The sequence $S_n = Y_1 + \frac{Y_2}{2} + \ldots + \frac{Y_n}{n}$ is a martingale.

To apply the martingale convergence theorem, we must show $E(S_n^2) < M$ for some $M$ and all $n$. We observe that

$$E(S_n^2) = E((Y_1 + \ldots + Y_n)^2) = E\left(\frac{Y_1^2}{n^2} + \ldots + \frac{Y_n^2}{n^2} + \sum_{i=1}^{n} \frac{Y_i Y_j}{ij}\right)$$

$$= \frac{1}{n^2} + \ldots + \frac{1}{n^2} \quad (\text{since } E(Y_i^2) = \frac{n^2}{4})$$

$$\leq \frac{\sum_{i=1}^{n} 1}{n^2} = \frac{n}{n^2} = \frac{1}{n}$$

Thus $S_n$ converges a.s. and in mean square to a r.v. $S$. $S$ is not defective (since its $L^2$ norm exists), so $P(S = 0) = 0$, and we're done.

2a) We'd like to look at $X + Y - Z$, but that could be negative (and so not have a great fin). So we'll look at $X + Y + (n+1-2)$ instead. Note that $X + Y = 2 \Leftrightarrow X + Y + (n+1-2) = n+1$, and that $n+1-2$ has the same distribution as $X$. Thus

$$G_{X+Y+(n+1-2)} = G_X \cdot G_Y \cdot G_{n+1-2} = G^3_X$$

$$= \left(\frac{1}{n} \left(1 - \frac{1}{n+1-2}\right)\right)^3 = \left(\frac{1}{n} \cdot \frac{n^2}{n+1-2}\right)^3 = \frac{n^3}{n^2} \cdot \left(-\frac{1}{n} \right)^3 - \left(-\frac{1}{n} \right)^3$$

We need to coeff of $S_n^{n+1}$, so we apply the binomial theorem twice to get

$$\frac{\Delta^3}{n^3} \left(-\frac{1}{n} \right)^3 \left(-\frac{1}{n} \right)^3 = \frac{\Delta^3}{n^3} \left(1 - 3s + 3s^2 - s^3\right)\left(1 + \frac{3}{3}s + \left(\frac{3}{3}\right)^2 + \ldots \right)$$

So the coeff of $S_n^{n+1}$ is $\frac{1}{n^3} \binom{n}{n-1} = \frac{n(n-1)}{n^3} = \frac{n-1}{dn^2}$.
(3) Since $Z \sim \mathcal{N}(0, \sigma^2)$ and $n+1-Z$ have the same distribution, this is exactly the same as part (a).

NOTE: The problem is essentially the same as § 5.12.1.

For the density of a Poisson process, we have that

\[ P(Q(t+h) = j \mid Q(t) = i) = \begin{cases} \frac{\lambda^j}{j!} e^{-\lambda t} & \text{if } j = 0, 1, \ldots, i \\ \lambda^j t^j e^{-\lambda t} / j! & \text{if } j = i+1, i+2, \ldots \end{cases} \]

so, by the definition of a birth-death process (p. 68), this is one, with birth rate $\lambda$ and death rates $\mu_i = \lambda$.

To find the stationary distribution $P \pi = \pi^T \pi_0$, we find that $P \pi_0 = \pi_0 + \pi_1 (\lambda \pi_0) + \pi_2 (\lambda \pi_0)^2 + \cdots = \pi_0 \left( 1 - \frac{\lambda}{\mu} \right) + \frac{\lambda^2}{\mu^2} + \frac{\lambda^3}{\mu^3} + \cdots$

so $\pi_0 = \frac{1}{e^\lambda - 1}$, and $\pi_n = \frac{(\lambda/\mu)^n}{n!} e^{-\lambda/\mu}$.

(4) We know (Th. 13.4.6) that $M(t) = \max \{ W(s) : 0 \leq s \leq t \}$ has the same distribution as $|W(t)|$. We have

\[ P(\sup_{0 \leq s \leq t} |W(s)| \geq w) = P(M(t) \geq w) + P(\min_{0 \leq s \leq t} |W(s)| \leq -w) \]

\[ = 2P(M(t) \geq w) \quad \text{by symmetry} \]

\[ = 2P(\max_{0 \leq s \leq t} W(s) \geq w) \]

Chebyshev's inequality gives $P(|W(t)| \geq w) \leq \frac{E(|W(t)|^2)}{w^2} = \frac{\epsilon}{w^2}$.
Intuitively, the original state \( \alpha \neq \beta \) must be indistinguishable, i.e., \( \alpha = \beta \). More formally:

If \( \alpha = \beta \), the \( Y_0, Y_1, \ldots \) is clearly a Markov chain with transition matrix

\[
\begin{pmatrix}
1 - \alpha & \alpha \\
\beta & 1 - \beta
\end{pmatrix}
\]

Conversely, assume that \( Y_0, Y_1, \ldots \) is a Markov chain, then it must be the case that

\[
P(Y_n = 2 \mid Y_{n-1} = 1, Y_{n-1} = 1) = P(Y_n = 2 \mid Y_{n-1} = 1)
\]

We have that

\[
P(Y_n = 2 \mid Y_{n-1} = 1) = \frac{\frac{1}{2} (1 - \alpha \beta) + \frac{1}{2} (1 - \beta \alpha)}{\frac{1}{2} (1 - \alpha) + \frac{1}{2} (1 - \beta)} = \frac{\frac{9 + \beta}{2} \alpha - \alpha \beta}{\frac{1}{2} (1 - \alpha \beta)} (\text{since } \alpha = \beta = 1, \text{ in which case we're done})
\]

So

\[
\frac{9 + \beta}{2} = \frac{\frac{9 + \beta}{2} \alpha - \alpha \beta}{\frac{1}{2} (1 - \alpha \beta)} \Rightarrow \text{Solve for } \alpha \text{ and } \alpha = \beta.
\]