Numerical Integration Techniques

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Advanced Topics: Calculus

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Calculator Commands

	TI-83/84	TI-85/86	TI-89
Definite Integrals	`fnInt( f(x), x, lower limit, upper limit)`		
	fnInt	MATH 9: fnInt	CALC fnInt	F3 B: nInt
Use function from Y=	To use a function entered as Y_1 in some other command (like fnInt or seq)		
	Y_1	VARS Y-VARS 1: Function 1:Y_1	Type y1 using 2nd-alpha for lowercase y	Type y1(x) from keyboard
Riemann Sums	`sum( seq( f(x), x, first, last, Δx))*Δx`		
	sum	LIST MATH 5: sum	LIST OPS MORE sum	MATH 3: List 6: sum
	seq	LIST OPS 5: seq	LIST OPS MORE seq	MATH 3: List 1: seq

Symbolic Riemann Sums $\int_a^b f(x)\,dx$ **f(x) a polynomial** **(TI-89 only)**	Left:
	`Σ( f(a+(k-1)*dx), k, 1, n)*dx | n = (b-a)/dx`
	
	Right:
	`Σ( f(a+k*dx), k, 1, n)*dx | n = (b-a)/dx`
	
	Midpoint:
	`Σ( f(a+(k-1/2)*dx), k, 1, n)*dx | n = (b-a)/dx`
	
	`expand(ans(1))`		

	Σ			F3 calc 4: Σ(sum
	expand			F2 algebra 3: expand(

Numerical Integration Worksheet

$$\int_0^2 x^4 - 3x^3 - 3x^2 + 30\,dx$$

n	Left sum	error	ratio	*n*	Right sum	error	ratio
2			XXXXXX	2			XXXXXX
4				4			
8				8			
16				16			
32				32			

n	Midpoint	error	ratio	*n*	Trapezoid	error	ratio
2			XXXXXX	2			XXXXXX
4				4			
8				8			
16				16			
32				32			

error = approximation – integral

"ratio" means take the error for $n = 2$ and divide by the error for $n = 4$. Put this in the box for $n = 4$. Then take the error for $n = 4$ and divide by the error for $n = 8$. Record this in the box for $n = 8$. Divide the error for $n = 8$ by the error for $n = 16$. Finally, take the error for $n = 16$ and divide by the error for $n = 32$.

Numerical Integration Worksheet

$$\int_0^2 x^4 - 3x^3 - 3x^2 + 30\,dx = \textbf{46.4}$$

n	Left sum	error	ratio	n	Right sum	error	ratio
2	**55**	**8.6**	XXXXXX	2	**35**	**–11.4**	XXXXXX
4	**51.0625**	**4.6625**	**1.84**	4	**41.0625**	**–5.3375**	**2.14**
8	**48.81641**	**2.41641**	**1.93**	8	**43.81641**	**–2.58359**	**2.07**
16	**47.62915**	**1.22915**	**1.97**	16	**45.12915**	**–1.27085**	**2.03**
32	**47.01979**	**0.61979**	**1.98**	32	**45.76979**	**–0.63021**	**2.02**

n	Midpoint	error	ratio	n	Trapezoid	error	ratio
2	**47.125**	**0.725**	XXXXXX	2	**45**	**–1.4**	XXXXXX
4	**46.57031**	**0.17031**	**4.26**	4	**46.0625**	**–0.33755**	**4.15**
8	**46.44189**	**0.04189**	**4.07**	8	**46.31641**	**–0.08359**	**4.04**
16	**46.41043**	**0.01043**	**4.02**	16	**46.37915**	**–0.02085**	**4.02**
32	**46.40261**	**0.00261**	**4.00**	32	**46.39479**	**–0.00521**	**4.00**

error = approximation – integral

"ratio" means take the error for $n = 2$ and divide by the error for $n = 4$. Put this in the box for $n = 4$. Then take the error for $n = 4$ and divide by the error for $n = 8$. Record this in the box for $n = 8$. Divide the error for $n = 8$ by the error for $n = 16$. Finally, take the error for $n = 16$ and divide by the error for $n = 32$.

Numerical Approximations of Integrals

If the graph is decreasing, then $\mathrm{Right}(n) < \int_a^b f(x)dx < \mathrm{Left}(n)$.

