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The Domain of Solutions To Differential Equations

Larry Riddle

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The Domain of Solutions to Differential Equations

Larry Riddle*

The 2006 AB Calculus Exam asked students to find the particular solution $y = f(x)$ to the differential equation

$$\frac{dy}{dx} = \frac{y+1}{x}, x \neq 0$$

with initial condition $f(-1) = 1$ and to state its domain. The domain of a function is always an important consideration when defining or working with a function. Take, for example, the composition of functions. Let $f(x) = \sqrt{x}$ and $g(x) = x^2$. Suppose $h(x) = g(f(x))$. It is easy to see that $h(x) = x$. What is the domain of h? As a function by itself, $h(x) = x$ would be considered to have the domain of all real x. As the composition of g and f, however, the domain of h is only $x \geq 0$. Similarly, the domain of a particular solution to a differential equation can be restricted for reasons other than the function formula not being defined, and indeed, may be a subset of what the domain would be when the solution is considered only as a function.

Two Reasons to Restrict the Domain

Consider the initial value problem $dy/dx = 2/\sqrt{y}$ with $y(0) = 9$. Separating variables and integrating produces the implicit solution $\frac{2}{3}y^{3/2} = 2x + C$, and the initial value gives $C = 18$. The explicit solution is therefore $y = (3x+27)^{2/3}$. But this solution is not valid for all x. The differential equation $dy/dx = 2/\sqrt{y}$ implies that the slopes along the solution curve are positive. Thus the solution $y = (3x + 27)^{2/3}$ only has domain $x > -9$ as illustrated in the slope field[1] in Figure 1.

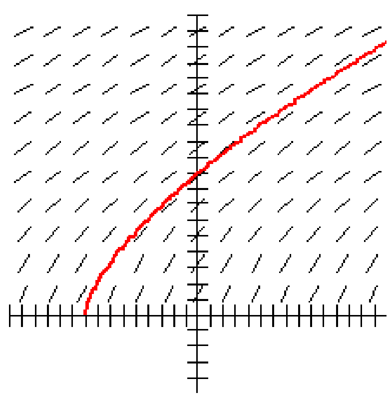

Figure 1: $\frac{dy}{dx} = \frac{2}{\sqrt{y}}$ with $y(0) = 9$

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[1]The illustrations of slope fields and solution curves for this article were drawn using the free Windows program Winplot available at http://math.exeter.edu/rparris/.

The 1985 BC Calculus exam contained the following problem: Given the differential equation

$$\frac{dy}{dx} = \frac{-xy}{\ln y}, \; y > 0$$

(a) Find the general solution of the differential equation.

(b) Find the solution that satisfies the condition that $y = e^2$ when $x = 0$. Express your answer in the form $y = f(x)$.

(c) Explain why $x = 2$ is not in the domain of the solution found in part (b).

The answer to part (b) is $y = e^{\sqrt{4-x^2}}$. The domain of this as a function is $-2 \leq x \leq 2$. However, as noted on that year's scoring rubric for BC 4, "if $x = 2$, then $y = 1$ and $\ln y = 0$ which causes $(-xy)/\ln y$ to be undefined" and hence $x = 2$ is not in the domain of the solution to the initial value problem. In fact the domain is $-2 < x < 2$. Students actually did fairly well on this problem with a mean score of 5.04 and with 585 of 9895 students earning all 9 points. But that was still the smallest number of students who received a perfect score on any of the six free-response problems that year.

