Complex Variables Practice Final

1. For each of the following, give an example of such a function, or explain why it's impossible.
(a) f is an analytic function on a domain D whose range is the set of all points (x, y) with $x \geq 0$.
(b) f is an analytic function and the maximum of $|f|$ on the set $|z| \leq 2$ is attained at $z = 0$.

2. Prove or give a counterexample for each of the following statements.
(a) The product of two analytic functions is analytic.
(b) The real part of an analytic function is analytic.

3. Find a conformal mapping that sends the unit disk $\{z : |z| < 1\}$ to the lower half-plane $\{z : Im(z) < 0\}$, or prove that no such mapping can exist.

4. Consider the function $f = u + iv$, where $u(x + iy) = x^2 + 2xy + y^2$ and $v(x + iy) = x^2 - 2xy - y^2$.
(a) Where does f' exist? Where is f analytic?
(b) Does there exist an analytic function g whose real part is equal to u? If so, give an example.
(c) Does there exist an analytic function h whose imaginary part is equal to v? If so, give an example.

5. Prove or disprove:
(a) Let D_1 and D_2 be simply connected domains with nonempty intersection. Then $D_1 \cup D_2$ is also simply connected.
(b) Let D be a domain and f be analytic on D. Then the image $f(D)$ is also a domain.

6. Evaluate the following integrals:
(a) $\displaystyle\int_0^\infty \frac{\cos(ax)}{(x^2 + b^2)^2}\, dx$, where a and b are positive real numbers.
(b) $\displaystyle\int_\gamma e^{\frac{2}{z}}\, dz$, where γ is the unit circle oriented counterclockwise.
(c) $\displaystyle\int_{\gamma_1} \overline{z}\, dz$ and $\displaystyle\int_{\gamma_2} \overline{z}\, dz$, where γ_1 is the semicircle in the upper half-plane going from -1 to 1, and γ_2 is the polygonal path from -1 to $-1 + i$ to $1 + i$ to 1. What can you conclude about the function $f(z) = \overline{z}$?
(d) $\displaystyle\int_\gamma \frac{\sin z}{e^{z^2}}\, dz$, where γ is the unit circle oriented counterclockwise.
(e) $\displaystyle\int_\gamma \frac{e^z}{z(z+4)}\, dz$, where γ is the unit circle oriented counterclockwise.

7. Find all solutions to the following equations.
(a) $z^2 + (1 + i)z = -5i$
(b) $e^z = \sqrt{3} - i$

8. Compute $(i^2)^i$ and $(i)^{2i}$, and conclude that the identity $(z^a)^b = z^{ab}$ does not always hold for complex z, a, and b.

9. How many solutions does the equation $z^5 + 6z^2 + 2z + 1 = 0$ have in the annulus $1 \leq r \leq 2$?

10. Let $f(z) = z^4 - z$, and let γ be the circle of radius 1/2, centered at the origin, oriented counterclockwise. How many times does the image $f(\gamma)$ wrap around the origin counterclockwise?

Answers:

1. (a) Impossible: the range must be an open set (Theorem 1, p. 191).
(b) f must be identically constant.

2. (a) True (product rule).
(b) False (one example: if $f(z) = z$, then the real part is $u(x + iy) = x$, which doesn't satisfy Cauchy-Riemann).

3. Many possibilities; one is $f(z) = \dfrac{iz + 1}{2z + 2i}$.

4. (a) f' exists only at 0, so f is nowhere analytic.
(b) No, because u is not harmonic.

(c) Yes, because v is harmonic. $h(x+iy) = (-x^2 - 2xy + y^2) + i(x^2 - 2xy - y^2) + C = (i-1)z^2 + C$, where C is any constant.

5. (a) False.

(b) False. Example: f is identically constant. The statement is true if f is nonconstant; try to prove that.

6. (a) $\frac{\pi}{4b^3}e^{-ab}(ab+1)$

(b) $4\pi i$

(c) $\int_{\gamma_1} \overline{z}\,dz = \pi i$, $\int_{\gamma_2} \overline{z}\,dz = -4i$; f can't be analytic everywhere in the region between the two curves.

(d) 0

(e) $\pi i/2$

7. (a) $1-2i$ and $-2+i$.

(b) $\ln 2 + i(-\pi/6 + 2n\pi)$, where n is an integer.

8. $(i^2)^i = e^{-(1+2n)\pi}$ and $(i)^{2i} = e^{-(1+4n)\pi}$, where n is an integer.

9. 3

10. 1