1. Solve the IVP

\[(2y + t)y' + y = 0, \quad y(-5) = \frac{9}{2}.\]

On what \(t\) interval does this solution exist?

Since \(\frac{d}{dt} \left(dy + t\right) = 1 = \frac{d}{dt}(y)\), the eqn. is \(\mathbb{R} \times t\):

\[
\begin{align*}
(2y + t)y' + y &= 0 \\
(y' + ty)' &= 0 \\
y' + ty &= C
\end{align*}
\]

Plug in \(y(-5) = \frac{9}{2}\):

\[
\begin{align*}
\frac{81}{4} - \frac{75y}{2} &= C, \\
\frac{-9}{4} &= C
\end{align*}
\]

\[
y = -t + \frac{\sqrt{t^2 - 9}}{2}
\]

Since \(y(-5) = \frac{9}{2}\), \(y = -t + \frac{\sqrt{t^2 - 9}}{2}\)

The soln. exists for \(-\infty < t < -3\).
2. The tank holding your dorm’s drinking water has a capacity of 100 gallons and is half full of fresh water. A pipe is opened that lets raw sewage enter the tank at the rate of 4 gallons per minute. At the same time, a drain is opened to allow the mixture to leave the tank at the rate of 2 gallons per minute. If the sewage contains 10 grams of potassium per gallon, what is the concentration of potassium in the tank the instant before it overflows?

The amount of water in the tank at time $t$ is $50 + 2t$ ($0 \leq t < 15$).

If $p(t)$ is the amount of potassium in the tank at time $t$, then we have

$$p' = 40 - \frac{dp}{ds} = 40 - \frac{p}{s+e},$$

or

$$p' + \frac{p}{s+e} = 40.$$ This is linear, so we look for an integrating factor $\mu(t)$.

We have $\mu p' + \frac{\mu p}{s+e} = (\mu p)'$, or $\mu' = \frac{\mu}{s+e}$, or

$$\frac{d\mu}{\mu} = \frac{1}{s+e},$$

or $\ln |\mu| = \ln |s+e| + C$

$$\mu = e^{C(s+e)}$$ simplest: $\mu = s+e$

So $(s+e)p' + p = (s+e)40$, or

$$(s+e)p' = 1000 + 40t,$$ so

$$(s+e)p = 1000t + 40t^2 + C.$$ Since $p(0) = 0$, $s \cdot 0 = C$, or $C = 0$.

So

$$(s+e)p = 1000t + 40t^2,$$

or

$$p = \frac{1000t + 40t^2}{s+e}$$

The tank overflows for $t > ds$, so we want

$$p(ds) = \frac{1000 \cdot ds + 40 \cdot ds^2}{s+e} = \frac{1000 + 40 \cdot ds}{s+e} = 500 + 10 \cdot ds$$

$$= 750.$$
3. Many chemical reactions can be viewed as interactions between two molecules that undergo a change and result in a new product. The rate of a reaction, therefore, depends on the number of interactions or collisions, which in turn depends on the concentrations of both types of molecules. Consider the simple reaction \( A + B = Y \), in which one molecule of substance \( A \) combines with one molecule of substance \( B \) to create one molecule of substance \( Y \).

Let's designate the initial number of molecules of \( A \) and \( B \) by constants \( \alpha \) and \( \beta \), respectively. Let \( y(t) \) denote the number of molecules of \( Y \) at time \( t \). Which of the following is the most reasonable model for \( y(t) \)? Explain. (Here, \( k \) is a positive constant.)

(a) \( y' = k\alpha\beta \)
(b) \( y' = ky \)
(c) \( y' = k(\alpha - y)(\beta - y) \)
(d) \( y' = k[\alpha - y + (\beta - y)] \)
(e) \( y' = k\alpha\beta y \)
(f) \( y' = k\alpha\beta y(1 - y) \)

(c) \( y' \) is jointly proportional to the ant. of \( A \) and the ant. of \( B \), since the same is true of the number of collisions between atoms of \( A \) and atoms of \( B \). Since each molecule of \( Y \) takes one of \( A \) and one of \( B \), the ant. of \( A \) at time \( t \) is \( \alpha - y(t) \), and similarly for \( B \).

4. If a solution of the ODE

\[ e^t y'' - t^2 y' + (\cos t^2) y = 0 \]

is tangent to the \( t \)-axis at any point, then it must be identically zero. Why?

We can rewrite the ODE as

\[ e^t y'' - \frac{t^2}{e^t} y' + \frac{\cos t^2}{e^t} y = 0 \]

The coefficients are continuous for all \( t \), so solutions exist and are unique for all \( t \).

If the soln. \( y \) is tangent to the \( t \)-axis at \( t = t_0 \), then \( y(t_0) = y'(t_0) = 0 \), and the unique soln. satisfying these initial conditions is \( y \equiv 0 \).
5. Find the general solution of the ODE
\[ y'' + 8y' + 25y = e^t. \]

What's the long-term behavior?

First, solve the associated homogeneous ODE
\[ y'' + 8y' + 25y = 0. \] The char. poly is \( r^2 + 8r + 25 \), with roots \( r = \frac{-8 \pm \sqrt{64 - 100}}{2} = \frac{-8 \pm \sqrt{-36}}{2} = -4 \pm 3i \).

So the gen. soln. is \( c e^{-4t} \cos 3t + d e^{-4t} \sin 3t \).

Next, find a particular soln. \( y_p \). Guess \( y_p = Ae^t \).

\[ Ae^t + 8Ae^t + 25 Ae^t = e^t \]
\[ 34Ae^t = e^t \]
\[ A = \frac{1}{34}, \quad y_p = \frac{1}{34} e^t. \]

So the gen. soln. is \( c e^{-4t} \cos 3t + d e^{-4t} \sin 3t + \frac{1}{34} e^t \).