GENERALIZED RECURRENCE AND THE NONWANDERING SET FOR PRODUCTS

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ABSTRACT. For continuous maps of compact metric spaces $f : X \to X$ and $g : Y \to Y$ and for various notions of topological recurrence, we study the relationship between recurrence for f and g and recurrence for the product map $f \times g : X \times Y \to X \times Y$. For the generalized recurrent set GR, we see that $\mathrm{GR}(f \times g) = \mathrm{GR}(f) \times \mathrm{GR}(g)$. For the nonwandering set NW, we see that $\mathrm{NW}(f \times g) \subset \mathrm{NW}(f) \times \mathrm{NW}(g)$ and give necessary and sufficient conditions on f for equality for every g. We also consider product recurrence for the chain recurrent set, the strong chain recurrent set, and the Mañé set.

1. INTRODUCTION

Let $f : X \to X$ and $g : Y \to Y$ be continuous maps of compact metric spaces. We are interested in the relationship between recurrence for f and g and recurrence for the product map $f \times g : X \times Y \to X \times Y$, and in how that relationship varies depending on which notion of recurrence we consider.

The strongest notion of recurrence is periodicity. It is clear that $\mathrm{Per}(f \times g)$, the set of periodic points for $f \times g$, is equal to $\mathrm{Per}(f) \times \mathrm{Per}(g)$. A slightly weaker condition is that a point is *(positively) recurrent* if it is in its own ω-limit set. The question of whether the positive recurrent set of a given product is equal to the product of the positive recurrent sets has been well studied and has led to some very deep and interesting mathematics; see [2] and [13] and the references therein. In this paper, we consider the corresponding question for several less restrictive notions of recurrent set, most importantly the generalized recurrent set and the nonwandering set.

The interesting dynamics occurs on the nonwandering set, so in order to understand the relationship between the dynamics of a product map and the dynamics of the original maps, we need to understand the nonwandering set; we give necessary and sufficient conditions (Theorem 3.12) for a point $x \in X$ to be product nonwandering, that is, for (x, y) to be nonwandering for $f \times g$ for any g and any nonwandering point $y \in Y$. Auslander's generalized recurrent set $\mathrm{GR}(f)$ (defined originally for flows (see [4]), and extended to maps (see [1,3])) is a larger and in many ways more dynamically natural set, particularly for understanding Lyapunov functions; see [8] and the references in [15]. We show that $\mathrm{GR}(f \times g) = \mathrm{GR}(f) \times \mathrm{GR}(g)$ (Theorem 3.1). The same is clearly true for the chain recurrent set, which reflects a still broader notion of recurrence.

We also consider product recurrence for Easton's strong chain recurrent set and Fathi and Pageault's Mañé set. These results come up for the most part in the study of the generalized recurrent set, but are also of independent interest.

In section 2, we give definitions and background information for the various notions of recurrence and for metrics on the product space. In section 3, we state and prove our results.

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2. Definitions and background

Throughout the paper, let $f : X \to X$ and $g : Y \to Y$ be continuous maps of compact metrizable spaces; unless stated otherwise, we will use the metrics d_X and d_Y respectively. Let $B_d(x; \varepsilon)$ be the closed ε-ball around x, $B_d(x; \varepsilon) = \{x' \in X : d(x, x') \leq \varepsilon\}$.

2.1. Recurrent sets.

Definition 2.1. A point $x \in X$ is *nonwandering* for f if for any neighborhood U of X, there exists an $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. We denote by $\mathrm{NW}(f)$ the set of nonwandering points.

Definition 2.2. An (ε, f, d_X)-*chain* (or (ε, d_X)-*chain*, if it is clear what the map is, or ε-*chain*, if the metric is also clear) of length n from x to x' is a sequence $(x = x_0, x_1, \ldots, x_n = x')$ such that $d_X(f(x_{i-1}), x_i) \leq \varepsilon$ for $i = 1, \ldots, n$. A point x is *chain recurrent* if for every $\varepsilon > 0$, there is an ε-chain from x to itself. We denote by $\mathrm{CR}(f)$ the set of chain recurrent points. (Chain recurrence is independent of the choice of metric; see, for example, [9].)

The following definition is due to Easton [7].

Definition 2.3. A *strong* (ε, f, d_X)-*chain* (or *strong* (ε, d_X)-*chain* or *strong* ε-*chain*) from x to x' is a sequence $(x = x_0, x_1, \ldots, x_n = x')$ such that the sum of the errors is bounded by ε, that is, $\sum_{i=1}^{n} d_X(f(x_{i-1}), x_i) \leq \varepsilon$. A point x is d_X-*strong chain recurrent* (or *strong chain recurrent*) if for every $\varepsilon > 0$, there is a strong (ε, d_X)-chain from x to itself. We denote the set of strong chain recurrent points by $\mathrm{SCR}_{d_X}(f)$.

We write $x_1 \sim_{d_X} x_2$ if for any $\varepsilon > 0$ there is a strong (ε, f, d_X)-chain from x_1 to x_2 and one from x_2 to x_1; then $\sim_{d_X}$ is a closed equivalence relation on $\mathrm{SCR}_{d_X}(f)$, and each equivalence class is a closed invariant set.

The strong chain recurrent set does depend on the choice of metric; see, for example, [16]. (Note, however, that it does not change if the new metric is bi-Lipschitz equivalent to the original [16, Prop. 3.2].) One way to eliminate this dependence is to take the intersection over all possible choices. This leads to the following definition.

Definition 2.4 ([8]). The *generalized recurrent set* $\mathrm{GR}(f)$ is $\bigcap_{d'_X} \mathrm{SCR}_{d'_X}(f)$, where the intersection is over all metrics d'_X compatible with the topology of X.

There are other, equivalent definitions of the generalized recurrent set; see [1, 3, 4, 8, 15]. In particular, $\mathrm{GR}(f)$ was originally defined as the set of points $x \in X$ such that all Lyapunov functions are constant on the orbit of x [1, 3, 4]; see section 3.

Another way to eliminate the dependence of the strong chain recurrent set on the choice of metric is to take the union over all possible choices.

Definition 2.5 ([8])**.** The *Mañé set* $\mathrm{M}(f)$ is $\bigcup_{d'_X} \mathrm{SCR}_{d'_X}(f)$, where the union is over all metrics d'_X compatible with the topology of X.

We will need an equivalent definition of the Mañé set. We begin with some notation. Let Δ_X be the diagonal in $X \times X$, $\Delta_X = \{(x,x) : x \in X\}$. Let $\mathcal{V}_{d_X}(\varepsilon)$ (or $\mathcal{V}(\varepsilon)$) be the closed ε-neighborhood of the diagonal Δ_X in $X \times X$, $\mathcal{V}_{d_X}(\varepsilon) = \{(x_1,x_2) : d_X(x_1,x_2) \leq \varepsilon\}$.

