1. Let $\mathbb{F}$ be the set of all 2×2 matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where a and b are real numbers. With the usual matrix addition and multiplication, is $\mathbb{F}$ a field? Explain.

$\mathbb{F}$ is a field. We get most of the axioms for free, from the defn. of addition & multiplication, but we need to check that mult. commutes ~~and~~ that we have inverses, & closure under mult.

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \underbrace{\begin{bmatrix} ac-bd & -bc-ad \\ bc+ad & ac-bd \end{bmatrix}}_{\in \mathbb{F}} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Inverses: $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in F$

2. Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be an orthonormal set of vectors. Show that they are linearly independent.

Say $c_1\vec{v}_1 + \ldots + c_n\vec{v}_n = \vec{0}$.

Dot both sides with $\vec{v}_i$:

$$\vec{v}_i \cdot (c_1\vec{v}_1 + \ldots + c_n\vec{v}_n) = \vec{v}_i \cdot \vec{0}$$

$$c_1(\vec{v}_i \cdot \vec{v}_1) + \ldots + c_n(\vec{v}_i \cdot \vec{v}_n) = 0$$

$$c_i = 0 \quad \left(\text{since } \vec{v}_j \cdot \vec{v}_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}\right)$$

So all the c_i's are 0, so $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are lin. ind.

3. Find the equation $y = mx + b$ of the line that best fits the data points $(0,1)$, $(3,4)$, and $(6,5)$.

"Solve" $\begin{aligned} m\cdot 0 + b &= 1 \\ m\cdot 3 + b &= 4 \\ m \cdot 6 + b &= 5 \end{aligned}$, or $\begin{bmatrix} 0 & 1 \\ 3 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$.

Least squares: $\begin{bmatrix} 0 & 3 & 6 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 3 & 6 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} 45 & 9 \\ 9 & 3 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 42 \\ 10 \end{bmatrix}$$

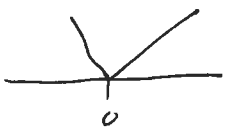
Row reduce $\left[\begin{array}{cc|c} 45 & 9 & 42 \\ 9 & 3 & 10 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1/5 & 14/15 \\ 9 & 3 & 10 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1/5 & 14/15 \\ 0 & 6/5 & 8/5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1/5 & 14/15 \\ 0 & 1 & 4/3 \end{array}\right]$.

$\to \left[\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & 4/3 \end{array}\right]$ $m = 2/3$, $b = 4/3$, or $y = 2/3\, x + 4/3$

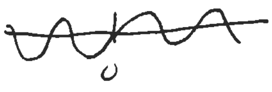
4. Let $C[-\pi,\pi]$ be the space of all continuous functions $f : [-\pi,\pi] \to \mathbb{R}$. Let V be the subspace of $C[-\pi,\pi]$ spanned by the set of functions $\{\sin(t), \sin(2t), \sin(3t), \ldots\}$. Show that the function $f(x) = |x|$ is an element of $V^{\perp}$. (We are using the usual inner product on $C[-\pi,\pi]$.)

$$\langle f, g \rangle = \int_{-\pi}^{\pi} fg \, dx$$

$f(x) = |x|$ is even



& $\sin(kx)$ is odd



So $f(x)\sin(kx)$ is odd. The integral of an odd function over a symmetric interval is 0 (cancellation), so $\langle f, \sin(kx) \rangle = 0$ for every k. Thus $f \in V^{\perp}$.

5. Let A and B be two 3×3 matrices such that $\det A = 7$ and $\det B = 2$. Compute the following, or prove that you don't have enough information:

(a) $\det A^T$

$$= \det A = 7$$

(b) $\det A^{-1}$ $= \dfrac{1}{\det A} = \dfrac{1}{7}$

(c) $\det A + B$ Not enough info. For example, if $A = \begin{bmatrix} \cancel{8}\,7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

then $\det A+B = \cancel{B}$. 9

But if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $\det A + B = \cancel{B}$ 24

(d) $\det AB$

$$= (\det A)(\det B) = 14$$

(e) $\det 5A$ $= 5^3 \det A$ (pull a 5 out of each row)

$= 125 \det A$

(f) $\det A_{\text{diag}}$ (A_{diag} is the matrix whose diagonal entries are identical to those of A, and whose off-diagonal entries are 0).

Not enough info. If $A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $\det A_{\text{diag}} = 7$.

But if $A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, then $\det A_{\text{diag}} = 0$.

6. Let V be the span in $\mathbb{R}^3$ of the vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$.

(a) Find a basis for $V^\perp$.

$\vec{x} \in V^\perp \iff \vec{x} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \vec{x} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 0$, ie, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfies

$\begin{matrix} x_1 + x_2 + x_3 = 0 \\ 2x_1 + 3x_2 + 4x_3 = 0 \end{matrix}$: row reduce $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

So $x_1 = x_3$ & $x_2 = -2x_3$, or $\vec{x} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

So the basis is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

(b) Find the matrix M_T for the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(\vec{x}) = \text{proj}_{V^\perp} \vec{x}$. (If you didn't get part (a), use the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, which is obviously not the right answer.)

The matrix for projection onto a unit vector $\vec{u}$ is $\vec{u}\,\vec{u}^T$.

$\vec{u} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a unit vector that spans $V^\perp$, so

$M_T = \left(\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right) \left(\frac{1}{\sqrt{6}} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \right) = \frac{1}{6} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

(If you used $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ you should get $\frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.)

(c) Use part (b) to find the matrix M_S for the transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ given by $S(\vec{x}) = \text{proj}_V \vec{x}$. (If you didn't get (b), use the matrix $M_T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$.)

Observe that $\vec{x} = \text{proj}_{\vec{v}} \vec{x} + \text{proj}_{V^\perp} \vec{x}$, or $\text{proj}_{\vec{v}} \vec{x} = \vec{x} - \text{proj}_{V^\perp} \vec{x}$,

so $M_S = Id - M_T = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{bmatrix}$

7. Compute the determinant of $\begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$.

$$= 2\det\begin{bmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} = 2\det\begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$$= 2\det\begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix} = 2\det\begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= 2 \cdot 3 \cdot -6 = -36$$

(EXTRA CREDIT) (5 pts) Let $\vec{x}$ and $\vec{y}$ be two vectors in a (real) inner product space. Find $\langle \vec{x}, \vec{y} \rangle$, given that $||\vec{x}+\vec{y}|| = 2$ and $||\vec{x}-\vec{y}|| = 3$.

Observe: $\cancel{\langle \vec{x}, \vec{y} \rangle^2 = \vec{x}, \vec{x}}$

$$||\vec{x}+\vec{y}||^2 = \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle = \langle \vec{x}, \vec{x} \rangle + 2\langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{y} \rangle$$

$$||\vec{x}-\vec{y}||^2 = \langle \vec{x}-\vec{y}, \vec{x}-\vec{y} \rangle = \langle \vec{x}, \vec{x} \rangle - 2\langle x, \vec{y} \rangle + \langle \vec{y}, \vec{y} \rangle$$

So $||\vec{x}+\vec{y}||^2 - ||\vec{x}-\vec{y}||^2 = 4\langle \vec{x}, \vec{y} \rangle$.

So $\langle \vec{x}, \vec{y} \rangle = \frac{1}{4}(4-9) = 5/4$