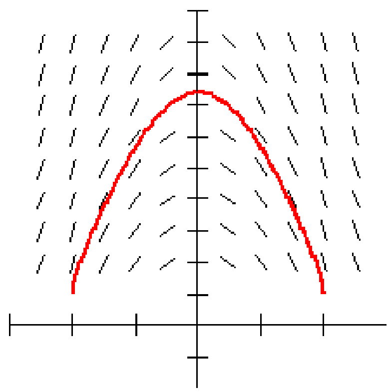


Figure 2: $\frac{dy}{dx} = \frac{-xy}{\ln y}$ with $y(0) = e^2$

In both of these examples, the domains have been intervals. In fact, the domain of a particular solution to a differential equation is the largest open interval containing the initial value on which the solution satisfies the differential equation. Some textbook authors call the domain of a solution the *interval of definition* of the solution or the *maximum interval of existence*. The reason domains are restricted to intervals will be discussed below, but first we present some more examples. Most of these are taken from differential equations textbooks because, unfortunately, most calculus texts have yet to provide a complete discussion about domains of solutions to initial value problems. See the appendix, however, for examples of how some calculus books have addressed the definition of the solution to a differential equation.

More Examples of Domains

Polking, Boggess, and Arnold discuss the following initial value problem in their textbook *Differential Equations*: find the particular solution to the differential equation $dy/dt = y^2$ that satisfies the initial value $y(0) = 1$. Solving the differential equation using separation of variables yields

$$y = \frac{1}{1-t}$$

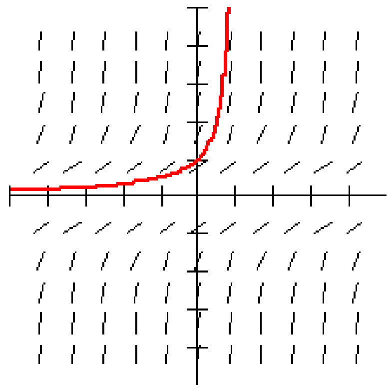


Figure 3: $dy/dt = y^2$ with $y(0) = 1$

as the solution. The solution is sketched on the slope field illustrated in Figure 3. The authors write: "The graph of y is a hyperbola with two branches....The function y has an infinite discontinuity at $t = 1$. Consequently, this function cannot be considered to be a solution to the differential equation $y' = y^2$ over the whole real line....Note that the left branch of the hyperbola...passes through the point $(0, 1)$ as required by the initial condition $y(0) = 1$. Hence, the left branch of the hyperbola is the solution curve needed. This particular solution curve extends indefinitely to the left, but rises to positive infinity as it approaches the asymptote $t = 1$ from the left. Any attempt to extend this solution to the right would have to include $t = 1$, at which point the function $y(t)$ is undefined. Consequently, the maximum interval on which this solution curve is defined is the interval $(-\infty, 1)$."

An initial value problem in *Elementary Differential Equations* by Boyce and DiPrima is the following: solve

$$y' + \frac{2}{x}y = 4x$$
$$y(1) = 2$$

and determine the interval in which the solution is valid. This differential equation is not separable but can be solved using the method of integrating factors. The authors determine that

$$y = x^2 + \frac{C}{x^2}$$

is the general solution, and then write: "To satisfy the initial condition it is necessary to choose $C = 1$; thus

$$y = x^2 + \frac{1}{x^2}, x > 0$$

is the solution of the initial value problem....Observe that the function $y = x^2 + (1/x^2)$ for $x < 0$ is not part of the solution for this initial value problem." In this case the differential equation has a singularity at $x = 0$ and, in addition, the solution approaches a vertical asymptote as $x \to 0^+$ as illustrated in Figure 4.

As another example, consider the initial value problem $dy/dx = y^2 + 1$ with $y(\pi) = 0$. Separating variables and integrating gives $\arctan(y) = x + C$. The initial condition implies that $C = -\pi$ and hence the solution satisfies $\arctan(y) = x - \pi$. But the range of the inverse tangent function is $(-\pi/2, \pi/2)$ which means that we must have $-\pi/2 < x - \pi < \pi/2$. Therefore the domain of the solution to this initial value problem is the interval $\pi/2 < x < 3\pi/2$. The solution, shown in Figure 5, can be written as $y = \tan(x - \pi) = \tan(x)$ on this interval. Again, the domain of the solution is

