You will have three hours to complete the actual exam, starting when you open it up. You may not use books, notes, the internet, friends, etc. – nothing but a calculator and a pen or pencil. It’s due at the beginning of class on Wednesday, April 1.

1. For each of the following, give an example of such a function, or explain why it’s impossible.
   (a) \( f \) is differentiable everywhere, but \( f''(0) \) does not exist.
   (b) \( f \) is differentiable everywhere except at 0.
   (c) \( f \) is differentiable everywhere except at 0, but has neither a pole nor a removable singularity at 0.

2. Find the Laurent series for the function \( f(z) = z^2 \sin(\frac{1}{z}) \) centered at the origin. Where is the series equal to the function?

3. Compute the following integrals.
   (a) \( \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 9)(x^2 + 4)^2} \, dx \)
   (b) \( \int_{0}^{\infty} \frac{\sin x}{x} \, dx \)

4. Let \( u(x, y) \) be a harmonic function on a domain \( D \). Show that \( u \) has no local maxima or minima on \( D \).

5. Determine how many zeros (counting multiplicities) the function \( f(z) = z^7 - 4z^3 + z - 1 \) has inside the unit circle.

6. Prove that an entire function whose imaginary part is bounded must be constant.

7. Determine the number of zeros (counting multiplicities) of \( f(z) = z^3 - z^2 + 2 \) in the first quadrant.

Answers:
1. (a) Impossible: the derivative of an analytic function is analytic.
   (b) One possibility: \( 1/z \).
   (c) It must have an essential singularity at 0. One possibility: \( e^{1/z} \).

2. \( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n + 1)!} \cdot \frac{1}{z^{4n}} \), equal to \( f \) on \( 0 < |z| < \infty \).

3. (a) \( \pi/100 \)
   (b) \( \pi/2 \)

4. Recall that a harmonic function is the real part of the analytic function \( f = u + iv \) (where \( v \) is a harmonic conjugate of \( f \); see p. 81). Then apply the discussion on p. 192.

5. 3.
6. Hint: apply Liouville’s theorem to the function \( e^{if} \).
7. 1.