Right Sum

Left Sum
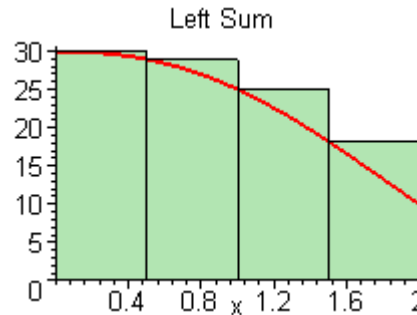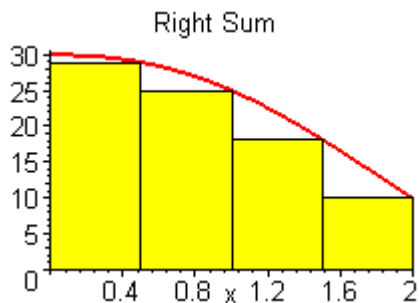
If the graph is concave down, then $\mathrm{Trap}(n) < \int_a^b f(x)dx < \mathrm{Mid}(n)$.
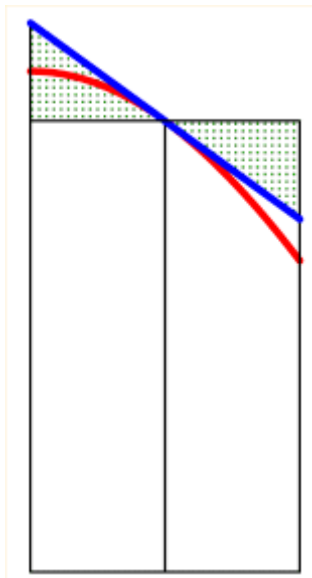
The blue line is the tangent line to the graph at the midpoint. This line lies above the graph because the graph is concave down.

The two shaded triangles have equal area.

The area of the midpoint rectangle is therefore equal to the area of the trapezoid with top given by the tangent line.

The area under the graph is therefore less than the area of the midpoint rectangle.

The trapezoid approximation uses the secant line as the top of the trapezoid, and the secant line lies underneath the graph when the graph is concave down.

If the graph is increasing or concave up, the respective inequalities are reversed.

Symbolic Riemann Sums (Maple)

```
f := x^4-3*x^3-3*x^2+30;
a := 0;
b := 2;
```

$$f := x^4 - 3\,x^3 - 3\,x^2 + 30$$

$$a := 0$$

$$b := 2$$

```
> with(student):
> ans := Int(f, x=a..b): % = value(%);
```

$$\int_0^2 x^4 - 3\,x^3 - 3\,x^2 + 30\ dx = \frac{232}{5}$$

```
> LS := leftsum(f, x=a..b, n);
```

$$LS := 2\,\frac{\displaystyle\sum_{i=0}^{n-1}\left(16\,\frac{i^4}{n^4} - \frac{24\,i^3}{n^3} - \frac{12\,i^2}{n^2} + 30\right)}{n}$$

```
> LEFT(dx) := subs(n=(b-a)/dx,expand(value(LS)));
```

$$\text{LEFT}(dx) := \frac{232}{5} + 10\,dx - \frac{4}{3}\,dx^2 - \frac{1}{15}\,dx^4$$

```
> RS := rightsum(f, x=a..b, n);
```

$$RS := 2\,\frac{\displaystyle\sum_{i=1}^{n}\left(16\,\frac{i^4}{n^4} - \frac{24\,i^3}{n^3} - \frac{12\,i^2}{n^2} + 30\right)}{n}$$

```
> RIGHT(dx) := subs(n=(b-a)/dx,expand(value(RS)));
```

$$\text{RIGHT}(dx) := -10\,dx - \frac{4}{3}\,dx^2 - \frac{1}{15}\,dx^4 + \frac{232}{5}$$

```
> TRAP(dx) := (LEFT(dx)+RIGHT(dx))/2;
```