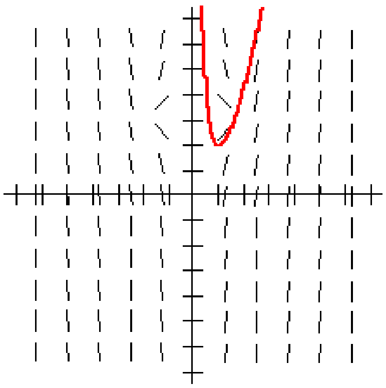


Figure 4: $y' + \frac{2}{x}y = 4x$ with $y(1) = 2$

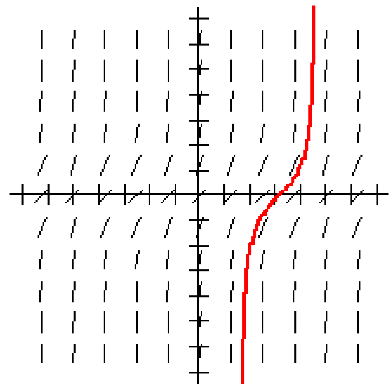


Figure 5: $dy/dx = y^2 + 1$ with $y(\pi) = 0$

a subset of the domain of the tangent function. As in the previous examples, the solution cannot jump over the vertical asymptotes.

A similar problem in *Differential Equations: A Modeling Approach* by Glenn Ledder involves the same differential equation but with the initial point $y(0) = 0$. Ledder remarks: "The tangent function solves the differential equation...on all intervals on which the function is defined. The tangent function solves the initial-value problem...only on the interval $(-\pi/2, \pi/2)$." He goes on to remark that "most initial-value problems have a unique solution, although not necessarily one that is valid for all values of the independent variable."

In *Calculus Problems for a New Century* from the Mathematical Association of America Notes series we find the following problem in section IV.2 (page 81, problem 8): Find the specific solution which satisfies $y(dy/dx) = x$ and the initial conditions (i) $y(2) = 1$, (ii) $y(2) = -1$, and (iii) $y(-2) = -1$. Solutions to this differential equation satisfy $y^2 = x^2 + C$. The specific solutions for the three initial conditions, as given in the commentary on this problem, are

(i) $y = \sqrt{x^2 - 3}\,,\ x > \sqrt{3}$

(ii) $y = -\sqrt{x^2 - 3}\,,\ x > \sqrt{3}$

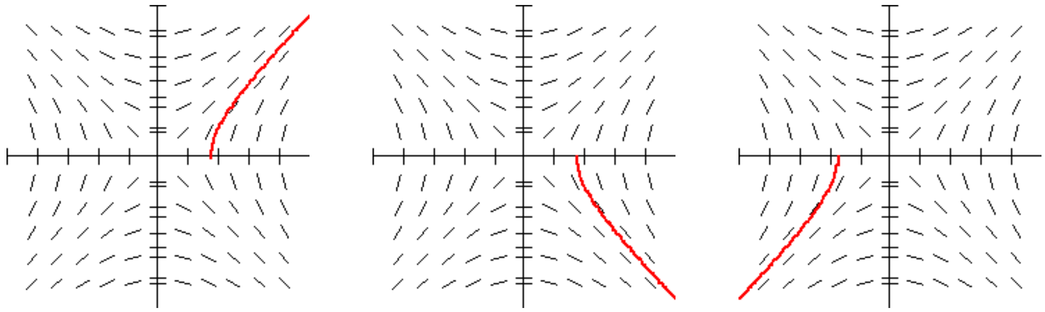
(iii) $y = -\sqrt{x^2 - 3}\,,\ x < -\sqrt{3}$

(The commentary actually gives the domains as closed intervals, but the derivatives do not exist at $x = \pm\sqrt{3}$.) Each solution curve is part of one of the branches of the hyperbola given by $y^2 = x^2 + C$. See Figure 6. Each of the solutions requires us to make two choices: the choice of the positive or negative sign for the square root when solving for y, and the choice of the interval satisfying $x^2 - 3 > 0$ for the domain of the solution. Both choices are made based on the initial value. The initial y determines which sign to use for the square root. The initial x determines which interval to use for the domain of the solution. In this case the solution cannot jump over the gap between the two branches of the hyperbola.

