Applications of Series to Differential Equations

Larry Riddle, Agnes Scott College, APAC 2007

(Based on excerpts from *Differential Equations: Graphics, Models, Data* by David Lomen and David Lovelock, John Wily & Sons, Inc., 1999)

Several free response problems in recent exams have given students an expression for $\frac{dy}{dx}$ in terms of *x* and *y*, and asked for $\frac{d^2y}{dx^2}$ in terms of *x* and *y*. This involved using implicit differentiation and the chain rule. Students were then asked questions that could be answered using these first and second derivative expressions. Extending the computation to additional derivatives, however, allows one to begin determining a Taylor series for $y(x)$. Some initial condition(s) must be provided, however, to allow for the evaluation of the derivatives.

In the following examples we illustrate this application of finding a series solution to a differential equation. This is a useful tool in many areas of physics where many of the special functions that are encountered are defined as power series solutions to differential equations.

Example 1: Solve the initial value problem $\frac{dy}{dx} = y$ with $y(0) = 1$.

Since the initial condition is given at $x = 0,$ let's construct a Taylor series for the solution $y(x)$ centered at $x = 0.$ To do this we need to compute the values of the derivatives at $x = 0.$ But the differential equation and initial value give us all the information we need to do that:

$$
\begin{aligned}
&y(0) = 1 \\
&y' = y \Rightarrow\ y'(0) = y(0) = 1 \\
&y'' = y' \ \Rightarrow\ y''(0) = y'(0) = 1 \\
&y''' = y'' \ \Rightarrow\ y'''(0) = y''(0) = 1
\end{aligned}
$$

and in general, $y^{(n)}(0) = 1$ for all *n*.

We quickly see that $y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, which is the Taylor series for e^x as we would expect for this solution.

Example 2 (2007 Form B, AB5): Solve $\frac{dy}{dx} = \frac{1}{2}x + y - 1$, $y(0) = 1$.

This is a linear differential equation that can be solved using the theory of linear equations with constant coefficients or using the technique of integrating factors (students did not need to solve the differential equation to answer the questions in AB5.) As an alternative, we construct a Taylor series for the solution $y(x)$ centered at $x = 0$. Again, we compute the derivatives:

$$y(0) = 1$$
$$y'(0) = \frac{1}{2}(0) + 1 - 1 = 0$$
$$y'' = \frac{1}{2} + y' \Rightarrow y''(0) = \frac{1}{2} + y'(0) = \frac{1}{2}$$
$$y''' = y'' \Rightarrow y'''(0) = y''(0) = \frac{1}{2}$$

And in general, $y^{(n)}(0) = \frac{1}{2}$ for $n \geq 2$. So

$$y(x) = 1 + 0x + \frac{1}{2} \cdot \frac{x^2}{2!} + \frac{1}{2} \cdot \frac{x^3}{3!} + \frac{1}{2} \cdot \frac{x^4}{4!} + \ldots = \frac{1}{2} - \frac{1}{2}x + \frac{1}{2} \cdot \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$$
$$= \frac{1}{2} - \frac{1}{2}x + \frac{1}{2}e^x$$

Example 3 (2007 Form B, BC5): Solve $\frac{dy}{dx} = 3x + 2y + 1$, $y(0) = -\frac{1}{4}$.

This is another linear differential equation that can be solved using the theory of linear equations with constant coefficients or using the technique of integrating factors (students did not need to solve the differential equation to answer the questions in BC5.) As an alternative, we construct a Taylor series for the solution $y(x)$ centered at $x = 0$. Again, we compute the derivatives:

$$y(0) = -\frac{1}{4}$$
$$y'(0) = 3(0) + 2y(0) + 1 = 2\left(-\frac{1}{4}\right) + 1 = \frac{1}{2}$$
$$y'' = 3 + 2y' \Rightarrow y''(0) = 3 + 2y'(0) = 3 + 2\left(\frac{1}{2}\right) = 4$$
$$y''' = 2y'' \Rightarrow y'''(0) = 2y''(0) = 2(4) = 8$$
$$y^{(4)} = 2y''' \Rightarrow y^{(4)}(0) = 2y'''(0) = 2(8) = 16$$

and in general, $y^{(n)}(0) = 2^n$ for $n \geq 2$. So

$$y(x) = -\frac{1}{4} + \frac{1}{2}x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \ldots = -\frac{5}{4} - \frac{3}{2}x + \left(1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \ldots\right)$$
$$= -\frac{5}{4} - \frac{3}{2}x + e^{2x}$$

This is the solution referred to in part (b) of BC5, when $m = -\frac{5}{4}$, $b = -\frac{3}{2}$, and $r = 2$.

