1. Let $B = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$. The reduced echelon form of B is $\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find a basis for the kernel of B. Is the transformation $T(\vec{x}) = B\vec{x}$ one-to-one?

$x_1 + 3x_2 + x_4 = 0 \Rightarrow x_1 = -3x_2 - x_4$

$x_3 + \frac{1}{3}x_4 = 0 \Rightarrow x_3 = -\frac{1}{3}x_4$

so $\ker B = \begin{bmatrix} -3x_2 - x_4 \\ x_2 \\ -\frac{1}{3}x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$, so $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix} \right\}$ is a basis

The kernel is non trivial, so T is not 1-1.

(b) Find a basis for the image of B. Is the transformation $T(\vec{x}) = B\vec{x}$ onto?

The pivot columns are the 1^{st} & 3^{rd}, so the 1^{st} & 3^{rd} columns of B form a basis, $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} \right\}$. The image is two-dimensional, so it can't be all of $\mathbb{R}^3$, so T is not onto

2. Give an example of three linearly dependent vectors in $\mathbb{R}^3$, any two of which are linearly independent.

Pick three vectors lying in the same plane, for example,

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

3. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix}$, and use it to solve the system $A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Row reduce

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 3 & 3 & 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & -3 & 1 & 0 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -1 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{-2}{3} & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & -2 & 3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -2 & 3 \end{array}\right]$$

A^{-1}

$A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, so $A^{-1}(A\vec{x}) = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\vec{x} = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

4. (15 pts) Which of the following are vector spaces? Justify your answers.

(a) The set of all (real) roots of the equation $x^5 - 2x^4 + 3x^2 = 0$.

The only subspaces of $\mathbb{R}$ are $\{0\}$ and $\mathbb{R}$. Clearly, not every real number is a root. Also, since $x^5 - 2x^4 + 3x^2 = x^2(x^3 - 2x^2 + 3)$, there are roots other than 0. So the set is <u>not</u> a vector space.

(b) The set of all 2×3 matrices M such that $M \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

This set is empty: you can't multiply a 2×3 matrix times $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The empty set is <u>not</u> a vector space (every vector space must contain a $\vec{0}$ element).

(c) The set of all differentiable functions that are solutions to the differential equation $y' + y = 0$.

This <u>is</u> a vector space. $f(x) = 0$ is clearly a soln. If f & g are two solns, then

$(f+g)' + (f+g) = f' + g' + f + g = (f'+f) + (g'+g) = 0 + 0 = 0$.

Similarly, $(kf)' + (kf) = kf' + kf = k(f'+f) = k \cdot 0 = 0$

5. (10 pts) Let $\mathbb{P}_2$ be the vector space of all polynomials of degree less than or equal to two. Find a basis for $\mathbb{P}_2$. What is its dimension?

Obvious choice is $\{1, x, x^2\}$. Dimension is 3.

6. Let $T : V \to W$ be a linear transformation. Show that $\operatorname{Im} T$ is a subspace of W..

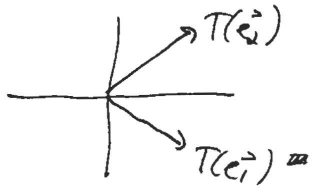
$\underline{\vec{0}}$ $T(\vec{0}) = \vec{0}$ ✓

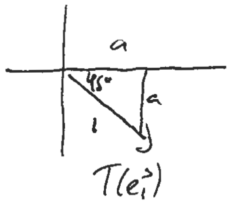
$\underline{+}$ Say $\vec{y_1}$ & $\vec{y_2}$ are in $\operatorname{Im} T$. Then we can write
$\vec{y_1} = T(\vec{x_1})$ & $\vec{y_2} = T(\vec{x_2})$ for some $\vec{x_1}, \vec{x_2} \in V$.
Then $T(\vec{x_1} + \vec{x_2}) = T(\vec{x_1}) + T(\vec{x_2}) = \vec{y_1} + \vec{y_2}$, so $\vec{y_1} + \vec{y_2} \in \operatorname{Im} T$.

$\underline{\text{Scalar mult}}$ Say $\vec{y} \in \operatorname{Im} T$. Then $\vec{y} = T(\vec{x})$ for some $\vec{x} \in V$, and
$T(k\vec{x}) = k\,T(\vec{x}) = k\vec{y}$, so $k\vec{y} \in \operatorname{Im} T$.

7. Find the matrix of the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates everything 45° clockwise around the origin.

$M_T = \left[T(\vec{e_1}) \; T(\vec{e_2}) \right]$!





Triangle is isosceles, & $a^2 + a^2 = 1^2$, $2a^2 = 1$, $a = \frac{1}{\sqrt{2}}$

So $T(\vec{e_1}) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$. Similarly, $T(\vec{e_2}) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

So $M_T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

8. Prove *one* of the following statements. Make sure that I can tell which one you're proving.

(a) Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a linearly independent set of vectors in $\mathbb{R}^m$, and A an invertible $m \times m$ matrix. Show that the set $\{A\vec{v}_1, \ldots, A\vec{v}_n\}$ is also linearly independent.

(b) Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for the vector space V. Show that every vector $\vec{v} \in V$ can be written in the form $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ in exactly one way.

(c) Show that any subset of a linearly independent set is linearly independent.

a) Say $c_1(A\vec{v}_1) + \ldots + c_n(A\vec{v}_n) = \vec{0}$. Then $A^{-1}(c_1(A\vec{v}_1) + \ldots + c_n(A\vec{v}_n)) = A^{-1}(\vec{0})$

$c_1\vec{v}_1 + \ldots + c_n\vec{v}_n = \vec{0}$

So $c_1 = \ldots = c_n = 0$, since $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is lin. ind.

b) Since $\{\vec{v}_1, \ldots, \vec{v}_n\}$ spans, we can write $\vec{v} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$. Say $\vec{v} = d_1\vec{v}_1 + \ldots + d_n\vec{v}_n$ also.

Then $\vec{0} = \vec{v} - \vec{v} = (c_1 - d_1)\vec{v}_1 + \ldots + (c_n - d_n)\vec{v}_n$

Since $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is lin. ind, $c_i - d_i = 0$ for all i, i.e., $c_i = d_i$ for all i.

c) Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be lin. ind., & $\{\vec{v}_{i_1}, \ldots, \vec{v}_{i_m}\}$ a subset.

If $c_{i_1}\vec{v}_{i_1} + \ldots + c_{i_m}\vec{v}_{i_m} = \vec{0}$, then, since ~~the~~ $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is lin. ind,

$c_{i_1} = \cdots = c_{i_m} = 0.$

(EXTRA CREDIT) Let V be a vector space. Give an example of a subset of V that is closed under addition and scalar multiplication, but is *not* a subspace.