For $\mathcal{N} \subset X \times X$, we denote by $\mathcal{N}^n$ the n-fold composition of $\mathcal{N}$ with itself, $\mathcal{N} \circ \mathcal{N} \cdots \circ \mathcal{N}$, that is,

$$\mathcal{N}^n = \{(x,x') : \text{there exists } z_0 = x, z_1, \ldots, z_n = x' \in X$$
$$\text{such that } (z_{i-1}, z_i) \in \mathcal{N} \text{ for } i = 1, \ldots, n\}.$$

Definition 2.6. Let $\mathcal{N}$ be a neighborhood of Δ_X. An $(\mathcal{N}, f)$-*chain* (or simply $\mathcal{N}$-*chain* if the map is clear) from x to x' is a sequence of points $(x = x_0, x_1, \ldots, x_n = x')$ in X such that $(f(x_{i-1}), x_i) \in \mathcal{N}$ for $i = 1, \ldots, n$.

Thus $(x, x') \in \mathcal{N}^n$ exactly when there is an $(\mathcal{N}, \mathrm{Id})$-chain of length n from x to x', where Id is the identity map.

Theorem 2.7 ([15, Theorem 3.3])**.** *A point x is in $\mathrm{M}(f)$ if and only if for any closed neighborhood $\mathcal{D}$ of the diagonal in $X \times X$, there exist a closed symmetric neighborhood $\mathcal{N}$ of the diagonal and an integer $n > 0$ such that $\mathcal{N}^{3^n} \subset \mathcal{D}$ and there is an $(\mathcal{N}, f)$-chain of length n from x to itself.*

Any nonwandering point is clearly strong chain recurrent for any metric d_X, and any strong ε-chain is clearly an ε-chain, so we have the inclusions $\mathrm{NW}(f) \subset \mathrm{GR}(f) \subset \mathrm{SCR}_{d_X}(f) \subset \mathrm{M}(f) \subset \mathrm{CR}(f)$.

2.2. **Metrics on the product space.** Given two metrizable spaces X and Y, we give the product space $X \times Y$ the product topology. We will need to be able to go from metrics on X and Y to a metric on $X \times Y$, and vice versa. There are many well-known ways of doing the former, all essentially equivalent; for convenience, we use the following definition.

Definition 2.8. Let d_X and d_Y be metrics on X and Y, respectively. Denote by D_{d_X, d_Y} the metric on $X \times Y$ given by $D_{d_X, d_Y}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$, and observe that D_{d_X, d_Y} induces the product topology on $X \times Y$.

Note that since the metrics $\max(d_X, d_Y)$ and D_{d_X, d_Y} are bi-Lipschitz equivalent, they yield the same strong chain recurrent set for $f \times g$, so we could just as well have used $\max(d_X, d_Y)$ (or in fact $||d_X, d_Y||$ for any norm $|| \cdot ||$ on $\mathbb{R}^2$).

Definition 2.9. Let D be a metric on $X \times Y$. Define a metric $\overline{D}_X$ on X by $\overline{D}_X(x_1, x_2) = \max_{y \in Y} D((x_1, y), (x_2, y))$ and a metric $\overline{D}_Y$ on Y by $\overline{D}_Y(y_1, y_2) = \max_{x \in X} D((x, y_1), (x, y_2))$.

Lemma 2.10. $\overline{D}_X$ *is a metric on X, and if D is compatible with the product topology on $X \times Y$, then $\overline{D}_X$ is compatible with the topology on X. The corresponding statements hold for $\overline{D}_Y$. Furthermore, $D((x_1, y_1), (x_2, y_2)) \leq \overline{D}_X(x_1, x_2) + \overline{D}_Y(y_1, y_2)$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.*

Proof. Define $i : X \to C(Y, X \times Y)$ by $i(x)(y) = (x, y)$, where $C(X, Y)$ is the space of continuous functions from Y to $X \times Y$. This is injective and continuous (any open set in $X \times Y$ containing $\{x_0\} \times Y$ contains $U \times Y$ for some open set U containing x_0, since Y is compact). Observe that $\overline{D}_X$ is the pullback of the sup metric on $C(X, Y)$. Fixing a $y_0 \in Y$ and composing the evaluation map $\mathrm{eval}_{y_0} : C(Y, X \times Y) \to X \times Y$ with projection onto X gives an inverse function for i, so the pullback is compatible with the topology on X.

Finally, we have that

$$D((x_1, y_1), (x_2, y_2)) \le D((x_1, y_1), (x_2, y_1)) + D((x_2, y_1), (x_2, y_2))$$
$$\le \overline{D}_X(x_1, x_2) + \overline{D}_Y(y_1, y_2).$$

$\square$

3. Recurrence for product maps

Our main result concerns the relationships, for the various notions of recurrent set, between those of f and g and that of the product map $f \times g$. The result for the chain recurrent set is easy, and the proof is included for completeness and to highlight the difference between ε-chains and strong ε-chains. The result for the nonwandering set is a consequence of examples in the literature (see below), although I have not been able to find an explicit statement elsewhere.

Theorem 3.1. *Let $f : X \to X$ and $g : Y \to Y$ be continuous maps of compact metrizable spaces. Let d_X and d_Y be compatible metrics on X and Y, respectively, and let D be a metric on $X \times Y$ compatible with the product topology. Then*

(1) $\mathrm{NW}(f \times g) \subset \mathrm{NW}(f) \times \mathrm{NW}(g)$, *and the inclusion can be strict.*
(2) $\mathrm{GR}(f \times g) = \mathrm{GR}(f) \times \mathrm{GR}(g)$.
(3) $\mathrm{M}(f \times g) \supset \mathrm{M}(f) \times \mathrm{M}(g)$, *and the inclusion can be strict.*
(4) $\mathrm{SCR}_{D_{d_X, d_Y}}(f \times g) = \mathrm{SCR}_{d_X}(f) \times \mathrm{SCR}_{d_Y}(g)$.
(5) $\mathrm{SCR}_D(f \times g) \supset \mathrm{SCR}_{\overline{D}_X}(f) \times \mathrm{SCR}_{\overline{D}_Y}(g)$, *and the inclusion can be strict.*
(6) $\mathrm{CR}(f \times g) = \mathrm{CR}(f) \times \mathrm{CR}(g)$.

We will prove the results in the order (1), (6), (4), (5), (2), and then, after some preliminary work, complete the proof of (3).

Proof of Theorem 3.1(1). It is easy to see that $\mathrm{NW}(f \times g) \subset \mathrm{NW}(f) \times \mathrm{NW}(g)$. Let (x, y) be a nonwandering point for $f \times g$, and let U be any neighborhood of x in X and V any neighborhood of y in Y. Then $U \times V$ is a neighborhood of (x, y) in $X \times Y$, and so $(f \times g)^n(U \times V) \cap U \times V \ne \emptyset$ for some $n > 0$; thus $f^n(U) \cap U \ne \emptyset$ and $g^n(V) \cap V \ne \emptyset$. So x is in $\mathrm{NW}(f)$ and y is in $\mathrm{NW}(g)$.