If we had instead specified $y(0) = -\frac{5}{4}$, the calculations would have given

$$y(0) = -\frac{5}{4}$$
$$y'(0) = 3(0) + 2y(0) + 1 = 2\left(-\frac{5}{4}\right) + 1 = -\frac{3}{2}$$
$$y'' = 3 + 2y' \Rightarrow y''(0) = 3 + 2y'(0) = 3 + 2\left(-\frac{3}{2}\right) = 0$$
$$y''' = 2y'' \Rightarrow y'''(0) = 2y''(0) = 2(0) = 0$$
$$y^{(4)} = 2y''' \Rightarrow y^{(4)}(0) = 2y'''(0) = 2(0) = 0$$

and in general, $y^{(n)}(0) = 0$ for $n \geq 2$. This would give the solution $y(x) = -\frac{5}{4} - \frac{3}{2}x$, which also fits the pattern for the solution in part (b) of BC5 with $m = -\frac{5}{4}$, $b = -\frac{9}{4}$, and $r = 0$.

In these examples we were able to get an expression for the general term of the Taylor series and also write the series in a closed form expression. This may not always happen, however, as shown in the following examples.

Example 4: Solve $\frac{d^2y}{dx^2} + \sin y = 0$, $y(0) = 0.2$, $y'(0) = 0.15$.

This is an initial value problem that governs the nonlinear motion of a pendulum. Here *y* measures the angle of the pendulum from the vertical and *x* represents time.

It is not possible to find a solution for this initial value problem in terms of familiar functions. But we can try to approximate the solution with a Taylor series centered at $x = 0$. To do this we write the differential equation in the form $y'' = -\sin y$. Then

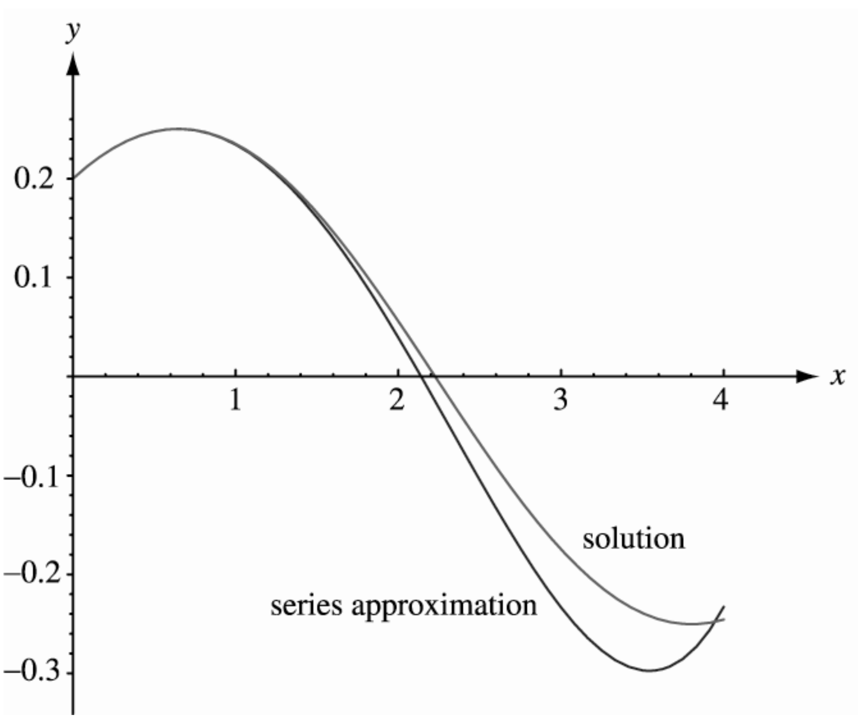
$$y(0) = 0.2$$
$$y'(0) = 0.15$$

$y''(0) = -\sin(y(0)) = -\sin(0.2) \approx -0.198669$
$y''' = -(\cos y)y' \Rightarrow y'''(0) = -(\cos 0.2)(0.15) \approx -0.147010$
$y^{(4)} = (\sin y)(y')^2 - (\cos y)y'' \Rightarrow y^{(4)}(0) \approx (\sin 0.2)(0.15)^2 - (\cos 0.2)(-0.198669) \approx 0.199179$

