# Math 16H

## Due in class Friday, 10/22.

(Remember, there's a definition of fields in the footnote on page 347 of Bretscher.)

**1.** Do the following problems from Bretscher: 7.5 (p. 350), #13-17. Optional: 7.5.37, 3.1.53, 3.1.54.

**2.** Does the set of all polynomials with integer coefficients form a field? What if the coefficients are allowed to be real numbers?

**3.** Let $\mathbb{F}$ be the set of all ordered pairs $(a, b)$ of real numbers.

(a) If addition and multiplication are defined by

$$(a, b) + (c, d) = (a + c, b + d)$$

and

$$(a, b)(c, d) = (ac, bd),$$

does $\mathbb{F}$ become a field?

(b) If addition and multiplication are defined by

$$(a, b) + (c, d) = (a + c, b + d)$$

and

$$(a, b)(c, d) = (ac - bd, ad + bc),$$

is $\mathbb{F}$ a field then? Does this remind you of anything?

**4.** If $p$ is prime, then $\mathbb{F}_p^n$ is a vector space over $\mathbb{F}_p$. (Remember, $\mathbb{F}_p$ is the set of integers modulo $p$.) How many vectors are there in this vector space?

**5.** Which of the following sets are linearly independent in the vector space $\mathbb{F}_2^3$?

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

**6.** What are the ranks of the following matrices with coefficients in $\mathbb{F}_2$?

(a) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$

**7.** Verify that the collection of rational functions

$$F(X) = \left\{ \frac{f(X)}{g(X)} \mid f(X) \text{ and } g(X) \text{ are polynomials with coefficients in } \mathbb{F},\ g(X) \neq 0 \right\}$$

is a field whenever $\mathbb{F}$ is a field.

**8.** Let $V$ be a complex vector space. Explain how to make $V$ into a real vector space with (almost) no effort. If $V$ has dimension $n$ as a complex vector space, what is the dimension of $V$ as a real vector space?

**9.** What is the dimension of $\mathbb{R}$ as a $\mathbb{Q}$-vector space?

**10.** (Optional) Let $\mathbb{F}$ be a field, and $S$ a subset of $\mathbb{F}$. Under what conditions is $S$ also a field?