In the text *Differential Equations* by Sanchez, Allen, and Kyner, the authors describe an example that has a similiar issue. Their example is to solve

$$\frac{dy}{dt} = \frac{t^4}{y^4}.$$

The authors show that the general solution is $y(t) = (t^5 + C)^{1/5}$. They go on to write: "What is the interval of definition of the solution? This question requires a careful answer. The function



$y(2) = 1$ $\quad$ $y(2) = -1$ $\quad$ $y(-2) = -1$

Figure 6: $y\,(dy/dx) = x$

$y(t)$...is defined and continuous for all t, but if $C \neq 0$, it is not differentiable at the point $t = b$, where $b^5 + C = 0$. Therefore, one solution will be defined on the interval $b < t < \infty$ and a second solution on the interval $-\infty < t < b$. This is not just a technical nicety; a solution is a *continuously differentiable* function that satisfies the differential equation on an interval. It is important that $(t, y(t))$, for t in the interval, be in the domain of definition of the differential equation. In this example, $y(b) = 0$ but since $f(t, y) = t^4/y^4$, the differential equation is not defined for $y = 0$, and the function is not a solution on any interval containing $t = b = (-C)^{1/5}$."

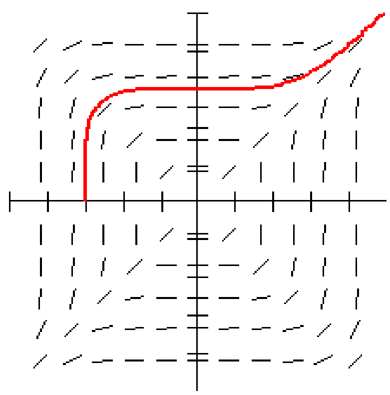


Figure 7: $\frac{dy}{dt} = \frac{t^4}{y^4}$ with $y(0) = 3$

The Domain in 2006 AB 5

One method to solve the initial value problem in 2006 AB 5 (see page 1) can be found in the scoring rubric available at AP Central. As an alternative method, we can separate variables and use the initial condition to write two definite integrals that must be satisfied by the particular solution.

$$\int_1^y \frac{1}{w+1}\,dw = \int_{-1}^x \frac{1}{t}\,dt$$

In order to write these two definite integrals, however, we must have $y > -1$ and $x < 0$ or the improper integrals will diverge. The solution then becomes

$$\ln(y+1) - \ln(2) \;\; = \;\; \ln(-x) - \ln(1) = \ln(-x)$$

$$\ln\left(\frac{y+1}{2}\right) = \ln(-x)$$
$$\frac{y+1}{2} = -x$$
$$y = -2x-1$$

but only for the interval $x < 0$. Notice that we will have $y > -1$ when $x < 0$. In this case the solution cannot jump over the singularity at $x = 0$ in the differential equation.

The general solution to the differential equation in 2006 AB 5 is $y = Cx - 1$. Figure 8 shows several solution curves in the slope field. None of these solutions can have $x = 0$ in the domain since the solutions do not satisfy the differential equation at $x = 0$; we cannot say that the slope C is equal to the indeterminate $0/0$. Each particular solution to this differential equation has domain either $x < 0$ or $x > 0$ even though the domain of the function $f(x) = Cx - 1$ is all real x.