This gives the first five nonzero terms in a Taylor series expansion of the solution:

$$\begin{aligned} y(x) &\approx 0.2 + 0.15x - \frac{0.198669}{2!}x^2 - \frac{0.147010}{3!}x^3 + \frac{0.199179}{4!}x^4 \\ &\approx 0.2 + 0.15x - 0.09933x^2 - 0.02450x^3 + 0.00830x^4 \end{aligned}$$

This polynomial should be a good approximation to the solution "close" to $x = 0$. It's graph is shown below compared with the actual solution. We can use this fourth degree polynomial to predict that the maximum angle that the pendulum attains (before swinging back) is approximately $y = 0.25$ which occurs when x is approximately 0.65. Moreover, the first time the pendulum is vertical ($y = 0$) occurs when x is approximately 2.14. A higher degree polynomial approximation would give better estimates, particularly for that first zero.



Example 5: Solve $y'' = xy$, $y(0) = 1$, $y'(0) = 0$.

This is Airy's equation, an important equation in optics used to model the diffraction of light.

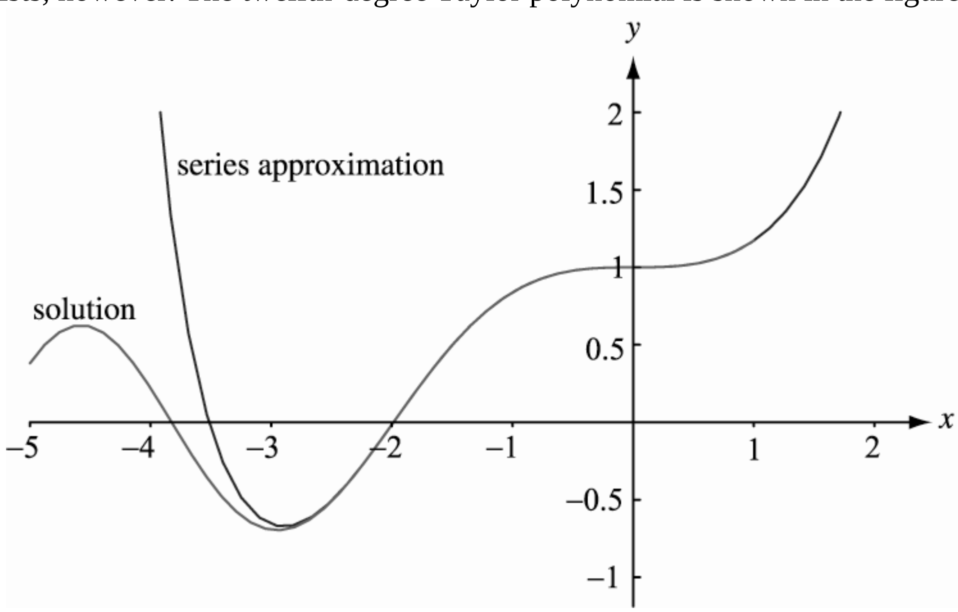
$y(0) = 1$
$y'(0) = 0$

$$y''(0) = 0y(0) = 0$$
$$y''' = y + xy' \Rightarrow y'''(0) = y(0) = 1$$
$$y^{(4)} = y' + y' + xy'' = 2y' + xy'' \Rightarrow y^{(4)}(0) = 2y'(0) = 0$$
$$y^{(5)} = 3y'' + xy''' \Rightarrow y^{(5)}(0) = 3y''(0) = 0$$
$$y^{(6)} = 4y''' + xy^{(4)} \Rightarrow y^{(6)}(0) = 4y'''(0) = 4$$

