Math 81 **FINAL EXAM** **May, 2004**

Due in my office 24 hours after you download it, and by 9:00 am, Sunday, May 16, at the latest. You may use only your text, notes, and old homework (in particular, no other books, calculators, or Mathematica). You may not talk to anyone about it. I will not answer any questions other than requests for clarification. Make sure that your answers are clear and legible. **Be sure to do both pages!**

1. Fourier tells us that we can write most functions of a variable x as an infinite sum of sines and cosines. But Taylor tells us that we can write most functions of x as an infinite sum of powers of x (that is, as a Taylor series, $a_0 + a_1x + a_2x^2 + \dots$, where $a_n = f^{(n)}(0)/n!$). **In one page or less**, explain why Fourier series are much more useful for solving, say, the heat equation on a bounded interval than are Taylor series.

2. (Minimal surfaces) Let Γ be a closed curve in space. We wish to find the surface with the smallest surface area among all smooth surfaces having Γ as their boundary. Assume that we can write this minimizing surface in the form $z = u(x, y)$.

(a) Show that u must satisfy the PDE

$$(1 + u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1 + u_x^2)u_{yy} = 0. \tag{1}$$

(HINT: The surface area is $\displaystyle\iint \sqrt{1 + u_x^2 + u_y^2}\ dx\ dy$.)

(b) Somewhat perversely, any surface that satisfies equation (1) is said to be a *minimal surface*, whether it actually minimizes area or not. (The problem of determining whether a given minimal surface minimizes area leads to another, even harder PDE.) As you might imagine, solving equation (1) is difficult, and finding examples of minimal surfaces has been an active area of research for the last 240 years. Clearly, a flat planar surface is minimal. One of the first more complicated minimal surfaces to be discovered was the helicoid, given by the equation $z = \arctan(y/x)$. Prove that the helicoid is in fact a minimal surface.

3. Solve the problem

$$tu_x + u_t = 1,\ \ u(x, 0) = e^{-x^2/2}.$$

In what region of the xt-plane is the solution uniquely determined?

4. Consider a piano string of length L. We can model its displacement $u(x, t)$ mathematically by

$$u_{tt} = c^2u_{xx},\ \ u(0, t) = u(L, t) = 0.$$

Sound is produced by striking the string with a hammer at a distance d from the near end of the string (i.e., at the point $x = d$). If we pretend that the hammer is infinitely narrow and that it contacts the string for only an instant, then the Dirac delta function is a good model for the hammer blow, and our initial conditions are

$$u(x, 0) = 0,\ \ u_t(x, 0) = \delta(x - d).$$

(a) Find the solution when $d = L/7$.

(b) It's roughly true that the human ear carries frequencies separated by more than a factor of 1.18 on separate neural paths, and those separated by less on the same paths. This gives rise to the sensation of dissonance in tones with frequency ratios less than a minor third (6:5). Given this, what about your solution to part (a) suggests that $d = L/7$ would be an especially good place for the hammer to hit the string?

(c) In fact, piano hammers generally strike near $L/7$ on any given string, but not exactly there, and the placement varies from string to string. What are some factors that could account for the discrepancy between your theoretical solution in part (b) and actual practice?

5. Let $u(r, \theta)$ be harmonic in the region D given by $1 < r < 2$, $0 < \theta < \pi/2$. Solve for u, given the following boundary conditions:

$$u(r, 0) = u(r, \pi/2) = 0, \; u(1, \theta) = f(\theta), \; u(2, \theta) = g(\theta).$$