Next we show that the inclusion can be strict. Sawada [14] and Coven and Nitecki [5] give examples of continuous maps f of compact metric spaces such that $\mathrm{NW}(f^2)$ is strictly contained in $\mathrm{NW}(f)$. Let f be such a map, and let x be a point in $\mathrm{NW}(f) \setminus \mathrm{NW}(f^2)$. Since $x \notin \mathrm{NW}(f^2)$, there exists a neighborhood U of x such that $f^{2m}(U) \cap U = \emptyset$ for all $m > 0$. Define $g : \{0, 1\} \to \{0, 1\}$ by $g(0) = 1$, $g(1) = 0$. Then $U \times \{0\}$ is a neighborhood of $(x, 0)$, but $(f \times g)^n(U \times \{0\}) \cap U \times \{0\} = \emptyset$ for all $n \ge 1$, since $g^n(0) = 1$ if n is odd and $f^n(U) \cap U = \emptyset$ if n is even. Thus $(x, 0)$ is not in $\mathrm{NW}(f \times g)$, even though $x \in \mathrm{NW}(f)$ and clearly $0 \in \mathrm{NW}(g)$.

(Note that for any $k > 0$, Proposition 3.13 gives examples such that $\mathrm{NW}(f^k)$ is strictly contained in $\mathrm{NW}(f)$. There are also functions constructed in [12] that can be adapted to give an example for which $\mathrm{NW}(f \times g) \subsetneq \mathrm{NW}(f) \times \mathrm{NW}(g)$.)

$\square$

Proof of Theorem 3.1(6). Recall that chain recurrence is independent of the choice of metric. If $((x_0, y_0), (x_1, y_1), \ldots, (x_m, y_m))$ is an $(\varepsilon, f \times g, D_{d_X, d_Y})$-chain, then $(x_0, \ldots, x_m)$ is an (ε, f, d_X)-chain and $(y_0, \ldots, y_m)$ is an (ε, g, d_Y)-chain; thus $\mathrm{CR}(f \times g) \subset \mathrm{CR}(f) \times \mathrm{CR}(g)$. Now let $(x_0, \ldots, x_m)$ be an $(\varepsilon/2, f, d_X)$-chain and $(y_0, \ldots, y_n)$ an $(\varepsilon/2, g, d_Y)$-chain. If $m \neq n$, we can concatenate the first chain with itself n times and the second with itself m times to get $\varepsilon/2$-chains of equal lengths, so we may assume that $m = n$. Then $((x_0, y_0), (x_1, y_1), \ldots, (x_m, y_m))$ is an $(\varepsilon, f \times g, D_{d_X, d_Y})$-chain. $\square$

The key observation in the preceding proof is that the concatenation of an ε-chain from p to q with an ε-chain from q to r gives an ε-chain from p to r, and, more generally, the concatenation of an arbitrary number of such ε-chains gives an ε-chain. This is not true for strong ε-chains: Concatenating N strong ε-chains gives an $(N\varepsilon)$-chain, so in order to control the sum of the errors in the concatenated chain, we must know in advance the number of chains to be concatenated. This is why we need Lemma 3.2 below.

Proof of Theorem 3.1(4). It is easy to see that $\mathrm{SCR}_{D_{d_X, d_Y}}(f \times g) \subset \mathrm{SCR}_{d_X}(f) \times \mathrm{SCR}_{d_Y}(g)$. Take any $(x, y) \in \mathrm{SCR}_{D_{d_X, d_Y}}(f \times g)$ and any $\varepsilon > 0$, and let $((x_0, y_0) = (x, y), (x_1, y_1), \ldots, (x_n, y_n) = (x, y))$ be a strong $(\varepsilon, f \times g, D_{d_X, d_Y})$-chain from (x, y) to itself. Then $\sum_{i=1}^{n} d_X(f(x_{i-1}), x_i) \leq \sum_{i=1}^{n} d_X(f(x_{i-1}), x_i) + d_Y(g(y_{i-1}), y_i) = \sum_{i=1}^{n} D_{d_X, d_Y}((f(x_{i-1}), g(y_{i-1})), (x_i, y_i)) \leq \varepsilon$, so $(x_0 = x, x_1, \ldots, x_n = x)$ is a strong (ε, f, d_X)-chain from x to itself. Since ε was arbitrary, we have $x \in \mathrm{SCR}_{d_X}(f)$. Similarly, we have $y \in \mathrm{SCR}_{d_Y}(g)$, and so $(x, y) \in \mathrm{SCR}_{d_X}(f) \times \mathrm{SCR}_{d_Y}(g)$.

Next we show that $\mathrm{SCR}_{d_X}(f) \times \mathrm{SCR}_{d_Y}(g) \subset \mathrm{SCR}_{D_{d_X, d_Y}}(f \times g)$. We must show that for $x \in \mathrm{SCR}_{d_X}(f)$, $y \in \mathrm{SCR}_{d_Y}(g)$, and $\varepsilon > 0$, there are strong ε-chains from x to x and y to y of the same length. We use the following lemmas.

Lemma 3.2. *For any $x \in \mathrm{SCR}_{d_X}(f)$ and any $\varepsilon > 0$, there exists an $n = n(x, \varepsilon) > 0$ such that for any x' with $x \sim_{d_X} x'$, there is a strong ε-chain $(x_0 = x, x_1, \ldots, x_{n(x, \varepsilon)} = x)$ of length $n(x, \varepsilon)$ from x to itself passing through x', that is, such that $x_i = x'$ for some i.*

Proof of Lemma 3.2. Restrict f to the space $X_x = \{x' : x \sim_{d_X} x'\}$. Since X_x is compact metric, it is separable; let $\{s_1, s_2, \ldots\}$ be a countable dense subset of X_x. For each i, choose $\delta_i \leq \frac{\varepsilon}{3 \cdot 2^i}$ such that if $d_X(z, w) \leq \delta_i$, then $d_X(f(z), f(w)) \leq \frac{\varepsilon}{3 \cdot 2^i}$. Since $x \sim_{d_X} s_i$, there is a strong $\delta_i/2$-chain from x to s_i, and one from s_i to x. Concatenate these chains to get a strong δ_i-chain from x to itself passing through s_i; call this chain L_i. If $d_X(x', s_i) \leq \delta_i$, then we can substitute x' for s_i in the chain to get a strong $\frac{\varepsilon}{2^i}$-chain from x to itself passing through x' of the same length; call this chain $L_i(x')$.

Pick a finite subcover of the open cover $\{B_{\delta_i}(s_i)\}_{i=1}^{\infty}$ of X_x; by renumbering if necessary, we can assume that the subcover is $\{B_{\delta_i}(s_i)\}_{i=1}^{N}$. Then x' is in $B_{\delta_i}(s_i)$ for some i. Concatenate the chains $L_1, L_2, \ldots, L_{i-1}, L_i(x'), L_{i+1}, \ldots, L_N$ to get a chain from x to itself passing through x'. Since each L_j is a strong $\frac{\varepsilon}{2^j}$-chain, and $\sum_{j=1}^{N} \frac{\varepsilon}{2^j} < \varepsilon$, the concatenation is a strong ε-chain. The length of the chain is the sum of the lengths of the chains L_j, which is independent of x', so define $n(x, \varepsilon) = \sum_{j=1}^{N} \mathrm{length}(L_j)$.