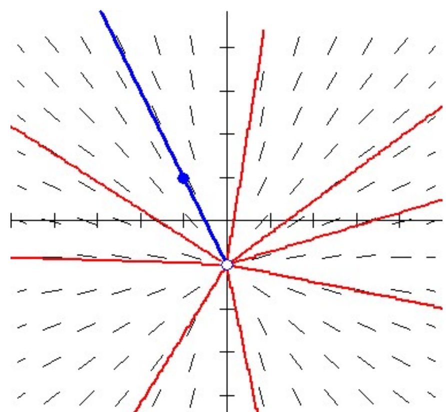


Figure 8: $\frac{dy}{dx} = \frac{y+1}{x}$

To see why this is so, consider what happens in the slope field. Imagine you were back in Newton's time and could march along the slope field in Figure 8 in infinitesimal steps. Starting at $x = -1$ and proceeding to the right, you would head towards the y-axis, marching along the line $y = -2x - 1$. But what would happen when you try to "arrive" at $x = 0$? There is suddenly no slope. The differential equation is undefined when $x = 0$. What direction should you continue? Since there is no direction specified, the solution curve must stop, again suggesting why the domain should only be $x < 0$. We have no information about what should happen for that particular solution on the other side of the singularity.

Why are Domains Intervals?

What do we tell students about sketching a solution to a DE with a slope field? Start at the initial point and sketch a smooth curve as far as the slope field can take you. This is somewhat the geometric interpretation of Euler's method. Of course, this is not a precise concept because the slope field is only given at discrete points and following those discrete slope lines can lead you off the actual solution and may disguise the existence of vertical asymptotes and other singularities. This geometric view of a solution to an initial value problem suggests, however, why such a solution should not cross over a vertical asymptote or a singularity in the differential equation, and thus why the domain must be restricted to an interval. Indeed, the reality is that most initial value problems cannot be solved analytically and solutions must be approximated either graphically or numerically. In this case, we want the differentiability of the solution to imply the intuitive concept of continuity that we often teach pre-calculus students: a function is continuous if you can draw its graph *without lifting the pencil*.

Another reason to work only on intervals is that those who work with differential equations to model physical situations want a unique solution that makes physical sense. The fundamental theorem of differential equations is the existence and uniqueness theorem. The existence theorem will only guarantee a solution on some interval containing the initial value. The initial value problem $dy/dx = g(x, y)$, $y(x_0) = y_0$ is equivalent to the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} g(t, y(t))dt$$

but to evaluate this definite integral we must restrict attention to where g is defined and continuous. This integral equation is the basis for the use of Picard's approximation method to generate a sequence of functions that, under appropriate conditions, can be shown to converge to a solution of the initial value problem. And if the partial derivative of g with respect to y is continuous (as in 2006 AB 5), then we are guaranteed a unique solution on some interval containing x_0.

In his textbook *Introduction to Ordinary Differential Equations*, Stephen Saperstone writes: "In the definition at the beginning of Section 1.2...we learned that an explicit solution of an ODE, in order to be meaningful and useful, must be defined on an interval. The restriction of t to an interval is motivated by the fact that ODEs model real phenomena. For instance, we observed that for the ODE $t^2\dot{x} + x = 0$...the integral curves (except for $x = 0$) appear to avoid the x-axis. In particular, the integral curve through $(\frac{1}{2}, \frac{1}{2})$ appears to approach the x-axis asymptotically and 'fly off' to infinity as t approaches zero from the right. If the domain of definition of the corresponding solution were to consist of $\mathbb{R}$ except for $t = 0$, then how could the solution ever 'get across the gap' at $t = 0$? Time would literally have to stop prior to $t = 0$ and resume thereafter. This certainly isn't the way things behave. We could argue that the system being modeled 'breaks down' at $t = 0$, but then the original ODE is not applicable if the model changes at $t = 0$."

The general form of a first-order differential equation is $F(x, y, \frac{dy}{dx}) = 0$. The form $dy/dx = g(x, y)$ that has been used in most of the examples and discussion above is usually preferred for theoretical and computational purposes. But it is important to realize that solutions, including their domains, can be different based on what form of the differential equation is used. For example, the differential equations $dy/dx = (y + 1)/x$ and $x(dy/dx) = y + 1$ are *different* equations. No solution of the former equation can contain $x = 0$ in its domain. But the solution $y = -2x - 1$ with domain all real x *does* satisfy the differential equation $x\,(dy/dx) = y+1$ at all values of x. Of course, a similar issue also arises in solving algebraic equations. The two equations $(x^2 - 4)/(x - 2) = x$ and $x^2 - 4 = x(x - 2)$ are not the same. The first equation has no solution while the second has $x = 2$ as a solution.