$$\text{TRAP}(dx) := \frac{232}{5} - \frac{4}{3}\,dx^2 - \frac{1}{15}\,dx^4$$

> `MS := middlesum(f, x=a..b, n);`

$$MS := 2\,\frac{\displaystyle\sum_{i=0}^{n-1}\left(16\,\frac{\left(i+\frac{1}{2}\right)^4}{n^4} - \frac{24\left(i+\frac{1}{2}\right)^3}{n^3} - \frac{12\left(i+\frac{1}{2}\right)^2}{n^2} + 30\right)}{n}$$
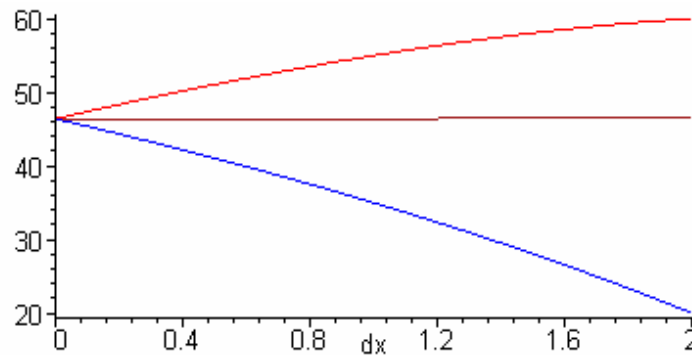
> `MID(dx) := subs(n=(b-a)/dx,expand(value(MS)));`

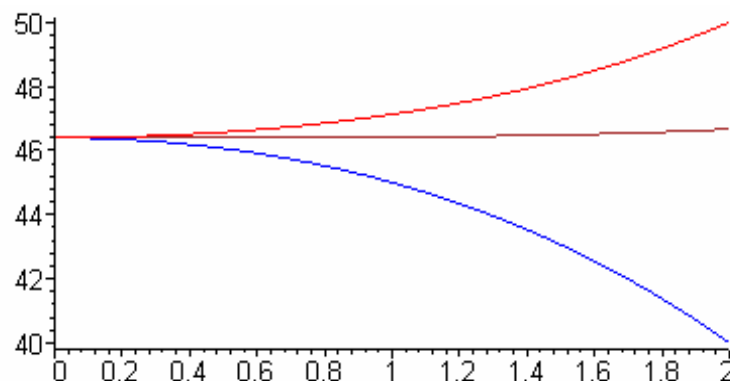$$MID(dx) := \frac{2}{3}\,dx^2 + \frac{232}{5} + \frac{7}{120}\,dx^4$$

> `SIMP(dx) := expand((2*MID(dx) + TRAP(dx))/3);`

$$SIMP(dx) := \frac{232}{5} + \frac{1}{60}\,dx^4$$

> `plot([LEFT(dx), RIGHT(dx), SIMP(dx)], dx=0..2, color=[red,blue,brown]);`

> `plot([MID(dx), TRAP(dx), SIMP(dx)], dx=0..2, color=[red,blue,brown]);`

For additional discussion, see L. Riddle, "Symbolic and Graphical Investigations of Riemann Sums with a Computer Algebra System", *PRIMUS*, Vol. VI, No. 4 (December 1996), 366-380. Also available at http://ecademy.agnesscott.edu/~lriddle/pub.htm.

Simpson's Rule approximates the area of a parabola

Suppose we have three data values as illustrated in the figure below
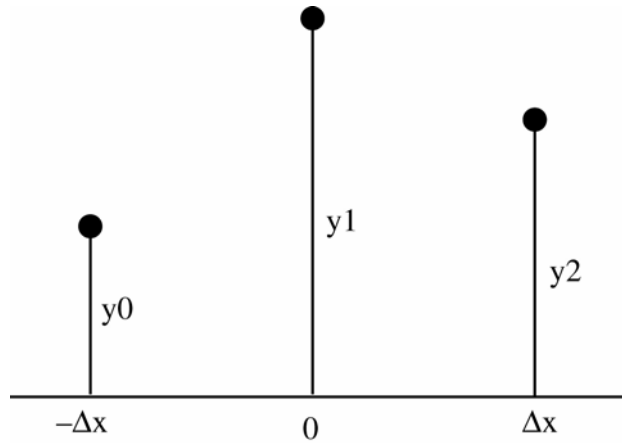
Let $y = Ax^2 + Bx + C$ be the equation of the parabola passing through the three points shown. Show that the area underneath this parabola is equal to

$$\frac{\Delta x}{3}\left(y_0 + 4y_1 + y_2\right)$$

which is the formula for the Simpson rule approximation with these three points.

