(1) (20 pts) For each of the following functions, compute the derivative with respect to x.

(a) $(x+1)^3 \sin x$ product rule: $3(x+1)^2 \sin x + (x+1)^3 \cos x$
+ chain rule

(b) $\dfrac{\ln x^2}{3x}$ $\dfrac{u}{v}$ quotient rule: $\dfrac{3x \cdot 2x \cdot \frac{1}{x^2} - (\ln x^2) \cdot 3}{(3x)^2}$
+chain rule

$$= \frac{6 - 3\ln x^2}{9x^2} = \frac{2 - \ln x^2}{x^2}$$

(c) $e^{\cos x^2}$ chain rule: $-2x \sin(x^2) \cdot e^{\cos x^2}$

(d) $\int_x^1 \ln(1+e^{t^4})\, dt$ $= -\int_1^x \ln(1+e^{t^4})dt$, so Fund. Th. of Calc.
says the derivative is $-\ln(1+e^{x^4})$

(e) $\int_1^2 e^{\cos x}\, dx$ This is the area under the curve $e^{\cos x}$ from $x=1$ to $x=2$
so it's a constant, so it's derivative is 0

(2) (20 pts) Compute the following integrals:

(a) $\int_0^1 x^{11}\,dx$ $= \frac{1}{12} x^{12}\Big|_0^1 = \frac{1}{12}$

(b) $\int \sin \pi t\, dt$ $= \frac{-1}{\pi} \cos \pi t + C$

(c) $\int (e^x + 2x + 2)\, dx$ $= e^x + x^2 + 2x + C$

(d) $\int_{-3}^{-2} \frac{1}{x}\, dx$ $= \ln|x| \Big|_{-3}^{-2} = \ln 2 - \ln 3$

(3) (15 pts) Use the definition of derivative to compute $f'(2)$, where $f(t) = 5t^2 + 7$.

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{5(2+h)^2+7-(5\cdot 2^2+7)}{h}$$

$$= \lim_{h\to 0} \frac{5(4+4h+h^2)+7-5\cdot 4-7}{h} = \lim_{h\to 0} \frac{20h+5h^2}{h}$$

$$= \lim_{h\to 0} 20+5h = 20$$

(4) (15 pts) Let $f(T)$ be the number of minutes it takes for an ice cube to melt if the temperature of the air around it is $T°$ Fahrenheit.

(a) (10 pts) Explain, in words, the meaning of the derivative $\frac{df}{dT}$.

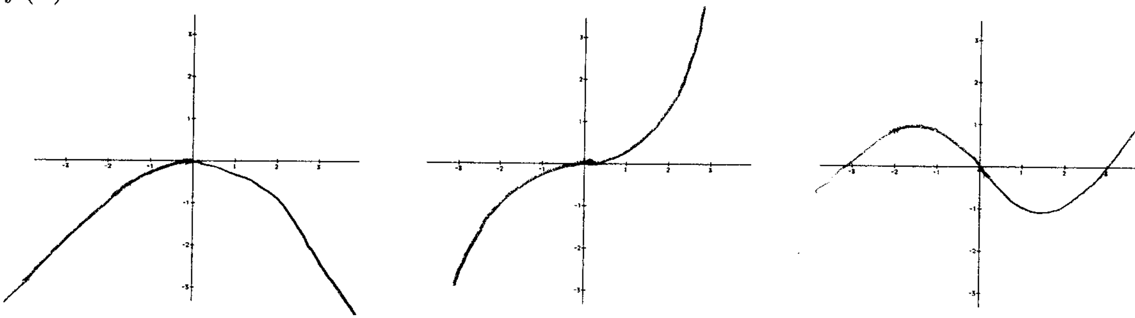
Roughly, it's the number of extra minutes it would take for the ice cube to melt if the air temp. went up 1°F.

(b) (5 pts) Do you expect $\frac{df}{dT}$ to be positive or negative? Why?

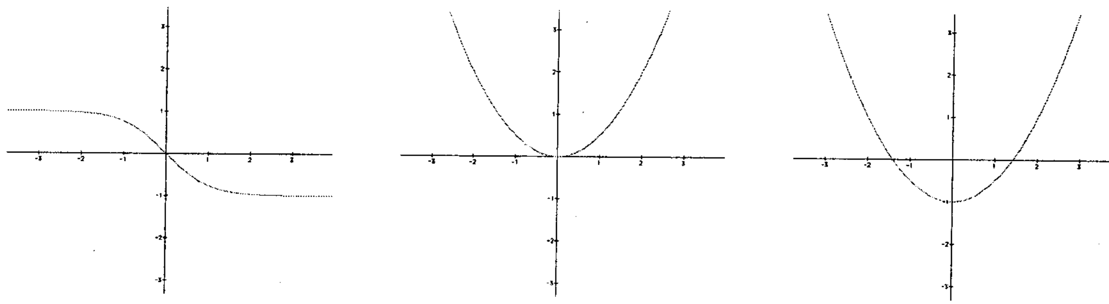
Negative - if the temp. goes up, the time it takes to melt should go down.

