Problem of the Week #12
Solution

Recall that the formula $f(t)$ to convert from Fahrenheit to Celsius is

$$f(t) = \frac{5}{9}(t - 32).$$

Suppose the actual temperature is T degrees F. Then the first conversion gives

$$f(T) = \frac{5}{9}(T - 32)$$

and the second is

$$f(f(T)) = f\left(\frac{5}{9}(T-32)\right) = \frac{5}{9}\left(\frac{5}{9}(T-32) - 32\right) = \left(\frac{5}{9}\right)^2 T - \left(\frac{5}{9}\right)^2 \cdot 32 - \frac{5}{9}\cdot 32$$

and the third is

$$\begin{aligned} f(f(f(T))) &= f\left(\left(\frac{5}{9}\right)^2 T - \left(\frac{5}{9}\right)^2 \cdot 32 - \frac{5}{9}\cdot 32\right) \\ &= \frac{5}{9}\left(\left(\frac{5}{9}\right)^2 T - \left(\frac{5}{9}\right)^2 \cdot 32 - \frac{5}{9}\cdot 32 - 32\right) \\ &= \left(\frac{5}{9}\right)^3 T - \left(\frac{5}{9}\right)^3 \cdot 32 - \left(\frac{5}{9}\right)^2 \cdot 32 - \frac{5}{9}\cdot 32. \end{aligned}$$

More generally, completing this process n times gives

$$\begin{aligned} (f \circ f \circ \cdots \circ f)(T) &= \left(\frac{5}{9}\right)^n T - \left[\frac{5}{9}\cdot 32 + \left(\frac{5}{9}\right)^2 \cdot 32 + \cdots + \left(\frac{5}{9}\right)^n \cdot 32\right] \\ &= \left(\frac{5}{9}\right)^n T - \frac{5}{9}\cdot 32 \left[1 + \frac{5}{9} + \left(\frac{5}{9}\right)^2 + \cdots + \left(\frac{5}{9}\right)^n\right] \\ &= \left(\frac{5}{9}\right)^n T - \frac{5}{9}\cdot 32 \cdot \frac{1 - \left(\frac{5}{9}\right)^{n+1}}{1 - \frac{5}{9}} \\ &= \left(\frac{5}{9}\right)^n T - \frac{5}{9}\cdot 32 \cdot \frac{1 - \left(\frac{5}{9}\right)^{n+1}}{\frac{4}{9}} \\ &= \left(\frac{5}{9}\right)^n T - 40 \cdot \left(1 - \left(\frac{5}{9}\right)^{n+1}\right). \end{aligned}$$

As n gets large, $(5/9)^n$ tends towards zero, making the long-term behavior independent of the actual air temperature. Also, $(5/9)^{n+1}$ tends towards zero, of course, making the long-run result of the calculation

$$0 - 40(1 - 0) = -40.$$

This makes sense since $-40°F$ and $-40°C$ are the same.

Emily Piff and John Snyder submitted correct solutions.