$$y_0 = A(\Delta x)^2 - B(\Delta x) + C$$
$$y_1 = C$$
$$y_2 = A(\Delta x)^2 + B(\Delta x) + C$$

$$y_0 + y_2 = 2A(\Delta x)^2 + 2y_1 \quad \Rightarrow \quad A = \frac{y_0 - 2y_1 + y_2}{2(\Delta x)^2}$$

$$y_2 - y_0 = 2B(\Delta x)^2 \quad \Rightarrow \quad B = \frac{y_2 - y_0}{2\Delta x}$$

$$\int_{-\Delta x}^{\Delta x} Ax^2 + Bx + C \, dx = \frac{1}{3}Ax^3 + \frac{1}{2}Bx^2 + Cx \Big|_{-\Delta x}^{\Delta x}$$
$$= \frac{2}{3}A(\Delta x)^3 + 2C(\Delta x)$$
$$= \frac{1}{3}(y_0 - 2y_1 + y_2)\Delta x + 2y_1\Delta x$$
$$= \frac{1}{3}\Delta x(y_0 - 2y_1 + y_2 + 6y_1)$$
$$= \frac{1}{3}\Delta x(y_0 + 4y_1 + y_2)$$

Trapezoid Rule for Equally Spaced Data Points $y_0, y_1, y_2, \ldots, y_n$

$$\text{Trap}(n) = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right)$$

Simpson's Rule for Equally Spaced Date Points $y_0, y_1, y_2, \ldots, y_n$ (odd number of points)

$$\text{Simpson}(n) = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

In both cases, Δx (= width of intervals) is the same for all data points.

WARNING!! These two rules for numerical approximations of definite integrals require **EQUALLY SPACED** intervals of uniform width Δx. Many free response problems involving trapezoid approximation on recent AP Calculus Exams have presented data in tables with intervals of unequal width. In such problems students must go back to the basic principle of trapezoid approximations and compute the area of each individual trapezoid for each interval (and should show their work for these computations!)
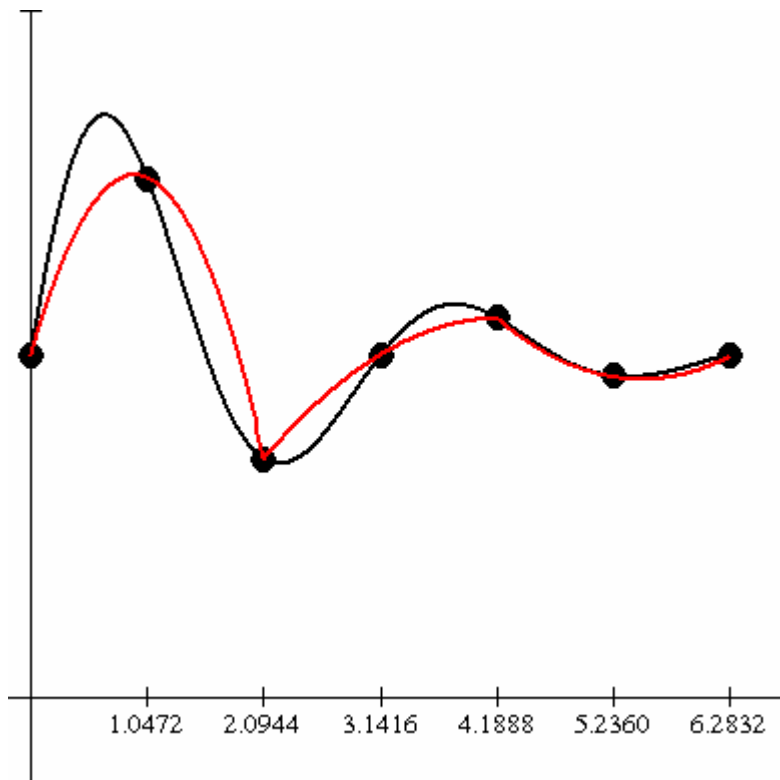
The following data was collected about a function $f(x)$.

Estimate $\int_0^{2\pi} f(x)\,dx$

x	$f(x)$
0	1.000
$\pi/3$	1.513
$2\pi/3$	0.696
π	1.000
$4\pi/3$	1.107
$5\pi/3$	0.937
2π	1.000

The following data was collected about a function $f(x)$.

