Math 119 Practice Second Midterm

The actual midterm will of course be shorter than this. It will contain at least one question taken directly off your homework. You may use a calculator, but **not any of its calculus functions.**

1. Section 8.8 (p. 426): #5. Chapter 8 Review (p. 428): #17, 21, 37, 55.

2. Sarah Series would like to set up a scholarship for Math 119 students, to be awarded each year, starting in one year (sorry). How much money would she have to deposit in an account which earns 7% interest a year (compounded continuously; that is, if she deposits $\$D$, then after t years she'll have $\$De^{0.07t}$) in order to award a $1,000 scholarship each year? (Hint: As is usually the case with word problems, it helps to break the problem into small pieces. First compute how much Sarah would need to deposit now in order to award a $1,000 scholarship next year. Then add in the amount that Sarah would need to deposit in order to award a $1,000 scholarship in two years. And so on...)

To have enough to give a scholarship in 1 year, she needs to deposit $\$D_1$ **now, where** $D_1e^{.07} = 1000$**, or** $D^1 = 1000e^{-.07}$**. To give one in two years as well, she must now deposit an additional** $\$D_2$**, where** $D_2e^{2\cdot.07} = 1000$**, or** $D_2 = 1000(e^{-.07})^2$**. So to give one every year, she must now deposit** $D_1+D_2+D_3\cdots = 1000e^{-.07}+1000(e^{-.07})^2+1000(e^{-.07})^3+\cdots = 1000e^{-.07}(\frac{1}{1-e^{-.07}}) \approx \$13,791.50$**.**

3. Determine if the following series converge or diverge. Be sure to give reasons!

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^4 - 6n^3 + 13n}{n^5 + n^2 + 4}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^{n-1} + 1}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{(n!)(n!)}{(2n)!}$

(d) $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{2}{n}\right)^n$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2^n}$

(f) $\displaystyle\sum_{n=1}^{\infty} a_n$, if the nth partial sum of this series is given by $s_n = \dfrac{n-1}{2n+1}$.

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n}\, a_n$, if you know that $\displaystyle\sum_{n=1}^{\infty} a_n$ is a positive series that converges.

(a) diverges - limit comparison to $\sum 1/n$ **(b) converges - ratio test, or comparison to geometric** $\sum 1/(3^{n-1})$ **(c) converges - ratio test (d) diverges - divergence test (**n**th term doesn't go to 0) (e) converges (absolutely) - ratio test (f) converges, by definition of series convergence, to** $\lim_{n\to\infty} s_n = 1/2$**. (g) converges, by comparison to** $\sum a_n$

4. Find positive numbers A and B such that

$$0 < A \le \sum_{n=0}^{\infty} \frac{1}{5^n + n^3} \le B.$$

We know that $1 < \frac{1}{1+0} + \frac{1}{5+1} + \frac{1}{5^2+8} + \cdots = \sum_{n=0}^{\infty} \frac{1}{5^n+n^3} < \sum_{n=0}^{\infty} \frac{1}{5^n} = \frac{1}{1-\frac{1}{5}} = 5/4$**, so** $1 < \sum_{n=0}^{\infty} \frac{1}{5^n+n^3} < 5/4$**.**

5. Use a third-degree Taylor polynomial to estimate $\sqrt{1.1}$.

We'll compute the Taylor polynomial at $a = 1$**.** $\mathbf{P_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3}$**, so** $\sqrt{1.1} \approx P_3(1.1) = 1 + \frac{1}{2}(0.1) - \frac{1}{8}(0.1)^2 + \frac{1}{16}(0.1)^3 = 1.0488125$**.**

6. Compute $1/e$ to within $1/10$ of its actual value. Be sure to explain how you can be sure of the accuracy of your estimate.

(HINT: $1/e = e^{-1}$, and $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$)

$\mathbf{e^{-1} = 1 + (-1) + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \cdots = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \ldots}$ **this is a decreasing alternating series, so the error** $|e^{-1} - S_n| \leq a_{n+1}$**, where** S_n **is the** n**th partial sum and** a_{n+1} **is the first term of the series that we didn't add. To make** $\frac{1}{n!} \leq \frac{1}{10}$**, we need** $n = 4$ **(**$1/3! = 1/6$ **and** $1/4! = 1/24$**). So,** $1 - 1 + 1/2 - 1/6$ **is within 1/24 of** e^{-1}**, i.e.** $e^{-1} \approx 1/3$**.**

7. Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{4^n}$.

Use the ratio test: $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{\frac{x^{2n+2}}{4^{n+1}}}{\frac{x^{2n}}{4^n}} = \lim_{n\to\infty} \frac{x^{2n+2}}{4^{n+1}} \cdot \frac{4^n}{x^{2n}} = \lim_{n\to\infty} \frac{x^2}{4} = \frac{x^2}{4}$**.**

So the series converges absolutely if $\frac{x^2}{4} < 1$**, i.e. if** $x^2 < 4$**, i.e. if** $-2 < x < 2$**, and diverges if** $\frac{x^2}{4} > 1$**, i.e. if** $x < -2$ **or** $x > 2$**. Thus the radius of convergence is 2.**

8. A rod of length 3 meters and density $\delta(x) = 2 + \cos x$ grams/meter is positioned along the x-axis with its left end at the origin.

(a) Where is the rod most dense?
(b) Where is the rod least dense?
(c) Is the center of mass of this rod closer to the origin, or closer to x = 3?
(d) What is the total mass of the rod?
(e) Where is the center of mass of the rod?

(a) The function describing the density is decreasing for $0 \leq x \leq 3$**. (Note that its derivative,** $-\sin x$**, is negative for this interval.) So it is most dense at** $x = 0$**. (b) It is least dense at** $x = 3$**. (c) Because the density is decreasing as you go from left to right, the center of mass is closer to the origin. (d) The total mass of the rod is given by** $\int_0^3 (2 + \cos x)\,dx = (2x + \sin x)|_0^3 = 6 + \sin 3 \approx 6.14$ **gm. (e) The center of mass is given by** $\frac{1}{\textbf{mass}} \int_0^3 x(2 + \cos x)\,dx = \frac{1}{6.14}(x^2 + x\sin x + \cos x)|_0^3 \approx 1.21$ **m.**