$\square$

For any $x \in \mathrm{SCR}_{d_X}(f)$ and $\varepsilon > 0$, define the set $N(x, \varepsilon)$ to be $\{n : \text{for every } x'$ with $x \sim_{d_X} x'$, there is a strong ε-chain of length n from x to itself passing through $x'\}$. A set of natural numbers is an *IP-set* [10] if it consists of all finite sums of some infinite set.

Lemma 3.3. *For any $x \in \mathrm{SCR}_{d_X}(f)$ and $\varepsilon > 0$, the set $N(x, \varepsilon)$ contains an IP-set.*

Proof of Lemma 3.3. Observe that $N(x, \varepsilon)$ contains all finite sums of the set $\{n(x, \frac{\varepsilon}{2^i})\}_{i=1}^{\infty}$, with $n(x, \frac{\varepsilon}{2^i})$ as in Lemma 3.2, since a concatenation of strong $\frac{\varepsilon}{2^i}$-chains (with distinct i's) is a strong ε-chain. If X_x does not consist of a single periodic orbit, then necessarily $n(x, \frac{\varepsilon}{2^i}) \to \infty$ as $i \to \infty$, so the set $\{n(x, \frac{\varepsilon}{2^i})\}_{i=1}^{\infty}$ is infinite. If X_x is a single periodic orbit, $\{x, f(x), \ldots, f^p(x) = x\}$, then we can choose $n(x, \frac{\varepsilon}{2^i}) = ip$, and again the set $\{n(x, \frac{\varepsilon}{2^i})\}_{i=1}^{\infty}$ is infinite. $\qquad\square$

We need the following result from [10]

Theorem 3.4 ([10, Theorem 12(ii)])**.** *Any IP-set R is a set of topological recurrence. That is, for every compact metric space (Z, d), continuous map $T : Z \to Z$, and $\varepsilon > 0$, there are $z \in Z$ and nonzero $n \in R$ such that $d(z, T^n(z)) < \varepsilon$.*

Now take any point (x, y) in $\mathrm{SCR}_{d_X}(f) \times \mathrm{SCR}_{d_Y}(g)$ and any $\varepsilon > 0$, and note that Lemmas 3.2 and 3.3 apply to y and g as well. Consider g restricted to the set $Y_y = \{y' : y \sim_{d_Y} y'\}$. Since $N(x, \varepsilon/4)$ contains an IP-set, the theorem above applied to $g|_{Y_y}$ guarantees that there exist a $y' \in Y_y$ and an $m_y \in N(x, \varepsilon/4)$ such that $d_Y(y', g^{m_y}(y')) \le \varepsilon/4$. Thus $(y', g(y'), g^2(y'), \ldots, g^{m_y-1}(y'), y')$ is a strong $(\varepsilon/4, g)$-chain of length m_y from y' to itself. Similarly, there exist an $x' \in X_x$ and an $m_x \in N(y, \varepsilon/4)$ such that $(x', f(x'), \ldots, f^{m_x-1}(x'), x')$ is a strong $(\varepsilon/4, f)$-chain.

Since $m_y \in N(x, \varepsilon/4)$, there is a strong $(\varepsilon/4, f)$-chain $(x_0 = x, x_1, \ldots, x_i = x', \ldots, x_{m_y} = x)$ of length m_y from x to itself passing through x'. Similarly, there is a strong $(\varepsilon/4, g)$-chain $(y_0 = x, y_1, \ldots, y_j = y', \ldots, y_{m_x} = y)$ of length m_x from y to itself passing through y'. Then

$$L_x = (x_0 = x, x_1, \ldots, x_i = x', f(x'), \ldots, f^{m_x-1}(x'), x', x_{i+1}, \ldots, x_{m_y} = x)$$

is a strong $(\varepsilon/2, f)$-chain of length $m_x + m_y$ from x to itself, and similarly $L_y = (y_0 = y, y_1, \ldots, y_j = y', g(y'), \ldots, g^{m_y-1}(y'), y', y_{j+1}, \ldots, y_{m_x} = y)$ is a strong $(\varepsilon/2, g)$-chain of length $m_x + m_y$ from y to itself. Thus the product of the chains L_x and L_y (that is, the chain $((x_0, y_0) = (x, y), (x_1, y_1), \ldots, (x, y)))$ is a strong $(\varepsilon, f \times g, D_{d_X, d_Y})$-chain of length $m_x + m_y$ from (x, y) to itself. Since ε was arbitrary, we have $(x, y) \in \mathrm{SCR}_{D_{d_X, d_Y}}(f \times g)$.

$$\square$$

Proof of Theorem 3.1(5). Let (x, y) be a point in $\mathrm{SCR}_{\overline{D}_X}(f) \times \mathrm{SCR}_{\overline{D}_Y}(g)$. As in the proof of Theorem 3.1(4), for any $\varepsilon > 0$ we can construct a strong $(\varepsilon/2, f, \overline{D}_X)$-chain $(x_0 = x, x_1, \ldots, x_n = x)$ and a strong $(\varepsilon/2, f, \overline{D}_Y)$-chain $(y_0 = y, y_1, \ldots, y_n = y)$ of the same length. Since for all i, $1 \le i \le n$, we have $D((f(x_{i-1}), g(y_{i-1})), (x_i, y_i)) \le \overline{D}_X(f(x_{i-1}), x_i) + \overline{D}_Y(g(y_{i-1}), y_i)$, the product chain $((x_0, y_0) = (x, y), \ldots, (x_n, y_n) = (x, y))$ is a strong $(\varepsilon, f \times g, D)$-chain. Since ε was arbitrary, we have $(x, y) \in \mathrm{SCR}_D(f \times g)$.

We prove that the inclusion can be strict by contradiction. Assume that $\mathrm{SCR}_D(f \times g) = \mathrm{SCR}_{\overline{D}_X}(f) \times \mathrm{SCR}_{\overline{D}_Y}(g)$ for every metric D on $X \times Y$ compatible with the product topology, and let (x, y) be a point in $\mathrm{M}(f \times g)$. Thus there is a metric D for which $(x, y) \in \mathrm{SCR}_D(f \times g)$, which implies that $(x, y) \in \mathrm{SCR}_{\overline{D}_X}(f) \times \mathrm{SCR}_{\overline{D}_Y}(g)$,

which implies that $(x, y) \in \mathrm{M}(f) \times \mathrm{M}(g)$. But in Example 3.7 below, we construct f and g such that $\mathrm{M}(f) \times \mathrm{M}(g) \subsetneq \mathrm{M}(f \times g)$, which means that the inclusion must be strict for some metric on $X \times Y$. (In fact, the metric derived from the Minkowski ?-function discussed in the example will work.)

