Complex Variables Practice Midterm #2

You will have three hours to complete the actual exam, starting when you open it up. You may not use books, notes, the internet, friends, etc. – nothing but a calculator and a pen or pencil. It's due at the beginning of class on Wednesday, April 1.

1. For each of the following, give an example of such a function, or explain why it's impossible.
  (a) f is differentiable everywhere, but $f''(0)$ does not exist.
  (b) f is differentiable everywhere except at 0.
  (c) f is differentiable everywhere except at 0, but has neither a pole nor a removable singularity at 0.

2. Find the Laurent series for the function $f(z) = z^2 \sin(\frac{1}{z^2})$ centered at the origin. Where is the series equal to the function?

3. Compute the following integrals.
  (a) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2}\,dx$
  (b) $\displaystyle\int_{0}^{\infty} \frac{\sin x}{x}\,dx$

4. Let $u(x, y)$ be a harmonic function on a domain D. Show that u has no local maxima or minima on D.

5. Determine how many zeros (counting multiplicities) the function $f(z) = z^7 - 4z^3 + z - 1$ has inside the unit circle.

6. Prove that an entire function whose imaginary part is bounded must be constant.

7. Determine the number of zeros (counting multiplicities) of $f(z) = z^3 - z^2 + 2$ in the first quadrant.

Answers:

1. (a) Impossible: the derivative of an analytic function is analytic.
  (b) One possibility: $1/z$.
  (c) It must have an essential singularity at 0. One possibility: $e^{1/z}$.

2. $\displaystyle 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}}$, equal to f on $0 < |z| < \infty$.

3. (a) $\pi/100$
  (b) $\pi/2$

4. Recall that a harmonic function is the real part of the analytic function $f = u + iv$ (where v is a harmonic conjugate of f; see p. 81). Then apply the discussion on p. 192.

5. 3.

6. Hint: apply Liouville's theorem to the function e^{if}.

7. 1.