Math 325 Practice Midterm Answers

1. Recall that the Jefferson apportionment method works as follows. Begin by giving each state one representative. Then, give the next representative to the state with the smallest value of $\frac{a_i+1}{p_i}$, where a_i is the number of seats that state i has gotten so far, and p_i is the population of state i. Keep going until all the representatives are apportioned.

(a) Why does this method favor the populous states more than the Adams method, which gives the next representative to the state with the smallest value of $\frac{a_i}{p_i}$?

(b) What is the Alabama Paradox, and why can't it occur with the Jefferson method?

(c) What problem does occur with the Jefferson method (other than favoring the large states)?

(a) Because we're adding $\frac{1}{p_i}$, which is small if the population p_i is big and big if the population is small, to $\frac{a_i}{p_i}$. Thus a large state is more likely to have the smallest value of $\frac{a_i+1}{p_i}$, and get the next seat with the Jefferson method, than it is to have the smallest value of $\frac{a_i}{p_i}$, and get the next seat with the Adams method.

(b) The AP occurs when a state has a larger apportionment at house size h than at house size $h+1$. It can't happen with the Jefferson method because that hands out the first n seats the same way regardless of the total house size. Thus the apportionment at house size $h+1$ is the same as at h, except that one state gets an extra seat.

(c) The Jefferson method can violate quota. For example, if Georgia is entitled to 11.38 seats, it might end up getting 10 or 13.

2. Exercise 3.13.7 (page 138).

(a) State 1: broken but fixable. State 2: working. State 3: unfixable. Then 1 and 2 are transient, 3 is absorbing. (b) $\begin{pmatrix} 0.5 & 0.3 & 0 \\ 0.498 & 0.699 & 0 \\ 0.002 & 0.001 & 1 \end{pmatrix}$ **(c)** $T = \begin{pmatrix} 273.64 & 272.72 \\ 452.73 & 454.54 \end{pmatrix}$. **Depending on which state it starts in, it'll spend about 273 days under repair and 453 days working, for a total of about 726 days before it dies.**

3. (a) Consider a rental car company with branches in Orlando and Tampa. Each rents cars to Florida tourists. The company specializes in catering to travel agencies that want to arrange tourist activities in both Orlando and Tampa. Consequently, a traveler will rent a car in one city and drop it off in either city. This can lead to an imbalance in available cars to rent. Historical data on the percentage of cars rented and returned to these locations are reflected in the transition matrix below:

	Orlando	Tampa
Orlando	0.5	0.2
Tampa	0.5	0.8

(Thus 20% of the cars rented in Tampa are returned to Orlando, etc.) Assuming that all of the cars are originally in Orlando, figure out the long-term behavior of the percentages of cars available at each location. What happens if you start with different initial conditions (that is, not all the cars start in Orlando)?

(b) Now consider a Markov chain with transition matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. What's the long-term behavior? How does it depend on the initial conditions?

(c) Same questions as (2), for the chain with matrix $\begin{pmatrix} 0.5 & 0.2 & 0.2 & 0 \\ 0.5 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.8 & 0 \end{pmatrix}$. (Hint: Draw the transition diagram and ponder it before starting any long computations.)

(a) dominant eigenvalue is 1, with normalized eigenvector $\begin{pmatrix} 2/7 \\ 5/7 \end{pmatrix}$**, so the long-term behavior is 2/7 of the cars in Orlando, 5/7 in Tampa. (b) Everything cycles from 1 to 3 to 2 and back to 1, so the initial condition repeats every three steps. (c) States 3 and 4 are transient, so the long-term behavior is the same as in (a): 2/7 in state 1, 5/7 in state 2 (and 0 in states 3 and 4).**

4. Suppose that there are currently 25,000 unemployed workers in Atlanta. Each month 8% of all those unemployed find jobs, but another 1500 become unemployed.

(a) How many people will be unemployed 6 months from now?

(b) Will the number of unemployed workers ever stabilize? If so, at what level?

(a) 22,540 (b) Yes, at the attracting equilibrium 18,750.

5. During a typical year, the monkey population grows by a factor of X, where X is a normal random variable with mean 1.1 and variance 0.2. However, with probability 0.05, a given year can produce a catastrophically bad banana harvest. In that case, subtract 0.4 from the growth factor.

Let M_n be the monkey population after n years, and assume that $M_0 = 100$. You may also assume that all of the random variables are independent.

(a) Compute the mean and variance of M_1.

(b) Compute the mean of M_n.

(a) mean (1.1 - .05*.4)100=108, variance $0.2 + (.4)^2 .05(.95)$ **= 0.2076 (b)** $(1.08)^n 100$