find interior critical pts., then check the boundary

**Crit pts:** set \( \nabla f = \mathbf{0} \). \( \nabla f = (4, 6) \): never \( \mathbf{0} \), so no crit. pts.

**Boundary:** Use Lagrange multipliers. The constraint is \( g(x, y) = x^4 + y^2 = 1 \).

\( \nabla g = (2x, 2y) \), which is \( \mathbf{0} \) only at \((0, 0)\), which is not on the curve \( g = 1 \). So the max & min occur at a pt.

Satisf g \( \nabla f = \lambda \nabla g \)

\[
(4, 6) = \lambda (2x, 2y)
\]

Get 3 eqns:

\[
\begin{align*}
4 &= 2\lambda x \\
6 &= 2\lambda y \\
x^4 + y^2 &= 1
\end{align*}
\]

\( \lambda = \frac{x}{2} \)

\( \lambda = \frac{3}{2y} \)

So \( \frac{x}{2} = \frac{3}{2y} \), or \( dy = \frac{3}{2} x \), or \( y = \frac{3}{2} x \)

Then \( x^4 + \left(\frac{3}{2} x\right)^2 = 1 \)

\( x^4 + \frac{9}{4} x^4 = 1 \)

\( \frac{13}{4} x^4 = 1 \)

\( x^4 = \frac{4}{13} \), \( x = \pm \frac{2}{\sqrt{13}} \)

So the possible extreme are: \( \left( \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right) \) & \( \left( \frac{-2}{\sqrt{13}}, \frac{-3}{\sqrt{13}} \right) \)

max value: \( 4 \cdot \frac{2}{\sqrt{13}} + 6 \cdot \frac{3}{\sqrt{13}} = \frac{26}{\sqrt{13}} \)

min value: \( 4 \left( \frac{-2}{\sqrt{13}} \right) + 6 \left( \frac{-3}{\sqrt{13}} \right) = \frac{-26}{\sqrt{13}} \)
2. Evaluate $\iint_R y \, dA$, where $R$ is the region bounded by the line $y = x - 1$ and the parabola $y^2 - 2x + 6$.

\[ y = 2x + 6, \quad \text{or} \quad x = \frac{1}{2} y - 3 \]
\[ y = x - 1, \quad \text{or} \quad x = y + 1 \]

At pts. of intersection, \( \frac{1}{2} y^2 - 3 = y + 1 \)
\[ y^2 - 6 = dy + 2 \]
\[ y^2 - dy - 8 = 0 \]
\[ y = 4, -2 \]

So pts. of intersection are \((-2, -1) \& (5, 4)\)

Easiest as a Type II region: \( \frac{1}{2} y^2 - 3 \leq x \leq y + 1 \)
\[-2 \leq y \leq 4 \]

\[
\iint_R y \, dA = \int_{-2}^{4} \int_{\frac{1}{2} y^2 - 3}^{y + 1} y \, dx \, dy
\]
\[
= \int_{-2}^{4} \left( xy \bigg|_{\frac{1}{2} y^2 - 3}^{y + 1} \right) \, dy
= \int_{-2}^{4} \left( y^2 + y - \frac{1}{2} y^3 + 3y \right) \, dy
= \int_{-2}^{4} \left( -\frac{1}{2} y^3 + y^2 + 3y \right) \, dy
= \left. -\frac{1}{8} y^4 - \frac{1}{3} y^3 + \frac{3}{2} y^2 \right|_{-2}^{4}
= -32 + \frac{64}{3} + \frac{16}{3} + 2 + \frac{8}{3} - 8
= 18
\]
3. Set up, but do not solve, the following problem:
Find the parabola $ax^2 + bx + c$ that best fits the data points (1, 0), (2, 0), (4, 10), and (-1, -3).
(That is, write down the function to be optimized, but don’t optimize it.)

\[
\begin{align*}
\text{Minimize } & \quad d_1^4 + d_2^4 + d_3^4 + d_4^4 = (8 + c + 3)^2 + (8 + c + 5)^2 + (16 + 4c + 4)^2 + (16 + 4c + 10)^2 \\
& = f(\alpha, \beta, c)
\end{align*}
\]

4. Find the region $R$ for which the triple integral
\[
\iiint_R (1 - x^2 - y^2 - z^2) \, dV
\]
is a maximum.

Want $R$ to include all the pts where $1 - x^2 - y^2 - z^2 \geq 0$,
none of the pts, where it’s negative:
\[1 - x^2 - y^2 - z^2 < 0 \iff x^2 + y^2 + z^2 < 1\]

Thus $R$ is the solid ball of radius 1.
5. Set up, but do not evaluate, an iterated integral giving the average value of the function \( f(x, y, z) = x^2z + y^2z \) over the region enclosed by the paraboloid \( z = 1 - x^2 - y^2 \) and the plane \( z = 0 \). 

![3D diagram](image)

In rectangular coordinates, \( \iiint_R f(x, y, z) \, dV \) is

\[
\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^1 \left( x^2z + y^2z \right) \, dz \, dy \, dx \\
\text{or} \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^1 \left( x^2z + y^2z \right) \, dz \, dx \, dy
\]

Better in cylindrical coordinates:

\[
\int_0^1 \int_{-\pi}^{\pi} \int_0^1 r^2 \, r \, dr \, d\theta \, dz
\]

\[
\text{or} \int_0^1 \int_{-\pi}^{\pi} \int_0^1 r^2 \, r \, d\theta \, dr \, dz
\]

So the average value of \( f \) over \( R \) is \( \frac{\iiint_R f \, dV}{\iiint_R dV} \), where we evaluate the triple integral as above.

**EXTRA CREDIT** Draw the monkey in his saddle.