(1) (10 points) (This question is for section 2 (with linear algebra) only. If you’re in section 1, you’ve got the wrong version of the exam.) True or false: When numerically integrating a vector field using Runge-Kutta, we can always get better accuracy by decreasing the step size (i.e., by making $\Delta t$ smaller). Explain.

(2) (10 points) Find the general solution of the ODE $y'' - 3y' + 2y = 3e^t$. 
(3) (15 points) Solve the ODE \( \frac{dy}{dt} = ry - ky^2 \), \( r > 0 \) and \( k > 0 \). (HINT: Make the substitution \( v = 1/y \).)

(4) (15 points) Find the general solution of the system \( \mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \mathbf{x} \).
(5) (20 points) Suppose two similar countries $Y$ and $Z$ are engaged in an arms race. Let $y(t)$ and $z(t)$ denote the size of the stockpiles of arms of $Y$ and $Z$, respectively. We model this situation with the system of differential equations
\[
\begin{align*}
y' &= h(y, z) \\
z' &= k(y, z)
\end{align*}
\]
Suppose that all we know about the functions $h$ and $k$ are the two assumptions:
(i) If country $Z$’s stockpile of arms is not changing, then any increase in the size of $Y$’s stockpile of arms results in a decrease in the rate of arms building in country $Y$. Similarly, if country $Y$’s stockpile of arms is not changing, then any increase in the size of $Z$’s stockpile of arms results in a decrease in the rate of arms building in country $Z$.
(ii) If either country increases its stockpile, the other responds by increasing its rate of arms production.

(a) What do the assumptions imply about $\partial h/\partial y$ and $\partial k/\partial z$?

(b) What do the assumptions imply about $\partial h/\partial z$ and $\partial k/\partial y$?

(c) What types of equilibrium points are possible for this system? Justify your answer.
(6) (15 points) Consider the system
\[\begin{align*}
x' &= x(x - 1) \\
y' &= x^2 - y.
\end{align*}\]
Sketch the $x$- and $y$-nullclines. Then find all equilibrium points. Using the direction of the vector field between the nullclines, describe the possible behavior of the solution corresponding to the initial condition $x(0) = -0.5, y(0) = 2$. (N.B.: Look at the whole plane, not just the first quadrant.)
(7) (10 points) Consider the initial value problem $y'' + y = 0$, $y(0) = y(a) = 0$. For what values of $a$ (if any) will there be more than one solution to the IVP?

(8) (15 points) Let $y(t)$ be the population of fish in a certain lake at time $t$. Assume that the population is governed by the ODE $y' = y^2 + b$ (where $b$ is a constant) and that initially there are 1000 fish in the lake. What’s the long-term behavior of the fish population? (HINT: Your answer will be different for different values of $b$.)
(9) (20 points) Solve the following ODEs:

(a) \( y' = (ty)^2 \). (Find the general solution.)

(b) \( x + yy' = 0, \ y(0) = -2 \). For what \( x \) values does the solution exist?
(10) (15 points) Suppose that the solution $y(t)$ of the IVP $y'' + py' + qy = \delta(t)$, $y(0) = y'(0) = 0$ (where $p$ and $q$ are constants) has a Laplace transform $\mathcal{L}\{y(t)\}$ whose value at $s = 0$ is $1/5$ and whose value at $s = 2$ is $1/17$.

(a) Find $p$ and $q$.

(b) Find $y(t)$. (If you couldn’t solve part (a), use the values $p = 2$ and $q = 5$.)
(11) (15 points) Find the first four nonzero terms of the series solution of Airy’s equation, $y'' = xy$, with initial conditions $y(0) = 1$, $y'(0) = 0$. 
(12) (20 points) The figures below show solutions in the $ty$-plane of equations of the form $y' = F(t, y)$.

(a) Which systems, if any, have some solutions which are not unique?

(b) Which systems, if any, are autonomous?
(13) (20 points) A 30-gallon tank initially contains 15 gallons of saltwater containing 6 pounds of salt. Suppose saltwater containing 1 pound of salt per gallon is pumped into the top of the tank at the rate of 2 gallons per minute, while a well-mixed solution leaves the bottom of the tank at the rate of 1 gallon per minute. How much salt is in the tank when the tank is full?

EXTRA CREDIT (5 points) True or false: Classification of mathematical problems as linear and nonlinear is like classification of the universe as bananas and non-bananas.