By this time you might observe the pattern that has started to develop with the derivatives. One can show that the remaining non-zero derivative values are $y^{(9)}(0) = 7\cdot 4$, $y^{(12)}(0) = 10\cdot 7\cdot 4$, etc. The solution has the Taylor series

$$y(x) = 1 + \frac{1}{3!}x^3 + \frac{4}{6!}x^6 + \frac{7\cdot 4}{9!}x^9 + \frac{10\cdot 7\cdot 4}{12!}x^{12} + \ldots$$
$$= 1 + \frac{1}{(2\cdot 3)}x^3 + \frac{1}{(2\cdot 3)(5\cdot 6)}x^6 + \frac{1}{(2\cdot 3)(5\cdot 6)(8\cdot 9)}x^9 + \frac{1}{(2\cdot 3)(5\cdot 6)(8\cdot 9)(11\cdot 12)}x^{12} + \ldots$$

While we can probably write an expression for the general term of the series, we cannot write the series itself in terms of familiar functions. This series has been well studied by mathematicians and physicists, however. The twelfth-degree Taylor polynomial is shown in the figure below.



Example 6: Solve $y'' + 2xy' + 2y = 0$, $y(0) = c_1$, $y'(0) = c_2$

This is a second order linear differential equation. By rewriting the equation as $y'' = -2xy' - 2y$ we can compute that

$$y(0) = c_1$$
$$y'(0) = c_2$$

$$y''(0) = -2(0)(c_2) - 2(c_1) = -2c_1$$
$$y''' = -2xy'' - 2y' - 2y' = -2xy'' - 4y' \Rightarrow y'''(0) = -4y'(0) = -4c_2$$
$$y^{(4)} = -2xy''' - 2y'' - 4y'' = -2xy''' - 6y'' \Rightarrow y^{(4)}(0) = -6y''(0) = 12c_1$$
$$y^{(5)} = -2xy^{(4)} - 2y''' - 6y''' = -2xy^{(4)} - 8y''' \Rightarrow y^{(5)}(0) = -8y'''(0) = 32c_2$$

This gives the first five terms in the series solution as

$$\begin{aligned} y(x) &= c_1 + c_2 x - \frac{2c_1}{2!}x^2 - \frac{4c_2}{3!}x^3 + \frac{12c_1}{4!}x^4 + \frac{32c_2}{5!}x^5 \ldots \\ &= c_1\left(1 - \frac{2}{2!}x^2 + \frac{12}{4!}x^4 - \ldots\right) + c_2\left(x - \frac{4}{3!}x^3 + \frac{32}{5!}x^5 - \ldots\right) \end{aligned}$$

There is an important question about the series solutions in these last three examples: for what values of *x* do they converge? An elegant theorem by Lazarus Fuchs (1833-1902) states the following:

Consider the differential equation $y'' + p(x)y' + q(x) = 0$ with initial conditions of the form $y(0) = c_1$, $y'(0) = c_2$. If both $p(x)$ and $q(x)$ have Taylor series that converge on the interval $(-r, r)$, then the differential equation has a unique power series solution $y(x)$ that also converges on the interval $(-r, r)$.

In other words, the radius of convergence of the series solution is at least as big as the minimum of the radii of convergence of $p(x)$ and $q(x)$. In particular, if $p(x)$ and $q(x)$ are both polynomials, as in examples 5 and 6, then the series solution solves the differential equation for all values of *x*. The first series in parentheses in the solution in example 6 actually converges to e^{-x^2}. You can check that $y = e^{-x^2}$ is a solution to the initial value problem in example 6 with $c_1 = 1$ and $c_2 = 0$. The second series in parentheses in example 6 does not come from any familiar function.

An alternative method for finding the coefficients in a power series solution to a differential equation is to substitute the power series and its derivative(s) into the differential equation, then match coefficients of similar powers of *x* to generate recurrence relations for the coefficients. These recurrence relations can sometimes be solved to find explicit formulas for the coefficients.

References:

1. *Differential Equations: Graphics, Models, Data* by David Lomen and David Lovelock, John Wiley & Sons, Inc., 1999.