<div align="right">□</div>

Proof of Theorem 3.1(2). This follows from Theorem 3.1(4) and (5). We have

$$\mathrm{GR}(f) \times \mathrm{GR}(g) \subset \cap_D \mathrm{SCR}_{\overline{D}_X}(f) \times \mathrm{SCR}_{\overline{D}_Y}(g)$$
$$\subset \cap_D \mathrm{SCR}_D(f \times g)$$
$$= \mathrm{GR}(f \times g),$$

where the intersection is over all metrics D on $X \times Y$ compatible with the product topology, and

$$\mathrm{GR}(f \times g) \subset \cap_{d_X, d_Y} \mathrm{SCR}_{D_{d_X, d_Y}}(f \times g)$$
$$= \cap_{d_X, d_Y} \mathrm{SCR}_{d_X}(f) \times \mathrm{SCR}_{d_Y}(g)$$
$$= \mathrm{GR}(f) \times \mathrm{GR}(g),$$

where the intersection is over all metrics d_X and d_Y compatible with the topologies on X and Y, respectively.

<div align="right">□</div>

To prove that the inclusion in Theorem 3.1(3) can be strict, we need an alternative description of the Mañé set $\mathrm{M}(f)$, in terms of ordinary chain recurrence.

Definition 3.5. We say that x is *chain-recurrent through A* if for every $\varepsilon > 0$, there is an ε-chain from x to itself lying entirely in A. We denote by $\mathrm{CR}_A(f)$ the set of points chain-recurrent through A. (Note that if A is a closed invariant set, then $\mathrm{CR}_A(f) = \mathrm{CR}(f|_A)$, where $f|_A : A \to A$ is the restriction of f to A.)

Fathi and Pageault originally proved the following result for homeomorphisms ([8, Theorem 3.5]). Here, we extend the result to continuous maps and give a somewhat more topological, less technical proof.

Theorem 3.6. $\mathrm{M}(f) = \mathrm{Fix}(f) \cup \mathrm{CR}_{X \setminus \mathrm{Int}(\mathrm{Fix}(f))}(f).$

Proof. We first prove that $\mathrm{M}(f) \subset \mathrm{Fix}(f) \cup \mathrm{CR}_{X \setminus \mathrm{Int}(\mathrm{Fix}(f))}(f)$. Take any point x not in $\mathrm{Fix}(f) \cup \mathrm{CR}_{X \setminus \mathrm{Int}(\mathrm{Fix}(f))}(f)$ and any metric d' compatible with the topology on X. We will show that $x \notin \mathrm{SCR}_{d'}(f)$; since d' was arbitrary, this implies that $x \notin \mathrm{M}(f)$.

First, we show that there exist an $\varepsilon_0 > 0$ and an $\alpha > 0$ such that for any $\varepsilon \leq \varepsilon_0$, any (ordinary, not strong) (ε, d')-chain from x to itself must pass through the α-interior of $\mathrm{Fix}(f)$. (The α-interior of $\mathrm{Fix}(f)$ is $\{z : B_{d'}(z; \alpha) \subset \mathrm{Fix}(f)\}$, which we will denote by $\mathrm{Int}_\alpha(\mathrm{Fix}(f))$.) Assume not. Then for any $\alpha > 0$, there exists a sequence $\{\varepsilon_j\}$ tending to 0 such that for each j there is an ε_j-chain from x to itself which stays entirely within α of $X \setminus \mathrm{Int}(\mathrm{Fix}(f))$. For a given ε, choose $\delta > 0$ such that if $d'(x', x'') < \delta$, then $d'(f(x'), f(x'')) < \varepsilon/3$; then choose $\alpha < \min(\varepsilon/3, \delta)$ and $\varepsilon_j < \alpha$ and let $(x_0 = x, x_1, \ldots, x_n = x)$ be an ε_j-chain which stays entirely within α of $X \setminus \mathrm{Int}(\mathrm{Fix}(f))$. For each $x_i, 0 < i < n$, there is a point $\tilde{x}_i \in X \setminus \mathrm{Int}(\mathrm{Fix}(f))$ with $d'(x_i, \tilde{x}_i) < \alpha$; then $(x_0 = x, \tilde{x}_1, \ldots, \tilde{x}_{n-1}, x_n = x)$ is an ε-chain that stays entirely in $X \setminus \mathrm{Int}(\mathrm{Fix}(f))$. Since ε was arbitrary, we have $x \in \mathrm{CR}_{X \setminus \mathrm{Int}(\mathrm{Fix}(f))}(f)$, contrary to assumption.

Thus there exist an $\varepsilon_0 > 0$ and an $\alpha > 0$ such that for any $\varepsilon \leq \varepsilon_0$, any ε-chain from x to itself must pass through $\mathrm{Int}_\alpha(\mathrm{Fix}(f))$. Now choose an $\varepsilon < \min(\varepsilon_0, \alpha)$ and let $(x_0 = x, x_1, \ldots, x_n = x)$ be an ε-chain from x to itelf. Let m be the smallest index such that $x_m \in \mathrm{Int}_\alpha(\mathrm{Fix}(f))$, and let $m + k$ be the smallest index greater than m such that $x_{m+k} \notin \mathrm{Fix}(f)$. Then $f(x_i) = x_i$ for $i = m, \ldots, m + k - 1$, so we have that $\sum_{i=1}^n d'(f(x_{i-1}), x_i) \geq \sum_{i=m+1}^{m+k} d'(f(x_{i-1}), x_i) = \sum_{i=m+1}^{m+k} d'(x_{i-1}, x_i) \geq d'(x_m, x_{m+k}) \geq \alpha$. Thus $x \notin \mathrm{SCR}_{d'}(f)$, as desired.

Next we show that $\mathrm{Fix}(f) \cup \mathrm{CR}_{X \setminus \mathrm{Int}(\mathrm{Fix}(f))}(f) \subset \mathrm{M}(f)$. Let d be a metric on X compatible with the topology. It is obvious that any fixed point is in $\mathrm{M}(f)$, so assume that for any $\varepsilon > 0$, there is an (ε, f, d)-chain from x to itself lying in $X \setminus \mathrm{Int}(\mathrm{Fix}(f))$. We will use the alternative definition for $\mathrm{M}(f)$ from Theorem 2.7. Let $\mathcal{D}$ be a closed neighborhood of the diagonal; we must find a closed symmetric neighborhood $\mathcal{N}$ of the diagonal and an integer $n > 0$ such that $\mathcal{N}^{3^n} \subset \mathcal{D}$ and there is an $(\mathcal{N}, f)$-chain of length n from x to itself. Choose ε such that $\mathcal{V}(4\varepsilon) \subset \mathcal{D}$, choose $\delta \leq \frac{\varepsilon}{2}$ such that if $d(z_1, z_2) \leq \delta$, then $d(f(z_1), f(z_2)) \leq \frac{\varepsilon}{2}$, and let $(x_0 = x, x_1, \ldots, x_n = x)$ be a δ-chain from x to itself contained in $X \setminus \mathrm{Int}(\mathrm{Fix}(f))$; by starting with a $\frac{\delta}{2}$-chain and perturbing the points slightly if necessary, we can assume that in fact $(x_0 = x, x_1, \ldots, x_n = x)$ is contained in $X \setminus \mathrm{Fix}(f)$.

