1. (20 pts) The Taylor series centered at 0 for the function $f(x) = \dfrac{1}{(1-x)^2}$ is $1+2x+3x^2+4x^3+\ldots$.

(a) What is the radius of convergence for this series?

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n+2} \right| = 1$$

(b) What is the interval of convergence for this series?

Check endpts: $x=1$: $1+2+3+4 \ldots$ diverges

$x=-1$: $1-2+3-4 \ldots$ diverges

So int. of conv. is $(-1, 1)$.

(c) Explain why you can't use this series to approximate $\dfrac{1}{36} = f(-5)$.

If we plug in $x=-5$, we get

$1 - 10 + 75 - 500 + \ldots$ - diverges

The more terms we add, the worse it gets.

(20 pts) Find the first three nonzero terms of the power series solution $\sum c_n x^n$ to the differential equation $y' + y = x$ with initial condition $y(0) = 1$.

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

$$1 = y(0) = c_0 + c_1 \cdot 0 + c_2 0^2 + \dots, \text{ so } c_0 = 1.$$

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + \dots$$

So: $\quad y' + y = x$

$$(c_1 + 2c_2 x + 3c_3 x^2 + \dots) + (\cancel{c_0}1 + c_1 x + c_2 x^2 + \dots) = x$$

$$(1 + c_1) + (2c_2 + c_1)x + (3c_3 + c_2)x^2 + \dots = x$$

Match up coefficients:

constant $\quad 1 + c_1 = 0$, so $c_1 = -1$

$x \quad 2c_2 + c_1 = 1$, so $2c_2 - 1 = 1$, $2c_2 = 2$, $c_2 = 1$

& we have the 1st three nonzero terms:

$$y = c_0 + c_1 x + c_2 x^2 + \dots$$

$$= 1 - x + x^2$$

3. (20 pts) Show that $\cos x$ is equal to its Maclaurin series $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$. (In other words, show that at any point x_0, the error $E_n(x_0) = |\cos x_0 - P_n(x_0)|$ associated with the nth degree Taylor polynomial goes to 0 as n goes to infinity.)

$E_n(x_0) \leq \frac{M x_0^{n+1}}{(n+1)!}$, where M is an upper bound on the $(n+1)$st derivative of $\cos x$ on the interval from 0 to x_0.

Every derivative of $\cos x$ is $\pm \sin x$ or $\pm \cos x$, all of which are bdd. by 1, so we can use $M=1$.

Then $E_n(x_0) \leq \frac{x_0^{n+1}}{(n+1)!}$. This goes to 0 as $n \to \infty$ regardless of what x_0 is.

4. (20 pts) Determine whether the following series converge or diverge. Be sure to explain your reasoning.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

Root test: $\displaystyle\lim_{n\to\infty} \left(\frac{1}{(\ln n)^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0$

so <u>converges</u>

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n^2}{n^{3/2}}$

$\displaystyle\sum \left| \frac{\sin n^2}{n^{3/2}} \right| \leq \sum \frac{1}{n^{3/2}} \leftarrow$ convergent p-series $(p = 3/2 > 1)$,

so <u>converges</u> (absolutely)

(c) $\displaystyle\frac{\ln 3}{3} - \frac{\ln 4}{4} + \frac{\ln 5}{5} - \frac{\ln 6}{6} \cdots$

Alternating series test – <u>converges</u>

(d) $\displaystyle\sum_{n=11}^{\infty} \frac{n2^n + n}{n^{3/2} - \ln n}$

n^{th} term goes to infinity – <u>diverges</u>

(20 pts) Decide whether each of the following statements is true or false. If one is false, explain why or give an example showing that it's false. (If it's true, you don't have to say anything else.)

(a) If $\sum a_n^2$ converges, then so does $\sum a_n$ (provided that each a_n is positive).

False - Ex: $a_n = \frac{1}{n}$. $\sum a_n^2 = \sum \frac{1}{n^2}$ converges, but

$\sum a_n = \sum \frac{1}{n}$ diverges.

(b) The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges conditionally.

True: It converges, but $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ diverges.

(c) The sum of the series $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$ is $\frac{2}{3}$.

False. The sum is $\frac{1}{2} \cdot \frac{1}{1-(\frac{-1}{2})} = \frac{1}{2} \cdot \frac{1}{\frac{3}{2}} = \frac{1}{2} \cdot \frac{2}{3} = 1/3$

(d) If the series $\sum c_n(x-4)^n$ diverges when $x = 5$, then it also diverges when $x = -\frac{7}{2}$.

True - radius of conv. is at most $5-4=1$. Since $-\frac{7}{2}$ is more than 1 away from 4, the series must diverge there.