Math 16H

Due in class Friday, 10/22.

(Remember, there’s a definition of fields in the footnote on page 347 of Bretscher.)

1. Do the following problems from Bretscher: 7.5 (p. 350), #13-17. Optional: 7.5.37, 3.1.53, 3.1.54.

2. Does the set of all polynomials with integer coefficients form a field? What if the coefficients are allowed to be real numbers?

3. Let \( \mathbb{F} \) be the set of all ordered pairs \((a, b)\) of real numbers.
   (a) If addition and multiplication are defined by
   \[
   (a, b) + (c, d) = (a + c, b + d)
   \]
   and
   \[
   (a, b)(c, d) = (ac, bd),
   \]
   does \( \mathbb{F} \) become a field?
   (b) If addition and multiplication are defined by
   \[
   (a, b) + (c, d) = (a + c, b + d)
   \]
   and
   \[
   (a, b)(c, d) = (ac - bd, ad + bc),
   \]
   is \( \mathbb{F} \) a field then? Does this remind you of anything?

4. If \( p \) is prime, then \( \mathbb{F}_p^n \) is a vector space over \( \mathbb{F}_p \). (Remember, \( \mathbb{F}_p \) is the set of integers modulo \( p \).) How many vectors are there in this vector space?

5. Which of the following sets are linearly independent in the vector space \( \mathbb{F}_2^3 \)? How many are there in this vector space?
   (a) \[
   \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
   \]
   (b) \[
   \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}
   \]
   (c) \[
   \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}
   \]

6. What are the ranks of the following matrices with coefficients in \( \mathbb{F}_2 \)?
   (a) \[
   \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}
   \]
   (b) \[
   \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}
   \]
   (c) \[
   \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}
   \]

7. Verify that the collection of rational functions
   \[
   F(X) = \left\{ \frac{f(X)}{g(X)} \mid f(X) \text{ and } g(X) \text{ are polynomials with coefficients in } \mathbb{F}, g(X) \neq 0 \right\}
   \]
   is a field whenever \( \mathbb{F} \) is a field.

8. Let \( V \) be a complex vector space. Explain how to make \( V \) into a real vector space with (almost) no effort. If \( V \) has dimension \( n \) as a complex vector space, what is the dimension of \( V \) as a real vector space?

9. What is the dimension of \( \mathbb{R} \) as a \( \mathbb{Q} \)-vector space?

10. (Optional) Let \( \mathbb{F} \) be a field, and \( S \) a subset of \( \mathbb{F} \). Under what conditions is \( S \) also a field?