Complex Variables Practice Midterm #1

You will have three hours to complete the actual exam, starting when you open it up. You may not use books, notes, the internet, friends, etc. – nothing but a calculator and a pen or pencil. It's due at the beginning of class on Wednesday, February 18. It will of course be shorter than this.

1. Prove or give a counterexample: The union of two open sets is open.
2. Where is the function $f(x+iy) = x^2 + iy^2$ differentiable?
3. Show that the function $u(x,y) = y^2 - x^2$ is harmonic, find its harmonic conjugate, and find an analytic function $f(z)$ whose real part is u.
4. Use the definition of the derivative to show that every real-valued entire function is constant.
5. Let γ be the arc of the circle $|z| = 2$ from $z_1 = 2$ to $z_2 = 2i$ that lies in the first quadrant. Show that $\left| \int_\gamma \frac{z+4}{z^3-1}\, dz \right| \leq \frac{6\pi}{7}$.
6. (a) To what region does the function $f(z) = e^z$ map the unit square with corners 0, 1, $1+i$, and i?
(b) Solve the equation $e^z = -1 - \sqrt{3}i$
7. Compute the following integrals.
(a) $\int_\gamma \frac{1}{z}\, dz$, where γ is the circle $|z| = 3$, oriented counterclockwise. (For practice, try doing this directly and using a clever theorem.)
(b) $\int_\gamma \cos(\sin z)\, dz$, where γ is the parallelogram with corners $2+i$, $5+2i$, $6+3i$, and $3+2i$, oriented counterclockwise.
(c) $\int_\gamma (z^3+1)\, dz$, where γ is the line segment from $z_1 = 0$ to $z_2 = 1+i$.

Answers:

1. See the solution to problem 1.3.13a from the textbook.
2. The partials are continuous everywhere, so it's differentiable exactly when it satisfies Cauchy-Riemann. $\partial f/\partial x = 2x$ and $\partial f/\partial y = 2iy$, so f is differentiable iff $2x = -i2iy$ iff $x = y$.
3. $u_{xx} + u_{yy} = -2 + 2 = 0$, so it's harmonic. The harmonic conjugate is $v(x,y) = -2xy + c$, so $f(z) = u + iv = y^2 - x^2 - i(2xy) + C = -z^2$.
4. $f = u + i \cdot 0$. Since f is differentiable, $f'(z) =$ the limit from the right $= \partial u/\partial x$, and $f'(z) =$ the limit from above $= \frac{1}{i}\partial u/\partial y$. A real number and an imaginary number can be equal only if they're both 0, so $\partial u/\partial x = \partial u/\partial y = 0$ everywhere, so u is constant. Since $f = u + 0$, f is constant as well.
5. Use the fact that $|\int_\gamma f| \leq C\cdot$(length of γ), if C is an upper bound for f on γ. Here, $|z+4| \leq |z| + 4 = 6$, $|z^3 - 1| \geq ||z|^3 - 1| = 7$, and length($\gamma$)$= \pi$.
6. (a) The annular region given in polar coordinates by $e^0 = 1 \leq r \leq e^1 = e$ and $0 \leq \theta \leq 1$.
(b) $-1 - \sqrt{3}i = 2e^{i(-\frac{2}{3}\pi \pm 2\pi n)}$, so $z = \ln 2 + i(-\frac{2}{3}\pi \pm 2\pi n)$.
7. (a) $2\pi i$ (directly, or using the Cauchy integral formula)
(b) 0 (Cauchy-Goursat)
(c) i (Fundamental Theorem of Calculus)