For $1 \leq i \leq n$, let $C_i = B_d(f(x_{i-1}); \delta_C) \cup B_d(x_i; \delta_C)$, where δ_C will be chosen later. To construct $\mathcal{N}$, we will need to ensure that the C_i's are pairwise disjoint; thus we need to ensure that for $i < j$, we have $x_i \neq x_j$, $x_i \neq f(x_{j-1})$, $f(x_{i-1}) \neq x_j$, and $f(x_{i-1}) \neq f(x_{j-1})$. First, in the cases $x_i = x_j$, $f(x_{i-1}) = x_j$, and $f(x_{i-1}) = f(x_{j-1})$, we can shorten the chain to $(x_0, \ldots, x_{i-1}, x_j, x_{j+1} \ldots, x_n)$ and still have a δ-chain from $x_0 = x$ to $x_n = x$. Now consider the case $x_i = f(x_{j-1})$; we must have $j > i + 1$, since x_{j-1} is not fixed. Let i' be the smallest index such that $x_i' = f(x_{j-1})$ for some $j > i + 1$, and let j' be the largest such j for i'. Then the shortened chain $(x_0 = x, x_1, \ldots, x_{i'-1}, x_{i'}, x_{j'+1}, x_{j'+2}, \ldots, x_n = x)$ is an ε-chain, since $d(f(x_{i'}), x_{j'+1}) \leq d(f(x_{i'}), f(x_{j'})) + d(f(x_{j'}), x_{j'+1}) = d(f^2(x_{j'-1}), f(x_{j'})) + d(f(x_{j'}), x_{j'+1}) \leq \frac{\varepsilon}{2} + \delta \leq \varepsilon$. We may have to perform this truncation several times, but, by construction, the consecutive terms $x_{i'}$, $x_{j'+1}$ will remain, and thus we will end up with an ε-chain, which we will still denote by $(x_0 = x, x_1, \ldots, x_n = x)$.

Choose $\delta_C \leq \varepsilon$ small enough that the collection $\{C_i : 1 \leq i \leq n\}$ is pairwise disjoint, and choose $\varepsilon_0 < \min_{i \neq j} d(C_i, C_j)$; we can assume that $\varepsilon_0 \leq \varepsilon$ as well. Now define $\mathcal{N} = \mathcal{V}(\frac{\varepsilon_0}{3^n}) \bigcup (\bigcup_{i=1}^n C_i \times C_i)$. Since $(f(x_{i-1}), x_i)$ is in $C_i \times C_i$, we have that $(x_0 = x, x_1, \ldots, x_n = x)$ is an $(\mathcal{N}, f)$-chain. To see that $\mathcal{N}^{3^n} \subset \mathcal{D}$, let $z_0, z_1, \ldots, z_{3^n}$ be a sequence with $(z_{j-1}, z_j) \in \mathcal{N}$ for $1 \leq j \leq 3^n$; we want to show that $d(z_0, z_{3^n}) \leq 4\varepsilon$. Since the C_i's are more than ε_0 apart, there exists at most one C_i such that there is a pair (z_j, z_{j+1}) in the chain in $C_i \times C_i$; thus any other two consecutive points must be within $\frac{\varepsilon_0}{3^n}$ of each other. So $d(z_0, z_{3^n}) \leq 3^n \cdot \frac{\varepsilon_0}{3^n} + \max\{\mathrm{diam}(C_i)\} \leq \varepsilon_0 + \varepsilon + 2\delta_C \leq 4\varepsilon$.

$\square$

Proof of Theorem 3.1(3). We first use Theorem 3.6 to show that $\mathrm{M}(f \times g) \supset \mathrm{M}(f) \times \mathrm{M}(g)$. (This also follows from Theorem 3.1(4).) Take any point $(x, y) \in \mathrm{M}(f) \times \mathrm{M}(g)$. If either x or y is fixed, then (x, y) is clearly in $\mathrm{M}(f \times g)$, so assume that $x \in \mathrm{CR}_{X \setminus \mathrm{Int}(\mathrm{Fix}(f))}(f)$ and $y \in \mathrm{CR}_{Y \setminus \mathrm{Int}(\mathrm{Fix}(g))}(g)$. For any $\varepsilon > 0$, there exist an $(\varepsilon/2, f, d_X)$-chain $(x_0 = x, \ldots, x_m = x)$ in $X \setminus \mathrm{Int}(\mathrm{Fix}(f))$ and an $(\varepsilon/2, g, d_Y)$-chain $(y_0 = y, \ldots, y_n = y)$ in $Y \setminus \mathrm{Int}(\mathrm{Fix}(g))$. We may assume that $m = n$ (if not, concatenate the first chain with itself n times and the second with itself m

times to get two chains of equal length). Then the product chain $((x_0, y_0) = (x, y), \ldots, (x, y))$ is an $(\varepsilon, f \times g, D_{d_X, d_Y})$-chain in $(X \times Y) \backslash \operatorname{Int}(\operatorname{Fix}(f \times g))$. Since ε was arbitrary, we have $(x, y) \in \operatorname{M}(f \times g)$.

The following example shows that the inclusion can be strict.

Example 3.7 ([15, Example 3.8]). Let X be the circle with the usual topology, and let $f : X \to X$ be a map fixing every point on the closed left semicircle and moving points on the open right semicircle counterclockwise. Then by Theorem 3.6, $\operatorname{M}(f)$ is the closed left semicircle. Let Y be the set $\{0, 1\}$ and $g : Y \to Y$ the permutation switching the two points; then $\operatorname{M}(g) = Y$. Since $f \times g$ has no fixed points, $\operatorname{M}(f \times g) = \operatorname{CR}(f \times g) = X \times Y$, which strictly contains $\operatorname{M}(f) \times \operatorname{M}(g)$. (One can show that if we define the metric D on $X \times Y$ by giving $X \times \{0\}$ the usual circle metric and $X \times \{1\}$ the usual metric on the left semicircle and the metric induced by the Minkowski ?-function on the right semicircle, we have $\operatorname{SCR}_D(f \times g) = X \times Y$.)

