(C)\right| \le \sum_{i=1}^{\infty} |\alpha_i| |\mu_i(C)|$$

$$= \sum_{i=1}^{n_0} |\alpha_i| |\mu_i(C)| + \sum_{i=n_0+1}^{\infty} |\alpha_i| |\mu_i(C)|$$

$$\le \sum_{i=1}^{n_0} |\alpha_i| \left(\frac{\varepsilon}{2|\alpha_i| n_0}\right) + M\left(\frac{\varepsilon}{2M}\right)$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

OK

Therefore μ is regular. $\blacksquare$

THEOREM: The Banach lattice l_∞ can be identified with $C(\beta\mathbb{N})$ where $\beta\mathbb{N}$ is the Stone-Čech compactification of the discrete space $\mathbb{N}$. Then the normal measures on $\beta\mathbb{N}$ correspond to $l_1 \subset l_\infty$

Proof. First observe that $\mathbb{N}$ is dense in $\beta\mathbb{N}$ and that each point of $\mathbb{N}$ is an isolated point in $\beta\mathbb{N}$. Then the same argument as in 17 (with $I = \mathbb{N}$) shows that if μ is a normal measure on $\beta\mathbb{N}$, then

$$\mu = \sum_{n=1}^{\infty} \lambda_n \varepsilon_{\{n\}}$$

where $(\lambda_n) \in l_1$. Hence we can define a map $\Phi : \eta(\beta\mathbb{N}) \to l_1$ by

$$\Phi(\mu) = (\lambda_n)$$

Then Φ is clearly linear and

$$\|\Phi(\mu)\| = \sum_{n=1}^{\infty} |\lambda_n| = \sum_{n=1}^{\infty} |\mu(\{n\})| \leq \sum_{n=1}^{\infty} |\mu|(\{n\})$$

$$\leq |\mu|(\beta\mathbb{N}) = \|\mu\|$$

and so Φ is continuous. Actually, if π is any partition of $\beta\mathbb{N}$,

$$\sum_{A\in\pi} |\mu(A)| = \sum_{A\in\pi} \left| \sum_{n=1}^{\infty} \lambda_n \varepsilon_{\{n\}}(A) \right| \leq \sum_{A\in\pi} \sum_{n=1}^{\infty} |\lambda_n| \varepsilon_{\{n\}}(A)$$

$$= \sum_{n=1}^{\infty} |\lambda_n| \sum_{A\in\pi} \varepsilon_{\{n\}}(A) = \sum_{n=1}^{\infty} |\lambda_n|$$

and so

$$\|\mu\| = |\mu|(\beta\mathbb{N}) = \sup_{\pi} \sum_{A\in\pi} |\mu(A)| \leq \sum_{n=1}^{\infty} |\lambda_n| = \|\Phi(\mu)\|$$

Therefore Φ is an isometry.

Now suppose $(\alpha_n) \in \ell_1$. Then by the lemma,

$$\mu = \sum_{n=1}^{\infty} \alpha_n \varepsilon_{\{n\}}$$

is a radon measure on $\beta\mathbb{N}$. Let $B \subset \beta\mathbb{N}$ be a closed nowhere dense set. Then $\operatorname{int} B = \phi$, so $\operatorname{int} B \cap \mathbb{N} = \phi$. Therefore

$$\mu(B) = \sum \alpha_n \varepsilon_{\{n\}}(B) = 0$$

By regularity, any nowhere dense set has zero measure, and so μ vanishes on all Borel sets of first category. Therefore $\mu \in \mathcal{N}(\beta\mathbb{N})$, and it is clear that $\Phi(\mu) = (\alpha_n)$. Hence Φ is an isometry onto ℓ_1.

Finally, if $\mu \in \mathcal{N}(\beta\mathbb{N})$, then

$$[\Phi(\mu)]^+ = (\mu(\{n\}))^+ = (\mu^+(\{n\})) = \Phi(\mu^+)$$

and so Φ is a lattice homomorphism. OK

$\blacksquare$

<u>Note</u>

Now that we have done the material on the monotone covergence property, you should be able to come up with a quick two-line proof of the above result!

(19) **THEOREM:** Suppose that E is a Banach lattice and that the set E^0 of order continuous linear functionals on E separates points of E. For each $0 \leq f \in E^0$, define a seminorm p_f on E by

$$p_f(x) := f(|x|)$$

Then the topology τ determined by the family $\mathcal{P} = \{p_f : 0 \leq f \in E^0\}$ of seminorms is a locally convex lattice topology on E such that the dual of E for τ is E^0

Proof. It is clear that p_f is a seminorm for each $f \geq 0$ since

$$p_f(x+y) = f(|x+y|) \leq f(|x|+|y|) = f(|x|)+f(|y|)$$

$$= p_f(x) + p_f(y).$$

Also, each p_f is a lattice seminorm since

$$|x| \leq |y| \Rightarrow f(|x|) \leq f(|y|) \Rightarrow p_f(x) \leq p_f(y).$$

Finally, suppose $p_f(x) = 0$ for all $f \in E^0$. Then $f(|x|) = 0$ for all $f \in E^0$ and so $|x| = 0$ since E^0 separates points of E. Therefore $x = 0$, so that the topology generated by the family $\mathcal{P}$ is Hausdorff. Hence $E(\tau)$ is a locally convex lattice space.

Observe that for each $f \geq 0$

$$p_f(x) = f(|x|) = \sup\{g(x) : g \in [-f, f]\}$$

Therefore τ is the topology of uniform convergence on order bounded sets in E°. Let γ be the collection of all order bounded sets in E°. Then γ is saturated and covers E° (we have used a similar argument in a proof in class) Also, γ consists of $\sigma(E^\circ, E)$ relatively compact sets since any order bounded set in E^* is relatively $\sigma(E^*, E)$ compact and $\sigma(E^\circ, E) \leq \sigma(E^*, E)$. Hence by the Mackey-Arens theorem the dual of E for τ is E°.

$\blacksquare$

OK