(5) (27 pts) Here are the graphs of 3 derivative functions $y = f'(x)$. In each case, make a rough sketch of the graph of $y = f''(x)$ below it and the graph of the particular antiderivative $y = f(x)$ for which $f(0) = 0$ above it.

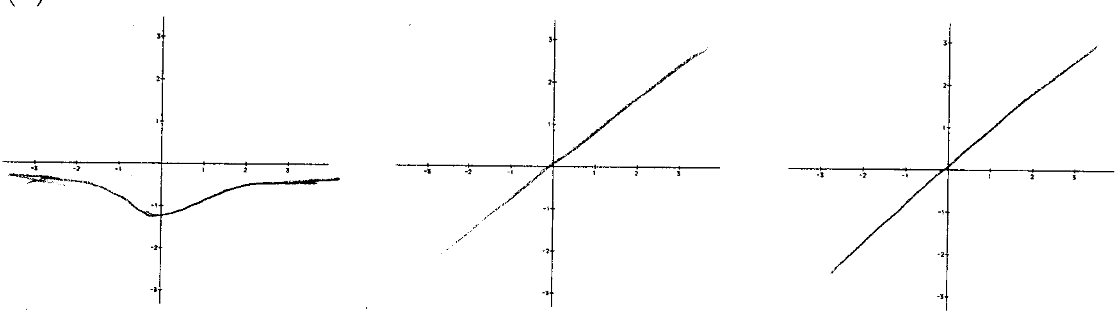
$f(x)$ below:



$f'(x)$ below:



$f''(x)$ below:



(6) (12 pts) I have \$500 in the bank now, and my balance after t days is $500e^{0.01t}$. What is my average balance over the first 30 days (i.e., from $t = 0$ to $t = 30$)?

$$= \frac{\int_0^{30} 500e^{0.01t}\,dt}{30-0} = \frac{500 \cdot 100 \cdot e^{0.01t}\Big|_0^{30}}{30} = \frac{50{,}000\,(e^{.3}-1)}{30}$$

$$= \frac{5000(e^{.3}-1)}{3}$$

(7) (10 pts) Consider the hyperbola $x^2 - y^2 = 7$.

(a) What is the slope of the tangent line to the hyperbola at the point $(4, 3)$?

Find slope $= \frac{dy}{dx}$ by implicit differentiation:

$$\frac{d}{dx}(x^2-y^2) = \frac{d}{dx}(7), \text{ so } 2x - 2y\frac{dy}{dx} = 0, \text{ or } \frac{dy}{dx} = \frac{x}{y}.$$

So at $(4,3)$, $\frac{dy}{dx} = \frac{4}{3}$

(b) Find an equation for the tangent line at that point.

$y - y_0 = m(x - x_0)$, so

$y - 3 = \frac{4}{3}(x-4)$, or $y = \frac{4}{3}x - 7/3$

(8) (16 pts) Compute the following limits, or explain why they do not exist. Quote any results that you use.

(a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x^2 + 2x}$

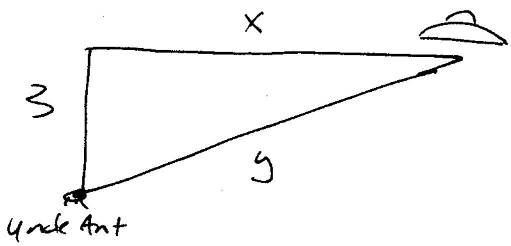
$\lim_{x\to 0} \sin x = 0 = \lim_{x\to 0} x^2+2x$. L'Hopital's rule says that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. So

$$\lim_{x\to 0} \frac{\sin x}{x^2+2x} = \lim_{x\to 0} \frac{\cos x}{2x+2} = \frac{1}{2}$$

(b) $\displaystyle\lim_{x\to \infty} \frac{\sin x}{x^2 + 2x}$

As $x \to \infty$, $x^2+2x \to \infty$ while $\sin x$ stays between -1 & 1, so the limit of their quotient is 0.

(9) (20 pts) A flying saucer traveling 500 miles per hour at an altitude of 3 miles flew directly over Uncle Ant. Let x and y be as labeled in the figure.