$\square$

The generalized recurrent set was originally defined in terms of Lyapunov functions, and Fathi and Pageault showed in [8] that the strong chain recurrent set can be defined in terms of Lipschitz Lyapunov functions, so we can restate some of our main results. Let $\theta : X \to \mathbb{R}$ be a Lyapunov function for f (that is, $\theta(f(x)) \leq \theta(x)$ for all x). Following the notation in [8], denote by $N(\theta)$ the set of neutral points, $N(\theta) = \{x \in X : \theta(f(x)) = \theta(x)\}$. Let $L(f)$ be the set of continuous Lyapunov functions for f, and let $L_{d_X}(f)$ be the set of Lipschitz (with respect to the metric d_X) Lyapunov functions for f. Since $\operatorname{GR}(f) = \bigcap_{\theta \in L(f)} N(\theta)$ [1,3,4] and $\operatorname{SCR}_{d_X}(f) = \bigcap_{\theta \in L_{d_X}(f)} N(\theta)$ [8], we have the following corollaries to Theorem 3.1.

Corollary 3.8. *Let $f : X \to X$ and $g : Y \to Y$ be continuous maps of compact metric spaces, with metric d_X and d_Y, respectively.*

(1) *A point (x, y) is in $\bigcap_{\Theta \in L(f \times g)} N(\Theta)$ if and only if x is in $\bigcap_{\theta \in L(f)} N(\theta)$ and y is in $\bigcap_{\phi \in L(g)} N(\phi)$.*

(2) *A point (x, y) is in $\bigcap_{\Theta \in L_{D_{d_X, d_Y}}(f \times g)} N(\Theta)$ if and only if x is in $\bigcap_{\theta \in L_{d_X}(f)} N(\theta)$ and y is in $\bigcap_{\phi \in L_{d_Y}(g)} N(\phi)$.*

Theorem 3.1 also gives an easy proof of the following well-known result from number theory. For real numbers w and z, denote by $|w - z|_1$ the difference in their fractional parts; that is, $|w - z|_1 = |w - z| \pmod 1$.

Corollary 3.9. *Let α and β be real numbers. Then for any $\varepsilon > 0$, there are infinitely many positive integers n such that $|n\alpha - n\beta|_1 < \varepsilon$.*

Proof. Let S^1 be the circle, considered as $\mathbb{R}/\mathbb{Z}$, with metric $d(x_1, x_2) = |x_1 - x_2|_1$, and let $R_\theta : S^1 \to S^1$ be rotation by some θ, $R_\theta(x) = x + \theta \pmod 1$. It is clear that x is strong chain recurrent for R_θ for every x and every θ, and so, by Theorem 3.1(4), $(0, 0) \in \operatorname{SCR}_{D_{d,d}}(R_\alpha \times R_\beta)$. Let $((x_0, y_0) = (0, 0), \ldots, (x_n, y_n) = (0, 0))$ be a strong $(\varepsilon, R_\alpha \times R_\beta, D_{d,d})$-chain; since $R_\alpha \times R_\beta$ is an isometry, we have that $D_{d,d}((x_n, y_n), (R_\alpha \times R_\beta)^n(x_0, y_0)) \leq \varepsilon$. Thus $\varepsilon \geq D_{d,d}((R_\alpha \times R_\beta)^n(0, 0), (0, 0)) = |n\alpha|_1 + |n\beta|_1 \geq |n\alpha - n\beta|_1$. Since this is true for any ε, there must be infinitely many such n. $\square$

A point x is *(positively) recurrent* for $f : X \to X$ if, for any neighborhood U of x, x returns to U, that is, $f^n(x) \in U$ for some $n > 0$. Thus any recurrent

point is nonwandering. A recurrent point x is *product recurrent* if for any recurrent point y of any map $g : Y \to Y$, the point (x, y) is recurrent for the product map $f \times g$. It is well known that a point is product recurrent if and only if it is distal [11, Theorem 9.11]. We have a somewhat analogous result for nonwandering points.

Definition 3.10. Let $f : X \to X$ be a continuous map of a compact metric space. A point x in X is *product nonwandering* if, for any continuous map $g : Y \to Y$ of a compact metric space and any nonwandering point y for g, the point (x, y) is nonwandering for $f \times g$. The map f is *locally (topologically) mixing at x* if for any neighborhood U of x, there exists an integer N such that $f^n(U) \cap U \neq \emptyset$ for all $n \geq N$.

Any topologically mixing map (such as the doubling map on the circle) is locally mixing at every point. The disjoint union of two circles, with the map given by doubling on each circle, is not topologically mixing, but it is locally mixing at every point; the same is true for the identity map on any nontrivial space. More generally, a map is locally mixing at any fixed point.

Example 3.11. Let X be the unit disk in $\mathbb{R}^2$ and let $f : X \to X$ be a map that fixes the center $(0, 0)$ and the north pole $(0, 1)$, moves other interior points in a clockwise spiral out towards the boundary circle, and moves points on the circle clockwise toward the north pole. Then the nonwandering set consists of the center and the boundary circle, and f is locally mixing at every nonwandering point.

Theorem 3.12. *Let $f : X \to X$ be a continuous map of a compact metric space to itself. A point x in X is product nonwandering if and only if f is locally (topologically) mixing at x.*

Proof. We prove the "if" direction first. Take any $y \in \mathrm{NW}(g)$ and any neighborhood W of (x, y) in $X \times Y$. Let U and V be neighborhoods of x in X and y in Y, respectively, such that $U \times V \subset W$, and choose N such that $f^n(U) \cap U \neq \emptyset$ for all $n \geq N$. The set of return times for V, $\{m > 0 : g^m(V) \cap V \neq \emptyset\}$, is infinite [6, Proposition 4.3.2]. Thus there exists a return time $m \geq N$, so we have $(f \times g)^m(W) \cap W \supset (f \times g)^m(U \times V) \cap (U \times V) = (f^m(U) \cap U) \times (g^m(V) \cap V) \neq \emptyset$. Since W was arbitrary, (x, y) is nonwandering for $f \times g$.

To prove the "only if" direction, we use the following proposition, which may be of independent interest.

Proposition 3.13. *Let $\mathcal{M}$ be any infinite set of natural numbers. Then there exist a continuous map $g_{\mathcal{M}} : Y_{\mathcal{M}} \to Y_{\mathcal{M}}$ of a compact metric space to itself, a point $y \in \mathrm{NW}(g_{\mathcal{M}})$, and a neighborhood V of y such that $\{m > 0 : g_{\mathcal{M}}^m(V) \cap V \neq \emptyset\} = \mathcal{M}$.*

Proof of Proposition 3.13. Let $\mathcal{M} = \{m_i\}_{i=1}^{\infty}$, with $m_1 < m_2 < \dots$. We construct the space $Y_{\mathcal{M}} = \bigcup_{i=0}^{\infty} Y_i$ as a subset of $\mathbb{R}^2$, where the spaces Y_i and the map $g_{\mathcal{M}}$ are defined as follows; see Figure 1 for an example with $m_1 = 3$, $m_2 = 5$, and $m_3 = 6$.