Other AP Domain Subtleties

There have been other subtle points about domains on the AP Exam. As discussed by Dan Kennedy in his article "Things I Have Learned at the AP Reading" in the *College Mathematics Journal* (November, 1999), problem AB 1 on the 1988 exam asked students to find the domain of the function f given by $f(x) = \sqrt{x^4 - 16x^2}$. The correct domain is $(-\infty, -4] \cup \{0\} \cup [4, \infty)$, but many students lost a point for not including the domain point $\{0\}$. The subtlety? The "identity"

$$\sqrt{x^4 - 16x^2} = |x|\,\sqrt{x^2 - 16}$$

is, in fact, invalid at $x = 0$ if only real arithmetic is considered. Kennedy calls this example "Radical lies that we tell students in algebra."

In the same article Kennedy also discusses another domain subtlety in problem 2 on the 1996 BC exam, which presented students with the Maclaurin series for $f(x)$ given by $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^n}{(n+1)!} + \cdots$. Part (d) of the problem asked students to write $f(x)$ in terms of a familiar function. This was essentially a "series manipulation" problem based on $(e^x - 1)/x$. As Kennedy writes, however: "There were only 5 perfect scores among the 21,020 BC exams that year. The problem, you see, is that $f(x) = (e^x - 1)/x$ is not even *defined* at $x = 0$, let alone infinitely differentiable, and is therefore not eligible to have a Maclaurin series. The correct f should be

$$f(x) = \begin{cases} \dfrac{e^x - 1}{x} & \text{for } x \neq 0, \\ 1 & \text{for } x = 0.\text{"} \end{cases}$$

For additional examples and discussion about the domain of solutions of differential equations, see the article "Solving Separable Differential Equations: Antidifferentiation and Domain are Both Needed" by David Lomen, available in the *Teaching Resource Materials* section of the Calculus AB and Calculus BC Course Home Pages at AP Central.

Appendix

Below are some examples of how a sample of calculus and differential equations textbooks discuss solutions to differential equations and solutions to initial value problems. The notation has been changed in a few cases to be consistent among the examples, but the emphasis and language in each excerpt is preserved from the original versions.

Edwards & Penney, Calculus with Analytic Geometry, Early Transcendentals (p412)

A first-order differential equation is a differential equation that can be written in the form

$$\frac{dy}{dx} = g(x, y)$$

where x denotes the independent variable and $y = y(x)$ is the unknown function. A solution...is a function $y = y(x)$ such that $y'(x) = g(x, y(x))$ for all x in some appropriate interval I.

Finney, Weir, and Giordano, Thomas' Calculus Early Transcendentals (p443)

A first-order differential equation is a relation

$$\frac{dy}{dx} = g(x, y)$$

in which $g(x, y)$ is a function of two variables defined on a region in the xy-plane. A solution...is a differentiable function $y = y(x)$ defined on an interval of x-values (perhaps infinite) such that

$$\frac{d}{dx}y(x) = g(x, y(x))$$

on that interval.

Swokowski, Calculus with Analytic Geometry (p822)

Recall that a function f (or $f(x)$) is a solution of a differential equation if substitution of $f(x)$ for y results in an identity for every x in some interval.

Larson, Hostetler, and Edwards, Calculus (p369)

These authors do not mention an interval in describing what is meant by a solution of a differential equation. However, they give an example with $dy/dx = y/x^2$ and $y(1) = 3$, and give the solution as $y = (3e)e^{-1/x}$ for $x > 0$.

Finney, Demana, Waits, and Kennedy, Calculus: Graphic, Numeric, Algebraic (p322)

An initial condition determines a particular solution by requiring that a solution curve pass through a given point. If the curve is continuous, this pins down the solution on the entire domain. If the curve is discontinuous, the initial condition only pins down the continuous *piece of the curve* that passes through the given point. In this case, the domain of the solution must be specified.