Estimate $\int_0^{1.25} f(x)\,dx$

x	$f(x)$
0	3.000
0.25	2.540
0.50	1.583
0.75	1.010
1.00	1.346
1.25	2.284

Simpson's rule uses parabolas with 3 points to approximate area

$$f(x) = 1 + e^{-x/2} \sin(2x)$$

1.0472 2.0944 3.1416 4.1888 5.2360 6.2832
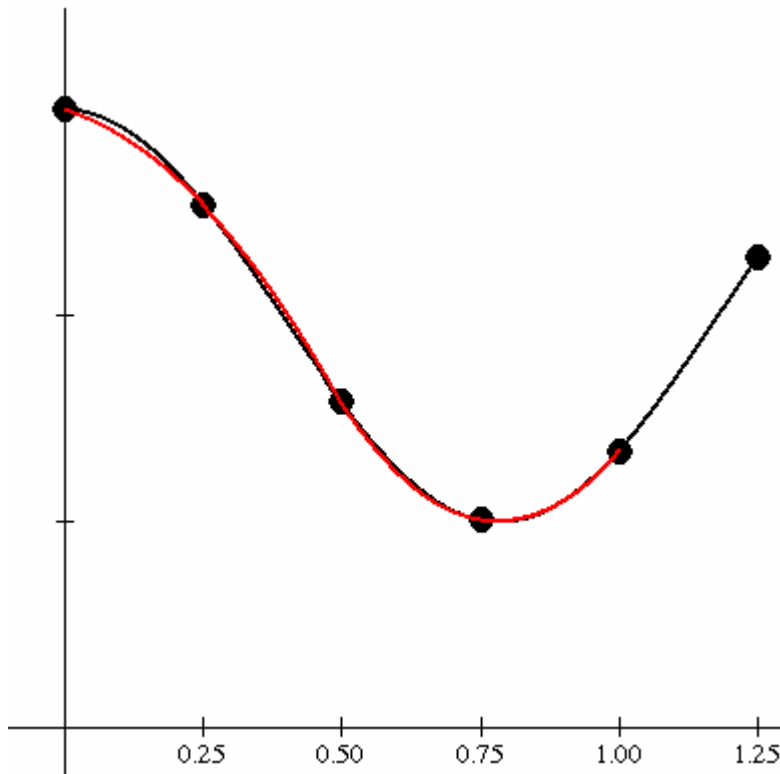
Simpson's rule uses parabolas with 3 points to approximate area

$$f(x) = \cos(4x) + 2$$

0.25   0.50   0.75   1.00   1.25

(Note: Simpson's rule can be used to approximate the integral from 0 to 1.0. The integral from 1 to 1.25 can be approximated by a trapezoid.)

