Math 81 **SECOND MIDTERM** **April 15, 2004**

Due in my office 24 hours after you download it, and by the beginning of class on Thursday, April 15, at the latest. You may use only your text, notes, and old homework (in particular, no other books, calculators, or Mathematica). You may not talk to anyone about it. I will not answer any questions other than requests for clarification. Make sure that your answers are clear and legible.

1. Let $f(x)$ be a function defined on the interval $[0, \pi]$ such that $\int_0^\pi |f(x)|^2\,dx = 7$. Let $\sum_{n=1}^\infty A_n \sin nx$ be its Fourier sine series. Show that $\lim_{n\to\infty} A_n = 0$.

2. The electric potential $v(x, y)$ in the space $0 \le x \le 1$, $y \ge 0$, satisfies the differential equation $\Delta v = 0$. Find the formula for v if the planes $x = 0$ and $x = 1$ are kept at zero potential and the plane $y = 0$ at the potential $f(x)$, if v is to be bounded as y goes to infinity.

3. A function $T(r, \theta)$ satisfying $\Delta T = 0$ is bounded within the circle $r = 4$ and has values

$$T(4, \theta) = \sin\theta - 3\cos\theta + 5\sin 4\theta.$$

Find T.

4. Let $\phi(x) = 2 - \dfrac{x}{5}$ for $0 \le x \le 10$, and let $\hat{\phi}(x)$ be its Fourier sine series on the interval $[0, 10]$.

(a) Does $\hat{\phi}$ converge to ϕ uniformly on $(0, 10)$? Pointwise? In L^2?

(b) For each x ($0 \le x \le 10$), what is the sum of the series $\hat{\phi}(x)$?

5. Consider the problem of heat conduction in a uniform rectangular plate with no internal source, perfect lateral insulation, a prescribed temperature of 0° on the sides $x = 0$ and $x = L$, and perfect insulation on the sides $y = 0$ and $y = H$. This problem is modeled mathematically as $u_t = k(u_{xx} + u_{yy})$, subject to the boundary conditions

$$u(0, y, t) = u(L, y, t) = u_y(x, 0, t) = u_y(x, H, t) = 0$$

and the initial condition $u(x, y, 0) = f(x, y)$. Find a formula for the general solution. (Be sure to explain how to find any constants that may appear in your solution.)

6. Suppose that $u(x, y)$ is harmonic on the unit disk $x^2 + y^2 < 1$, and also satisfies the boundary conditions $u(x, y) = 10$ on the part of the boundary of the disk that lies in the first quadrant (i.e., in polar coordinates, the set $r = 1$, $0 \le \theta \le \pi/2$), and $u(x, y) = 0$ on the rest of the boundary of the disk (i.e., $r = 1$, $\pi/2 < \theta < 2\pi$). What is $u(0, 0)$?

7. Give an example of a bounded set D in the plane and a function $u(x, y)$ that is harmonic in the interior of D and continuous on $\overline{D} = D \cup (\text{bdy } D)$, such that the maximum value of u is attained in the interior of D, but u is *not* a constant function. Why doesn't your example violate the maximum principle?