1. Suppose that the rhino population P shows seasonal growth modeled by the ODE $\frac{dP}{dt} = kP\cos(at)$, where k and a are positive constants. (The cosine factor suggest periodic fluctuations.) Solve for $P(t)$. What is the long-term behavior?

This is separable: $\frac{dP}{dt} = kP\cos(at)$

$$\frac{dP}{P} = k\cos(at)\,dt$$

$$\ln|P| = \frac{k}{a}\sin(at) + C$$

$$|P| = e^{C}e^{\frac{k}{a}\sin(at)}$$

Since P is popn, $P \geq 0$, so $P = De^{\frac{k}{a}\sin(at)}$, where $D \geq 0$.

Thus P oscillates between $De^{-\frac{k}{a}}$ and $De^{\frac{k}{a}}$,

where $D = P(0)$, the initial popn.

2. Solve the following ODEs:

(a) $6ty - y^3 + (4y + 3t^2 - 3ty^2)y' = 0$

$\frac{d}{dy}(6ty - y^3) = 6t - 3y^2 = \frac{d}{dt}(4y + 3t^2 - 3ty^2)$, so the eqn. is separable:

$$(2y^2 + 3t^2y - ty^3)' = 0,$$

or $2y^2 + 3t^2y - ty^3 = C$

(implicit soln)

(b) $8e^{t/3}y' - 8e^{t/3}y - 11 = 0, \ y(0) = -1$

This is linear: rewrite as $y' - y = \frac{11}{8}e^{-t/3}$

The integrating factor is $\mu(t) = e^{-t}$, so ~~write~~ write

$$e^{-t}y' - e^{-t}y = \frac{11}{8}e^{-4t/3}$$

$$(e^{-t}y)' = \frac{11}{8}e^{-4t/3}$$

$$e^{-t}y = \frac{-33}{32}e^{-4t/3} + C$$

$$y = \frac{-33}{32}e^{-t/3} + Ce^{t}$$

$y(0) = -1$, so $-1 = \frac{-33}{32} + C$, $C = \frac{1}{32}$.

so $y = \frac{-33}{32}e^{-t/3} + \frac{1}{32}e^{t}$

3. Find the general solution of the equation
$$y'' - 3y' + 2y = 3e^{-t}.$$

First, solve the assoc. homogeneous:

$$y'' - 3y' + 2y = 0$$

The char. poly. is $r^2 - 3r + 2 = (r-1)(r-2)$, so the soln is

$$c_1 e^{t} + c_2 e^{2t}.$$

Next, find a particular soln y_p. We guess $y_p = ae^{-t}$, & plug in: $(ae^{-t})'' - 3(ae^{-t})' + 2(ae^{-t}) = 3e^{-t}$

$$ae^{-t} + 3ae^{-t} + 2ae^{-t} = 3e^{-t}$$

$$6ae^{-t} = 3e^{-t}$$

$$a = \tfrac{1}{2}$$

$$\text{So } y_p = \tfrac{1}{2}e^{-t}.$$

The general soln is $c_1 e^{t} + c_2 e^{2t} + \frac{1}{2}e^{-t}$.

4. We wish to model an ecological system with two species, predators and prey. The model is

$$
\begin{aligned}
x' &= 7x - x^2 - xy \\
y' &= -5y + xy
\end{aligned}
$$

(units are thousands of individuals).

(a) Which term (x or y) represents the prey population? Which is the predators? Explain.

x is prey, because interaction (measured by xy) is bad for them – they get eaten. y is the predators, because interaction is good for them.

(b) What would happen to the x population in the absence of any y? Why does this make sense ecologically?

If $y=0$, then $x' = 7x - x^2 = x(7-x)$. This is logistic growth: if x is small, it grows to the carrying capacity of the environment ($x=7$). If x is large, the environment can't sustain the popn. and it shrinks to the carrying capacity.

(c) What would happen to the y population in the absence of any x? Why does this make sense ecologically?

If $x=0$, $y' = -5y$. This is negative exponential growth – the popn. drops to 0, which makes sense because they have nothing to eat.

(d) Is it possible for both species to coexist? If so, is coexistence a likely outcome? Explain.

Find the equilibria:

$x' = 0 = 7x - x^2 - xy$ $\qquad$ $y' = 0 = -5y + xy$

So $\underline{x=0}$ or $\underline{7-x-y=0}$ $\qquad$ $\underline{y=0}$ or $\underline{x=5}$

So $(5,2)$ is an equilibrium – it is possible for them to coexist.

To see if it's likely, classify the eq. point.

$$DF = \begin{bmatrix} 7-2x-y & -x \\ y & -5+x \end{bmatrix}, \text{ so } DF(5,2) = \begin{bmatrix} -5 & -5 \\ 2 & 0 \end{bmatrix}$$

The eigenvalues are $\dfrac{-5 \pm i\sqrt{15}}{2}$, so it is a spiral sink.

So coexistence $\underline{\text{is}}$ a likely outcome: if we start near the equilibrium pt, we get closer and closer to it.

5. A tank contains 1000 L of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of 5 L/sec, and the mixture – kept uniform by stirring – is pumped out at the same rate. How long will it be until only 10 kg of salt remains in the tank?

Let $y(t)$ = amt. of salt in tank at time t.

$$\frac{dy}{dt} = (\text{rate coming in}) - (\text{rate going out})$$

$$= 0 - \frac{5}{1000}y = \frac{-1}{200}y$$

So $\frac{dy}{dt} = \frac{-1}{200}y$.

The soln is $y(t) = Ce^{\frac{-1}{200}t}$.

Since $y(0) = 100$, $C = 100$.

Thus $y(t) = 100e^{-\frac{1}{200}t}$.

At time t_0, $y(t_0) = 10$

$$100e^{-\frac{1}{200}t_0} = 10$$

$$e^{-\frac{1}{200}t_0} = \frac{1}{10}$$

$$-\frac{1}{200}t_0 = -\ln 10$$

$$t_0 = 200 \ln 10$$

6. Use Laplace transform methods to solve the following ODEs, or explain why it can't be done.

(a) $y' + y = 3e^{2t}, \; y(0) = 0$

$$\mathcal{L}[y' + y] = \mathcal{L}[3e^{2t}]$$

$$sY(s) - y(0) + Y(s) = \frac{3}{s-2}$$

$$(s+1)\,Y(s) = \frac{3}{s-2}$$

$$Y(s) = \frac{3}{(s+1)(s-2)} = \frac{-1}{s+1} + \frac{1}{s-2}$$

so $y(t) = -e^{-t} + e^{2t}$

(b) $2y'' - y' + 4y = e^{t^2}, \; y(0) = -1, \; y'(0) = 0$

$$\mathcal{L}[e^{t^2}] = \int_0^\infty e^{-st} e^{t^2}\,dt = \int_0^\infty e^{t(t-s)}\,dt$$

$e^{t(t-s)} \to \infty$ as $t \to \infty$ for any s, so the integral blows up, and $\mathcal{L}[e^{t^2}]$ is undefined.

(c) $\cancel{y'' - y' = \sin t}$, $y(0) = 2, \; y'(0) = -3$

$y'' - \sin(y') = \sin t$

$$\mathcal{L}[\sin(y')] = \int_0^\infty e^{-st} \sin(y')\,dt = ?$$

Don't know how to write $\mathcal{L}[\sin(y')]$ in terms of $Y(s)$.