1. (a) Show that the following system of equations is inconsistent: 
\[
\begin{align*}
x & - 6y = -1 \\
x & - 2y = 2 \\
x + y & = 1 \\
z + 7y & = 6
\end{align*}
\]
(b) Find the least squares solution for the above equation.
(c) Find the least-squares error associated to the solution you found above.

2. The following rule gives an inner product on \( M_2 \), the set of all \( 2 \times 2 \) matrices:
\[
\left\langle \begin{bmatrix} a_{11} & a_{12} \\
a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\
b_{21} & b_{22} \end{bmatrix} \right\rangle = 2a_{11}b_{11} + (a_{11} + a_{12})(b_{11} + b_{12}) + (a_{11} + a_{21})(b_{11} + b_{21}) + a_{22}b_{22}.
\]
(a) Show that \( \left\{ \begin{bmatrix} 1 & 0 \\
0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\
0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix} \right\} \) is a linearly independent set from \( M_2 \).
(b) Let \( W = \text{Span} \left\{ \begin{bmatrix} 1 & 0 \\
0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\
0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix} \right\} \). Find an orthogonal basis for \( W \), using the above inner product.
(c) Find the projection of \( \begin{bmatrix} 1 & 8 \\
12 & 2 \end{bmatrix} \) onto \( W \).
(d) Find a nontrivial matrix in \( W^\perp \).

3. Let \( T : \mathcal{P}_2 \to \mathcal{P}_3 \) be defined as integration, with constant of integration zero. For example, \( T(1 + t + t^2) = t + \frac{t^2}{2} + \frac{t^3}{3} \).
(a) Show this is a linear transformation.
(b) Find the matrix representation of \( T \) relative to the standard bases for \( \mathcal{P}_2 \) and \( \mathcal{P}_3 \).
(c) Give an example of how this matrix relates to the transformation \( T \).

4. (a) Let \( A \) be a \( 12 \times 12 \) matrix. You are interested in showing that \( \bar{A} \bar{x} = \bar{b} \) has a solution for every \( \bar{b} \) in \( \mathbb{R}^{12} \). This is equivalent to which of the following?
Write the appropriate numbers here:

(i) \( \det A = 0 \).
(ii) the transformation \( \bar{x} \to A\bar{x} \) is one-to-one.
(iii) \( \text{Nul} \ A = \{ \bar{0} \} \).
(iv) \( A \) is diagonalizable.
(v) \( \text{rank} \ A = 12 \).
(vi) \( A = A^T \).
(vii) the columns of \( A \) span \( \mathbb{R}^{12} \).
(viii) \( A \) is invertible.
(b) Suppose \( \{\vec{u}_1, \ldots, \vec{u}_m\} \) is an orthonormal list of vectors in vector space \( V \). Prove that if \( \vec{v} \in \text{Span} \{\vec{u}_1, \ldots, \vec{u}_m\} \) then 
\[
||\vec{v}||^2 = |<\vec{v}, \vec{u}_1>|^2 + \cdots + |<\vec{v}, \vec{u}_m>|^2.
\]

5. Let \( W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\} \).

(a) Show that \( A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\} \) is a basis for \( W \).

(b) Show that \( B = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix} \right\} \) is another basis for \( W \).

(c) Find the change of basis matrix from \( A \) to \( B \), i.e. \( P_{B\leftarrow A} \).

(d) Find the coordinate vector of \( \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \) in both bases, and use your answer from the last question to relate them.

6. (a) Find the eigenvalues of \( A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} \), and find the eigenspace associated to each eigenvalue.

(b) Is matrix \( A \) diagonalizable?

(c) Let \( B \) be an arbitrary square matrix. If \( B \) is diagonalizable, what does this diagonal matrix represent?

(d) Prove that if a matrix \( B \) is invertible then 0 is not an eigenvalue of \( B \).

7. (a) Find the inverse to \( \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 3 & 8 \end{bmatrix} \) and use it to solve
\[
\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 3 & 8 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.
\]

(b) Find the determinant of \( \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 3 & 8 \end{bmatrix} \).

(c) Prove that the inverse of \( AB \) is \( B^{-1}A^{-1} \).