The graph below gives the rate of production of oil in thousands of barrels per day by a large refinery. Use Simpson's rule to estimate the total amount of oil produced through the end of the fourth day.
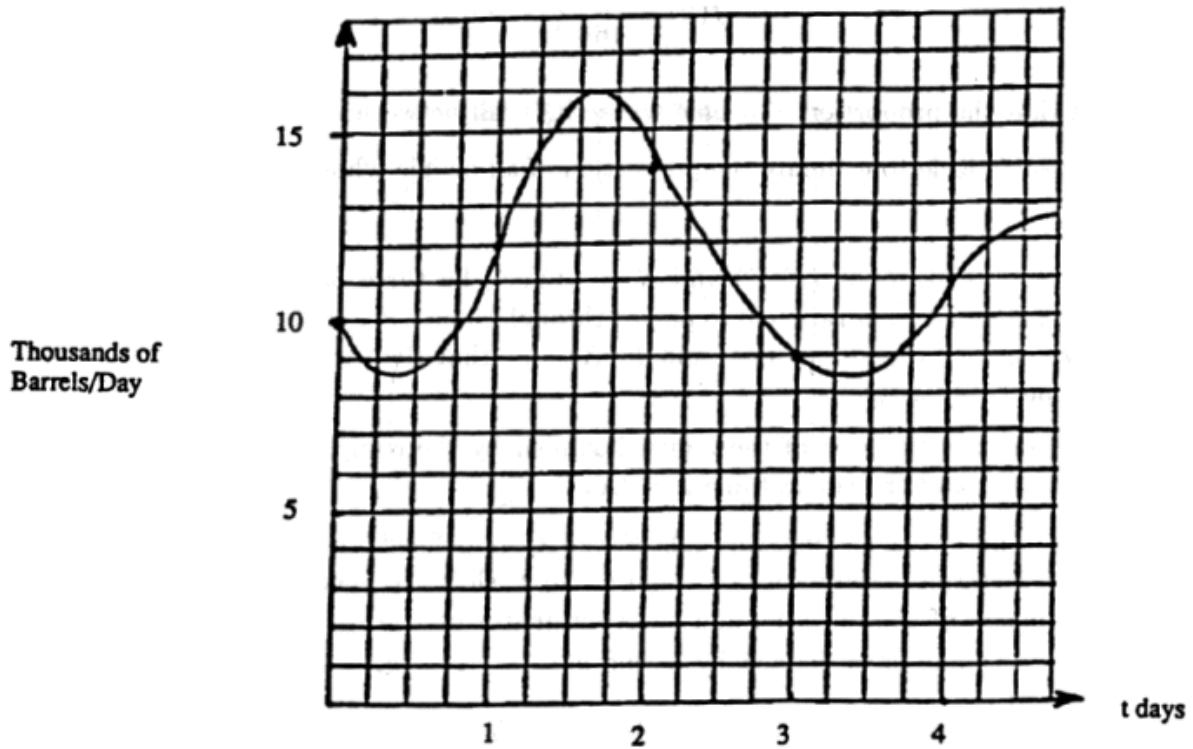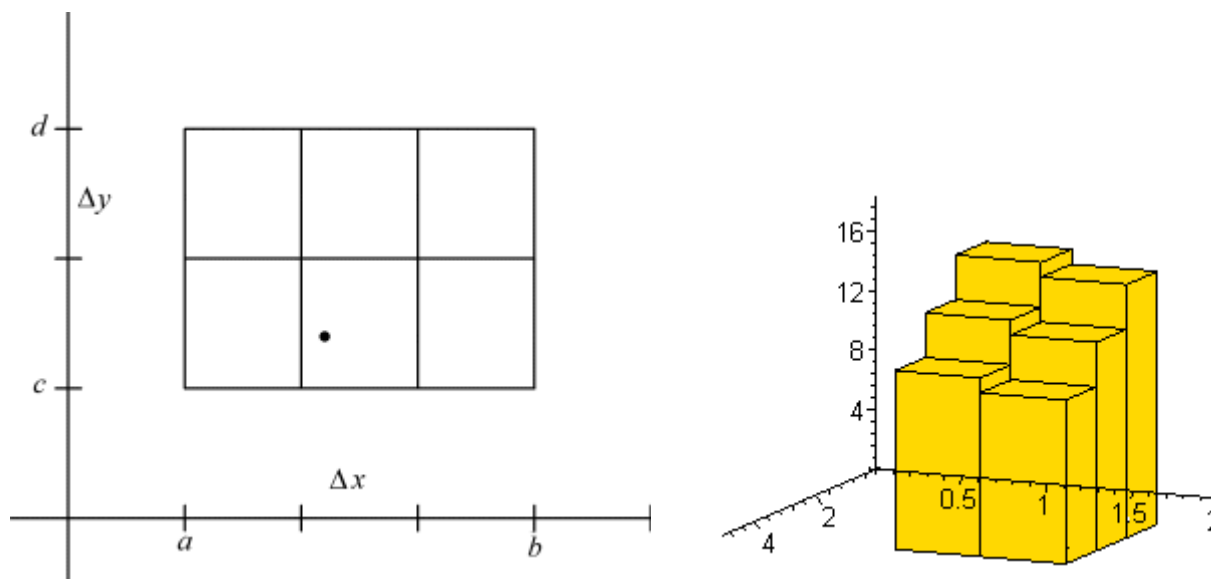
Riemann Sums for Double Integrals

Let $f(x, y)$ be a continuous function of two variables. Let $R = [a,b] \times [c,d]$ be a rectangle in the xy-plane.

Divide $[a,b]$ into n subintervals of width Δx and $[c,d]$ into m subintervals of width Δy. This divides R into nm subrectangles of area $\Delta A = \Delta x \Delta y$.

Pick a point (x_i, y_j) in each subrectangle. Common choices of points are
- lower left corner of each subrectangle
- any corner of each subrectangle
- the midpoint of each subrectangle

Form the Riemann sum $\displaystyle\sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i, y_j)\Delta A$. If $f(x, y) > 0$ on R, then each term in the sum has the geometric interpretation of the volume of a rectangular box of height $f(x_i, y_j)$ and base area ΔA. The double integral of the function f over the rectangle R is given by

$$\iint\limits_{R} f(x, y)\, dA = \lim_{n,m\to\infty} \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i, y_j)\Delta A = \lim_{\Delta x, \Delta y \to 0} \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i, y_j)\Delta x \Delta y$$

If $f(x, y) > 0$, this double integral has the geometric interpretation of the volume of the solid bounded above by the surface $z = f(x, y)$ and below by the rectangle R.

