1. Consider the system

\[ \begin{align*}
\dot{x} &= y + y(x^2 + y^2) \\
y &= -x - z(x^2 + y^2)
\end{align*} \]

The origin \((0,0,0)\) is the only equilibrium point. We want to classify the system's behavior near \((0,0)\).

(a) What does the local linearization tell us about the behavior near \((0,0)\)?

(b) What do the nullclines tell us about the behavior near \((0,0)\)?

(c) Consider the function \(f(z) = z^2 - y^2\). If \((x(t), y(t), z(t))\) is a solution of the system, what is \(\frac{d}{dt}(f(x(t), y(t), z(t)))\)? What does this tell you about the behavior of the system near \((0,0)\)?

- Hint: What is the physical meaning of the quantity \(x^2 + y^2\)?

\[ \mathbf{b}) \quad \begin{align*}
\dot{x} &= y + y(x^2 + y^2) \\
y &= -x - z(x^2 + y^2)
\end{align*} \]

2. Consider the system:

\[ \begin{align*}
\dot{x} &= y + y(x^2 + y^2) \\
\dot{y} &= -x - z(x^2 + y^2)
\end{align*} \]

\[ \mathbf{b}) \quad \begin{align*}
\dot{x} &= y + y(x^2 + y^2) \\
\dot{y} &= -x - z(x^2 + y^2)
\end{align*} \]

\[ \mathbf{c}) \quad \begin{align*}
\dot{x} &= y + y(x^2 + y^2) \\
\dot{y} &= -x - z(x^2 + y^2)
\end{align*} \]

\[ \mathbf{c}) \quad \begin{align*}
\dot{x} &= y + y(x^2 + y^2) \\
\dot{y} &= -x - z(x^2 + y^2)
\end{align*} \]
2. The Linearity Principle (also Principle of Superposition) states that if \( Y_1 \) and \( Y_2 \) are two solutions of a linear homogeneous system of differential equations, and \( a \) and \( b \) are constants, then \( aY_1 + bY_2 \) is also a solution.

(a) Give an example showing that the Linearity Principle is not true for linear nonhomogeneous equations.

\[
a) \text{Let } y \text{ possible } \frac{dy}{dx} = x. \text{ Then } y_1 = e^{x} \text{ and } y_2 = e^{-x}. \\
\quad \text{Let } y = e^{x} + e^{-x} \text{ is not.}
\]

\[
b) \text{Let } y \text{ possible } \frac{dy}{dx} = y^2. \text{ Then } y_1 = e^{-x} \text{ is a soln, but}
\quad \phi \cdot y = e \text{ is not.}
\]
3. Consider the second-order ODE \( y'' - 3y' - 2y = 3t \).
   (a) Find the general solution. What is the long-term behavior of solutions?
   (b) Find the solution satisfying the initial conditions \( y(0) = 3, y'(0) = -1 \).

   c) \[ \begin{align*}
   y'' + 2y' + 5y &= 0 \\
   
   \end{align*} \]
   \( x^2 + 2x + 5 = 0 \)
   \( x = -1 \pm 2i \)
   \( y = e^{-x}(C_1 \cos 2x + C_2 \sin 2x) \)

   Now, find a particular solution. Guess \( y_p = at + b \), so
   \( y_p' = a \), \( y_p'' = 0 \)

   \( y'' - 3y' - 2y = 3t \) \( \Rightarrow \)
   \( c(-2t) + 3(at + b) = 3t \)
   \( 2b = 3 \)
   \( b = \frac{3}{2} \)

   Gen soln: \( y = e^{-x}(C_1 \cos 2x + C_2 \sin 2x) + 3t/2 \)

   Subs \( y \to y_p \), class \( C_1 = 0 \), \( \beta = 3 \), \( B = \frac{3}{2} \)

b) \( y = C_1 e^{t} + C_2 e^{-2t} + 3t \), so \( y(0) = C_1 + C_2 = 1 \)
   \( y' = C_1 e^{t} - 2C_2 e^{-2t} - 3 \), so \( y'(0) = C_1 - 2C_2 = -1 \)
   \( C_1 = 3, C_2 = -1 \)
   \( y = e^{t} + e^{-2t} + 3t \)

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**Extra Credit** Which is better, a second-order equation or a system of two first-order equations?