Define $Y_0 = \{(0, 0)\} \cup \{(\frac{1}{n}, 0) : 1 \leq n < \infty\}$, and define

$$
\begin{aligned}
g_{\mathcal{M}}(\tfrac{1}{n}, 0) &= (\tfrac{1}{n-1}, 0) \ (n > 1) \\
g_{\mathcal{M}}(1, 0) &= (0, 0) \\
g_{\mathcal{M}}(0, 0) &= (0, 0).
\end{aligned}
$$
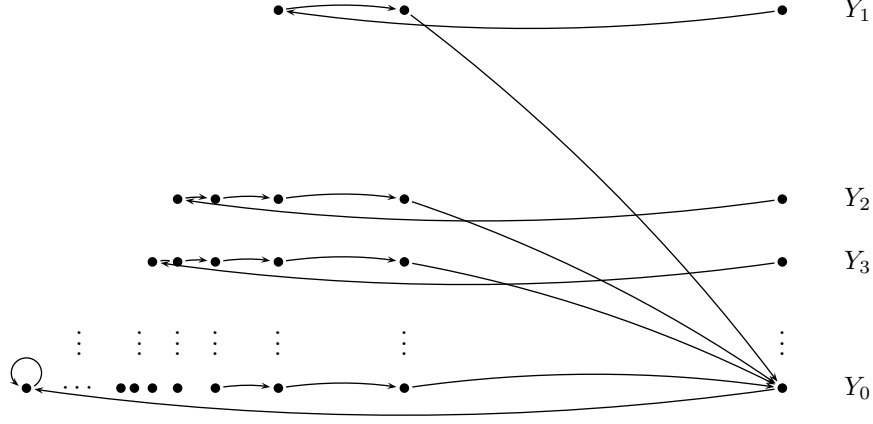
FIGURE 1. The map $g_{\mathcal{M}} : Y_{\mathcal{M}} \to Y_{\mathcal{M}}$ from Proposition 3.13

For $i \geq 1$, define $Y_i = \{(\frac{1}{n}, \frac{1}{i}) : 1 \leq n \leq m_i\}$, and define

$$
\begin{aligned}
g_{\mathcal{M}}(\frac{1}{n}, \frac{1}{i}) &= (\frac{1}{n-1}, \frac{1}{i}) \ (n > 2) \\
g_{\mathcal{M}}(\frac{1}{2}, \frac{1}{i}) &= (1, 0) \\
g_{\mathcal{M}}(1, \frac{1}{i}) &= (\frac{1}{m_i}, \frac{1}{i})
\end{aligned}
$$

(unless $m_i = 1$, which can happen only if $i = 1$; then $g_{\mathcal{M}}(1,1) = (1,0)$).

The set $Y_{\mathcal{M}}$ is compact and the map $g_{\mathcal{M}}$ is continuous. Observe that $g_{\mathcal{M}}^k(1,0) = (0,0)$ for $k \geq 1$, and that $g_{\mathcal{M}}^{m_i}(1, \frac{1}{i}) = (1,0)$. The point $y = (1,0)$ is nonwandering, because every neighborhood of y contains a point of the form $(1, \frac{1}{i})$. The set $V = \{y\} \cup \{(1, \frac{1}{i}) : 1 \leq i < \infty\}$ is a neighborhood of y such that $\{m > 0 : g_{\mathcal{M}}^m(V) \cap V \neq \emptyset\} = \mathcal{M}$.

$\square$

Now assume that f is not locally mixing at x; we will show that x is not product nonwandering. Since f is not locally mixing at x, there exists a neighborhood U of x such that the set of non-return times, $\{n > 0 : f^n(U) \cap U = \emptyset\}$, is infinite; let this set be $\mathcal{M}$. Consider the map $f \times g_{\mathcal{M}} : X \times Y_{\mathcal{M}} \to X \times Y_{\mathcal{M}}$. Since V returns to itself under $g_{\mathcal{M}}$ only at times in $\mathcal{M}$, and U returns to itself under f only at times not in $\mathcal{M}$, the neighborhood $U \times V$ of (x, y) never returns to itself under $f \times g_{\mathcal{M}}$. Thus (x, y) is not nonwandering, and so x is not product nonwandering.

$\square$

So, for example, if f is a nontrivial rotation of the circle, then every point is product recurrent but not product nonwandering. Conversely, in Example 3.11, every nonwandering point is product nonwandering, but only the fixed points are product recurrent.

References

[1] Ethan Akin, *The general topology of dynamical systems*, Graduate Studies in Mathematics, vol. 1, American Mathematical Society, Providence, RI, 1993. MR1219737 (94f:58041)

[2] ———, *Recurrence in topological dynamics: Furstenberg families and Ellis actions*, The University Series in Mathematics, Plenum Press, New York, 1997. MR1467479

[3] Ethan Akin and Joseph Auslander, *Generalized recurrence, compactifications, and the Lyapunov topology*, Studia Math. **201** (2010), no. 1, 49–63. MR2733274 (2012a:37013)

[4] Joseph Auslander, *Generalized recurrence in dynamical systems*, Contributions to Differential Equations **3** (1964), 65–74. MR0162238 (28 #5437)

[5] Ethan M. Coven and Zbigniew Nitecki, *Nonwandering sets of the powers of maps of the interval*, Ergodic Theory Dynamical Systems **1** (1981), no. 1, 9–31. MR627784 (82m:58043)

[6] J. de Vries, *Topological dynamical systems: an introduction to the dynamics of continuous mappings*, De Gruyter, Berlin, Boston, 2014.

[7] Robert Easton, *Chain transitivity and the domain of influence of an invariant set*, The structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ., Fargo, N.D., 1977), 1978, pp. 95–102. MR518550 (80j:58051)

[8] Albert Fathi and Pierre Pageault, *Aubry-Mather theory for homeomorphisms*, Ergodic Theory Dynam. Systems **35** (2015), no. 4, 1187–1207. MR3345168

[9] John Franks, *A variation on the Poincaré-Birkhoff theorem*, Hamiltonian dynamical systems (Boulder, CO, 1987), 1988, pp. 111–117. MR986260 (90e:58095)

[10] Nikos Frantzikinakis and Randall McCutcheon, *Ergodic theory: recurrence*, Mathematics of complexity and dynamical systems. Vols. 1–3, 2012, pp. 357–368. MR3220681

[11] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, N.J., 1981. M. B. Porter Lectures. MR603625

[12] José L. Gómez-Rueda, Alejandro Illanes, and Héctor Méndez, *Dynamic properties for the induced maps in the symmetric products*, Chaos Solitons Fractals **45** (2012), no. 9-10, 1180–1187. MR2979228

[13] Piotr Oprocha and Guohua Zhang, *On weak product recurrence and synchronization of return times*, Adv. Math. **244** (2013), 395–412. MR3077877

[14] Ken Sawada, *On the iterations of diffeomorphisms without $C^0 - \Omega$-explosions: an example*, Proc. Amer. Math. Soc. **79** (1980), no. 1, 110–112. MR560595 (81h:58055)

[15] Jim Wiseman, *The generalized recurrent set and strong chain recurrence*, Ergodic Theory Dynam. Systems (to appear).

[16] Katsuya Yokoi, *On strong chain recurrence for maps*, Annales Polonici Mathematici **114** (2015), 165–177.

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