Coddington and Levinson, Theory of Ordinary Differential Equations (pp1-2)

Let g be a continuous function on a rectangle D in the (x, y) plane. The central problem of this chapter may be phrased as follows: to find a differentiable function φ defined on a real x interval I such that $(x, \varphi(x)) \in D$ for all $x \in I$ and $\varphi'(x) = g(x, \varphi(x))$ for all $x \in I$. Suppose (τ, ξ) is a given point in D. Then an initial-value problem associated with the differential equation and this point is defined in the following way: To find an interval I containing τ and a solution φ satisfying $\varphi(\tau) = \xi$.

Rainville and Bedient, Elementary Differential Equations (pp3-4)

A function ϕ, defined on an interval $a < x < b$, is called a solution of the differential equation $y' = g(x, y)$, provided that the first derivative of the function exists on the interval $a < x < b$ and $\phi'(x) = g(x, \phi(x))$ for every x in $a < x < b$.

Ricardo, A Modern Introduction to Differential Equations (p7)

A solution of the differential equation $y' = g(x, y)$ on an interval (a, b) is a real-valued function $y = y(x)$ such that all necessary derivatives of $y(x)$ exist on the interval and $y(x)$ satisfies the equation for every value of x in the interval. Solving a differential equation means finding all possible solutions of a given equation. Note that we say "a" solution rather than "the" solution. A differential equation, if it has a solution at all, usually has more than one solution. Also, we should pay attention to the interval on which the solution may be defined.

Moore, Introduction to Differential Equations (pp1-3)

A first-order differential equation has for us the standard form $y' = g(x, y)$. The obvious problem is the finding of solutions, that is, functions which satisfy the equation identically. Specifically, a solution is a function $y(x)$, defined on an interval $a < x < b$, such that $y'(x) = g(x, y(x))$ holds for $a < x < b$.

Sanchez, Ordinary Differential Equations: A Brief Eclectic Tour (pp6-7)

Sanchez discusses the existence and uniqueness theorem for differential equations, and then talks about the continuation problem of extending the interval of existence guaranteed by the theorem. He writes: "Furthermore, it seems equally plausible that for the IVP, the solution $y(x)$ satisfying $y(x_0) = y_0$ will possess a maximum interval of convergence

$$x_0 + \alpha < x < x_0 + \beta, \qquad \alpha, \beta \text{ finite or infinite}$$

and that the interval will depend on x_0 and y_0."

He later (p41) gives the example $dy/dx = y/(x^2 - 1)$ and says that "solutions are defined for $-\infty < x < -1$, or $-1 < x < 1$, or $1 < x < \infty$; depending on the choice of x_0 in the IC $y(x_0) = y_0$." Note the use of the word *or*, not *and*.

Borrelli and Coleman, Differential Equations: A Modeling Perspective (p10)

Let's define what we mean by a solution of $y' = g(x, y)$ where $g(x, y)$ is a function defined on some portion (or all) of the xy-plane. A function $y(x)$ defined on an x-interval I is a solution of the ODE if $g(x, y(x))$ is defined and $y'(x) = g(x, y(x))$ for all x in I.

Blanchard, Devaney, and Hall, Differential Equations, Second Edition, Instructor's Manual

We also deal with questions of the domain of definition for solutions. The exceptional student will have wondered about domains of definition in Section 1.2 where restricted domains are the norm. We take the dynamical systems point of view that a solution that escapes to infinity or that encounters a singular point of the differential equation at a finite time cannot be extended beyond that time. For example, the function $y(t) = 1/t$ for $t > 0$ is a completely different solution to the equation $dy/dt = -y^2$ than the solution $y(t) = 1/t$ for $t < 0$.

Redheffer, Differential Equations: Theory and Applications (p34)

A *solution* of a differential equation $dy/dx = g(x, y)$ is a function $y = \phi(x)$ that satisfies the equation on an interval I. To know the solution one must know both ϕ and I. In nonlinear problems of interest in applications, however, the interval of existence usually depends on the initial value $y(x_0)$ and is difficult to determine.

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