2. *Elementary Differential Equations* by William Boyce and Richard DiPrima, Wiley & Sons, Inc., 2005 (8th edition).

Worksheet for Taylor Series Solutions to Differential Equations

1. (a) Find the first four nonzero terms in the Taylor series solution to $y' = 2xy$, $y(0) = 1$.
   (b) Solve the separable differential equation $y' = 2xy$, $y(0) = 1$. Show that your solution agrees with the beginning of the Taylor series solution you found in part (a).

2. Find the first four nonzero terms in the Taylor series solution to $y' = \cos(y) - \sin(x)$, $y(0) = 0$. [This will really test your ability to work with the product rule and chain rule for implicit differentiation!]

3. Find the Taylor series solution to $y'' = -y$, $y(0) = c_1$, $y'(0) = c_2$. Write the solution as the sum of two series as in example 6, then identify each of the two individual series in terms of familiar functions.

4. Find the first five nonzero terms in the Taylor series centered at $x = -2$ that is the solution to Airy's equation $y'' = xy$ with $y(-2) = 0$, $y'(-2) = 1$. [Hint: make the appropriate modifications in example 5.]

Solutions to worksheet problems:

1. (a) $y(0)=1$

$y'(0)=2(0)(1)=0$

$y''=2y+2xy' \Rightarrow y''(0)=2$

$y'''=2y'+2y'+2xy''=4y'+2xy'' \Rightarrow y'''(0)=0$

$y^{(4)}=4y''+2y''+2xy'''=6y''+2xy''' \Rightarrow y^{(4)}(0)=6(2)=12$

$y^{(5)}=8y'''+2xy^{(4)} \Rightarrow y^{(5)}(0)=0$

$y^{(6)}=10y^{(4)}+2xy^{(5)} \Rightarrow y^{(6)}(0)=10(12)=120$

$$y(x)=1+\frac{2}{2!}x^2+\frac{12}{4!}x^4+\frac{120}{6!}x^6+\ldots=1+x^2+\frac{x^4}{2}+\frac{x^6}{6}+\ldots$$

(b) Separation of variables gives the solution as $y=e^{x^2}$. Write out the series for this function.

2. $y'=\cos(y)-\sin(x) \Rightarrow y'(0)=1$

$y''=-\sin(y)y'-\cos(x) \Rightarrow y''(0)=-1$

$y'''=-\cos(y)(y')^2-\sin(y)y''+\sin(x) \Rightarrow y'''(0)=-1$

$y^{(4)}=-\sin(y)(y')^3-2\cos(y)y'y''-\cos(y)y'y''-\sin(y)y'''+\cos(x) \Rightarrow y^{(4)}(0)=4$

$$y(x)=x-\frac{1}{2!}x^2-\frac{1}{3!}x^3+\frac{4}{4!}x^4+\ldots$$

3. $y(0)=c_1$

$y''(0)=-c_1$

$y^{(4)}=-y'' \Rightarrow y^{(4)}(0)=c_1$

$y^{(6)}=-y^{(4)} \Rightarrow y^{(6)}(0)=-c_1$

etc.

$y'(0)=c_2$

$y'''=-y' \Rightarrow y'''(0)=-c_2$

$y^{(5)}=-y''' \Rightarrow y^{(5)}(0)=c_2$

$y^{(7)}=-y^{(5)} \Rightarrow y^{(7)}(0)=-c_2$

etc.

$$y(x)=c_1\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)+c_2\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right)=c_1\cos x+c_2\sin x$$

4. The derivatives are the same as in example 5. Because the initial conditions are given for $x=-2$, however, we must center our Taylor series at $x=-2$. This gives

$$y(-2)=0,\ y'(-2)=1,\ y''(-2)=0,\ y'''(-2)=-2,\ y^{(4)}(-2)=2,\ y^{(5)}(-2)=4,\ y^{(6)}(-2)=-12$$

$$y(x)=(x+2)-\frac{2}{3!}(x+2)^3+\frac{2}{4!}(x+2)^4+\frac{4}{5!}(x+2)^5-\frac{12}{6!}(x+2)^6+\ldots$$