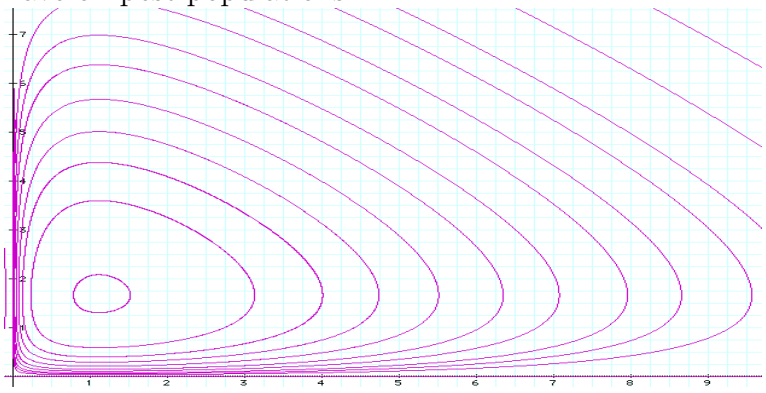
(1) (20 pts) Pesticides that kill an insect species not only are bad for the environment, but they can also be inefficient at controlling pest species. Suppose that a pest insect species in a particular field has population $R(t)$ at time t, and suppose that its primary predator is another insect species with population $F(t)$ at time t. Suppose that the populations of these species are accurately modeled by the system

$$\begin{aligned} R' &= 2R - 1.2RF \\ F' &= -F + 0.9RF \end{aligned}$$

Finally, suppose that at time $t = 0$ a pesticide is applied to the field, reducing both the pest and predator populations to very small but nonzero numbers.

(a) Using the figure below, which shows the phase portrait for this system, predict what will happen to the population of the pest species as t increases.

(b) Senator Haddock needs your help again. Write a paragraph, in nontechnical language, explaining to him the possibility of the paradoxical effect that pesticide application can have on pest populations.



(2) (15 pts) Find the general solution of $\mathbf{x}' = \begin{pmatrix} -8 & -1 \\ 16 & 0 \end{pmatrix} \mathbf{x}$.

(3) (10 pts) Consider the system of ODEs $\mathbf{x}' = \mathbf{P}\mathbf{x}$. Let $\mathbf{x}_1(t)$ be the solution satisfying the initial condition $\mathbf{x}_1(7) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{x}_2(t)$ the solution satisfying $\mathbf{x}_2(7) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$. Do you have enough information to find a solution $\mathbf{x}(t)$ satisfying $\mathbf{x}(7) = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$? If so, what is it? If not, why not?

(4) (10 pts) Consider the initial value problem

$$y'' + ay' + by = \sin t,\ y'(0) = 0,\ y(0) = 0.$$

You are given that $f(t)$ is the solution of the initial value problem

$$y'' + ay' + by = \delta_0(t),\ y'(0) = 0,\ y(0) = 0,$$

where δ_0 is the Dirac delta "function." What is the solution of the original problem?

(5) (10 pts) Consider the two systems of differential equations

(a) $\begin{aligned} x' &= 0.3x - 0.1xy \\ y' &= -0.1y + 2xy \end{aligned}$ (b) $\begin{aligned} x' &= 0.3x - 3xy \\ y' &= -2y + 0.1xy \end{aligned}$

One of these systems refers to a predator-prey system with very lethargic predators – predators who seldom catch prey but who can live for a long time on a single prey (for example, boa constrictors). The other system refers to a very active predator that requires many prey to stay healthy (such as a small cat). The prey in each case is the same. Identify which system is which and justify your answer.

(6) (10 pts) Consider the initial value problem

$$e^t y''' + (\ln t)y' + (\sin t)y = 0,\ y'(1) = 0,\ y(1) = 7.$$

For what values of t can you be sure that the solution exists? Why?

(7) (10 pts)

(a) Rewrite $\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$ as a series whose generic term involves x^n rather than x^{n-2}.

(b) Assume that the solution to $y'' - xy = 0$ is

$$a_0 + a_1 x + \frac{a_0}{2 \cdot 3} x^3 + \frac{a_1}{3 \cdot 4} x^4 + \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 + \dots$$

Find two independent solutions to this ODE and explain how you know that they're independent.

(8) (15 pts) Find the general solution of the ODE $y^{(4)} - y = 30e^{2t}$.

EXTRA CREDIT (2 pts) Which is more fun, an nth-order ODE or a system of n first-order ODEs?