(a) (4 pts) Find an equation relating x and y.

$$3^2 + x^2 = y^2$$
$$9 + x^2 = y^2$$

(b) (4 pts) Find the value of x when y is 5 miles.

$$9 + x^2 = 5^2$$
$$9 + x^2 = 25$$
$$x^2 = 16$$
$$x = 4 \text{ or } \cancel{-4}$$

(c) (12 pts) How fast is the distance from Uncle Ant to the flying saucer changing at the time when the saucer is 5 miles from Uncle Ant? That is, what is $\frac{dy}{dt}$ at the time when $y = 5$?

Implicit differentiation:

$$\frac{d}{dt}(9 + x^2) = \frac{d}{dt}(y^2)$$
$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

We know that $\frac{dx}{dt} = 500$, and at the given instant $y = 5$ and $x = 4$. Plugging in, we get

$$2 \cdot 4 \cdot 500 = 2 \cdot 5 \cdot \frac{dy}{dt}$$
$$400 = \frac{dy}{dt}$$

mph

(10) (15 pts)

(a) (10 pts) Use linear approximation or one iterate of Newton's method to estimate $\sqrt{10}$.

linear approximation: $y = f(x) = \sqrt{x}$

$f(x) \approx f(a) + f'(a)\cdot(x-a)$. Take $a=9$. $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$,

so $f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$, and we get

$$f(x) \approx f(9) + f'(9)(x-9) = 3 + \frac{1}{6}(x-9).$$

So $\underline{\sqrt{10} = f(10) \approx 3 + \frac{1}{6}(10-9) = 3 + 1/6}$

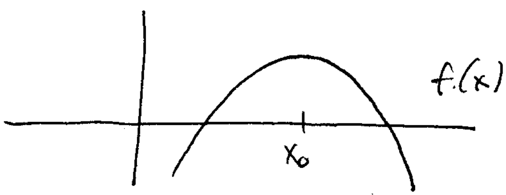
OR, Newton's method: Find a solution of $g(x) = x^2 - 10 = 0$.

Given a guess x_0, Newton's method gives the next guess

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)} = x_0 - \frac{x_0^2 - 10}{2x_0}.$$

Guess $x_0 = 3$. Then $x_1 = 3 - \frac{9-10}{2\cdot 3} = 3 - \frac{-1}{6} = 3 + 1/6$

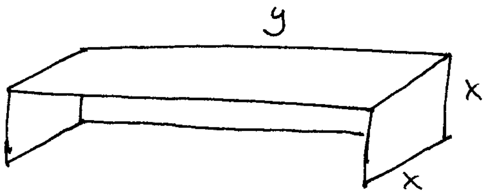
(b) (5 pts) Use the picture to explain why Newton's method will fail to find a solution of $f(x) = 0$ if you start with the initial guess $x = x_0$.



In Newton's method, you get your next guess x_1 from your original guess x_0 by finding where the tangent line to $y = f(x)$ at the point $(x_0, f(x_0))$ intersects the x-axis. Here, since the tangent line at $(x_0, f(x_0))$ is horizontal, it never intersects the x-axis, so you can't find x_1.

(Algebraically, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, which doesn't exist since $f'(x_0) = 0$.)

(11) (30 pts) A canvas wind shelter for the beach is to have a back, two *square* sides, and a top. You have 96 square feet of canvas available to build the shelter.



(a) (10 pts) Let x be the height of the shelter and y the length.

(i) What is the surface area of the shelter in terms of x and y?

Area of sides + Area of top + Area of back $= 2x^2 + xy + xy$

$= 2x^2 + 2xy$

(ii) What is the volume of the shelter in terms of x and y?

x^2y

(b) (20 pts) Find the dimensions of the shelter with maximum space inside (i.e., with maximum volume).

Maximize ~~f(x)~~ x^2y given that $\underline{2x^2 + 2xy = 96}$

solve for y: $2xy = 96 - 2x^2$

$y = \frac{48}{x} - x$ (x can't be 0).

So we want to maximize the function $f(x) = x^2y = x^2\left(\frac{48}{x} - x\right) = 48x - x^3$

Find the critical points: $f'(x) = 48 - 3x^2$. This exists everywhere, so the crit. pts. are where it's 0: $48 - 3x^2 = 0$, $x^2 = 16$, $x = 4$ or $\cancel{-4}$

The "end points" are when $x \to 0$ and $y \to 0$, but in both cases the volume goes to 0 as well. So the max. volume must be when $x = 4$, $y = \frac{48}{4} - 4 = 8$, and the volume is $4 \cdot 4 \cdot 8 = 128$.