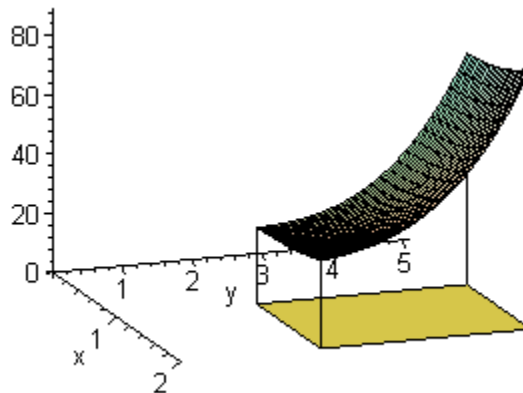# Double Integrals with Riemann Sums over a Rectangle (Maple)

Here is the function we want to integrate.
```
>f := (x,y) -> 30+2/15*x^4*y-3*y+1/10*y^4;
```

$$f := (x, y) \rightarrow 30 + \frac{2}{15} x^4 y - 3y + \frac{1}{10} y^4$$

Set up the rectangle in the form [a,b] x [c,d]
```
>a := 1 : b := 2 : c := 2 : d := 5:
```


The Riemann Sums
First we do the Riemann sum using the lower left corner of each sub-rectangle. We set up the sum, then find its value. We must then substitute for n and m in terms of dx and dy. The resulting expression must be expanded to multiply out all the terms. Once you get a simplified result, you can take the limit as dx and dy go to 0.

```
>lowerleftSum := Sum(Sum(f(a+(k-1)*dx,c+(j-1)*dy)*dx*dy,k=1..n),j=1..m):
>lleft := expand(subs({n=(b-a)/dx,m=(d-c)/dy},value(lowerleftSum)));
```

$$lleft := \frac{3226}{25} - \frac{21}{2} dx - \frac{2719}{100} dy + \frac{3}{2} dx\, dy - \frac{7}{15} dx^2\, dy + \frac{1}{150} dx^4\, dy + \frac{39}{10} dy^2 - \frac{7}{150} dx^4$$
$$- \frac{1}{100} dy^4 + \frac{49}{15} dx^2$$

Now do the same thing for the Riemann sum using the upper right corner of each sub-rectangle.
```
>upperrightSum := Sum(Sum(f(a+k*dx,c+j*dy)*dx*dy,k=1..n),j=1..m):
>uright := expand(subs({n=(b-a)/dx, m=(d-c)/dy}, value(upperrightSum)));
```

$$uright := \frac{3226}{25} + \frac{21}{2} dx + \frac{2719}{100} dy + \frac{3}{2} dx\, dy + \frac{7}{15} dx^2\, dy - \frac{1}{150} dx^4\, dy + \frac{39}{10} dy^2 - \frac{7}{150} dx^4$$
$$- \frac{1}{100} dy^4 + \frac{49}{15} dx^2$$

Now we do the Riemann sum based on the upper left corner of each sub-rectangle.
```
> upperleftSum := Sum(Sum(f(a+(k-1)*dx,c+j*dy)*dx*dy,k=1..n),j=1..m):
> uleft := expand(subs({n=(b-a)/dx,m=(d-c)/dy},value(upperleftSum)));
```

$$uleft := \frac{3226}{25} - \frac{21}{2}\,dx + \frac{2719}{100}\,dy - \frac{3}{2}\,dx\,dy + \frac{7}{15}\,dx^2\,dy - \frac{1}{150}\,dx^4\,dy + \frac{39}{10}\,dy^2 - \frac{7}{150}\,dx^4$$
$$- \frac{1}{100}\,dy^4 + \frac{49}{15}\,dx^2$$

Lower right corner
```
> lowerrightSum := Sum(Sum(f(a+k*dx,c+(j-1)*dy)*dx*dy,k=1..n),j=1..m):
> lright := expand(subs({n=(b-a)/dx,m=(d-c)/dy},value(lowerrightSum)));
```

$$lright := \frac{3226}{25} + \frac{21}{2}\,dx - \frac{2719}{100}\,dy - \frac{3}{2}\,dx\,dy - \frac{7}{15}\,dx^2\,dy + \frac{1}{150}\,dx^4\,dy + \frac{39}{10}\,dy^2 - \frac{7}{150}\,dx^4$$
$$- \frac{1}{100}\,dy^4 + \frac{49}{15}\,dx^2$$

We can get a better estimate by averaging the sums at the 4 corners. This is the two dimensional equivalent of the trapezoid rule.
```
> trap := (uleft + uright + lleft + lright)/4;
```

$$trap := \frac{3226}{25} + \frac{39}{10}\,dy^2 - \frac{7}{150}\,dx^4 - \frac{1}{100}\,dy^4 + \frac{49}{15}\,dx^2$$

How about the midpoint of each sub-rectangle?
```
> midpointSum := Sum(Sum(f(a+(k-1/2)*dx,c+(j-1/2)*dy)*dx*dy,k=1..n),j=1..m):
> mid := expand(subs({n=(b-a)/dx,m=(d-c)/dy},value(midpointSum)));
```

$$mid := \frac{3226}{25} - \frac{39}{20}\,dy^2 + \frac{49}{1200}\,dx^4 + \frac{7}{800}\,dy^4 - \frac{49}{30}\,dx^2$$

Can you combine the midpoint and trapezoid approximations to get an even better approximation?
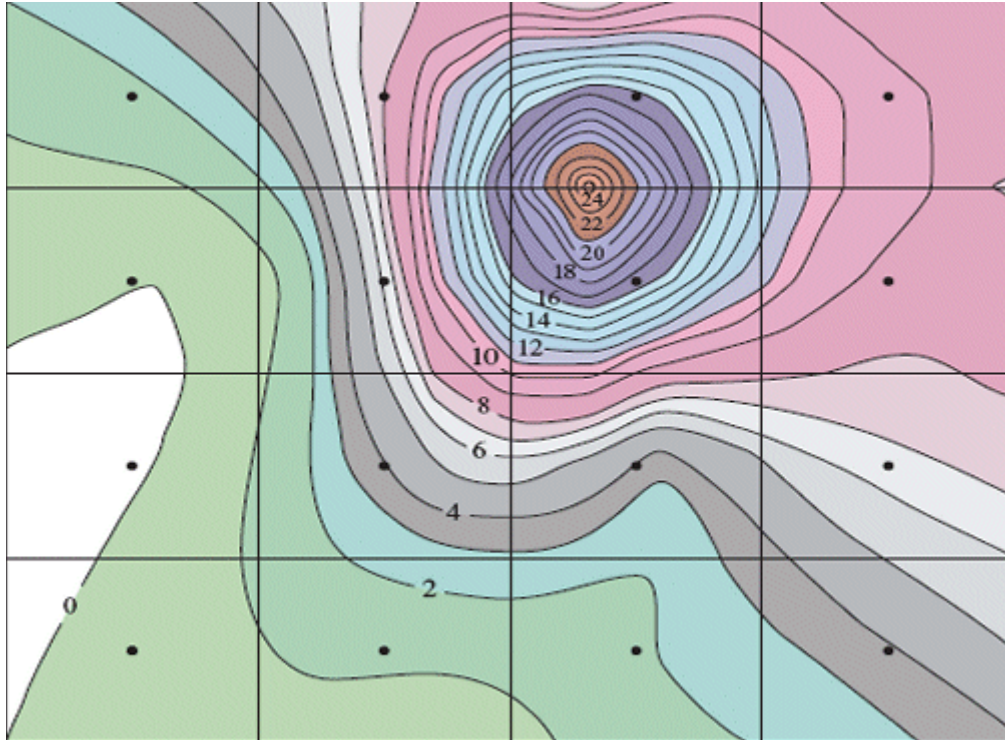```
> simpson := (2*mid+trap)/3;
```

$$simpson := \frac{3226}{25} + \frac{7}{600}\,dx^4 + \frac{1}{400}\,dy^4$$

The actual value of the double integral
```
> Int(Int(f(x,y),y=c..d),x=a..b): % = value(%);
```

$$\int_1^2 \int_2^5 30 + \frac{2}{15}\,x^4\,y - 3\,y + \frac{1}{10}\,y^4 \, dy \, dx = \frac{3226}{25}$$

The contour map below shows the snowfall, in inches, that fell on the state of Colorado on December 24, 1982 ("the Blizzard of 1982"). The state is approximately the shape of a rectangle that measures 388 miles west to east and 276 miles south to north. Use the contour map and a midpoint approximation to estimate the average snowfall for Colorado as a whole on that date.