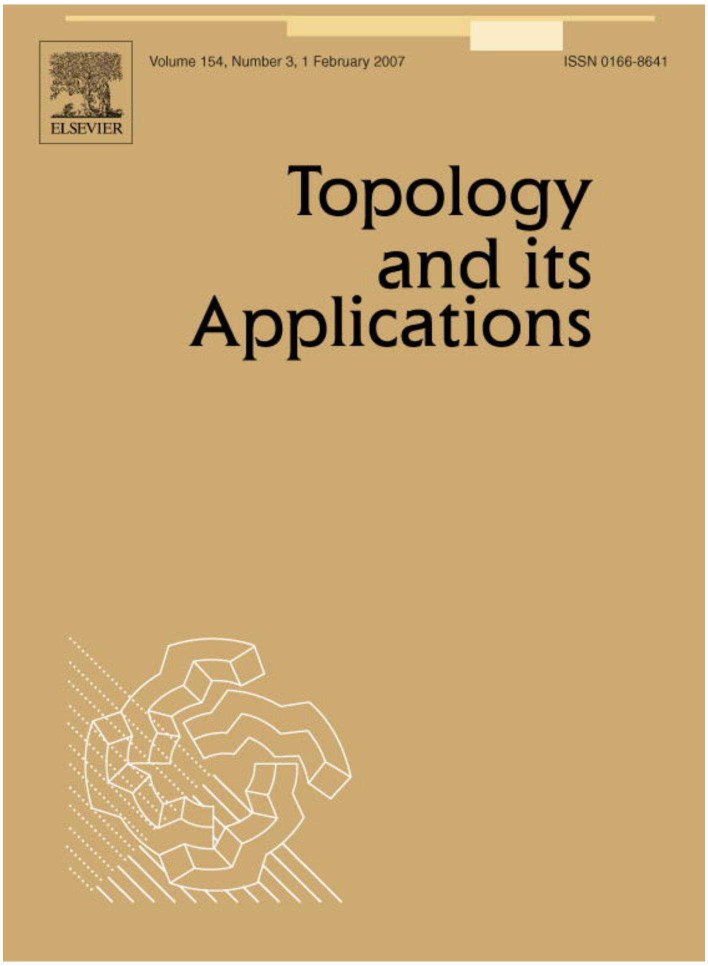
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Positively expansive dynamical systems

David Richeson [a], Jim Wiseman [b,*]

[a] *Dickinson College, Carlisle, PA 17013, USA*
[b] *Agnes Scott College, Decatur, GA 30030, USA*

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Abstract

We introduce the notions of weakly and strongly positively expansive (wPE and sPE, respectively) discrete dynamical systems. Both are topological generalizations of the well-known metric notion of positive expansiveness (PE). We prove that the three notions are identical on compact metrizable spaces, but not on noncompact spaces. We investigate properties of PE, wPE, and sPE dynamical systems and show how they are related. Finally, we show that the possible dynamics for wPE and sPE homeomorphisms are severely limited, and we classify sPE homeomorphisms.

MSC: primary 37B05; secondary 37B25

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1. Introduction

Certain dynamical systems, such as the doubling map on the circle, have the property that nearby points separate in positive time. Dynamicists have devised various ways of describing this separation of forward orbits. For instance, in the differentiable setting we may look for everywhere positive Lyapunov exponents. In the hyperbolic case there must be a trivial stable bundle. In our setting—a continuous function or a homeomorphism on a metric space—two notions dominate. When two nearby points are closer together than their images, the map is called expanding. When two points are eventually separated by more than some fixed distance it is called positively expansive. Specifically, a dynamical system $f : X \to X$ defined on a metric space (X, d) is *expanding* if there exist $\varepsilon > 0$ and $\lambda > 1$ such that whenever $d(x, y) < \varepsilon$, $d(f(x), f(y)) > \lambda d(x, y)$. It is *positively expansive* (PE) if there exists an *expansive constant* $\rho > 0$ such that for any distinct $x, y \in X$ there exists $n \geqslant 0$ such that $d(f^n(x), f^n(y)) > \rho$.

Unfortunately, both properties depend on the choice of metric. It is easy to concoct an example of a dynamical system $f : X \to X$ on a metrizable space X that is PE (or expanding) with respect to one metric, but not another (Example 7). Since both metrics generate the same topology, neither property is dynamical. That is, if (X, d) and (Y, δ) are metric spaces, $f : X \to X$ and $g : Y \to Y$ are topologically conjugate, and f is PE (or expanding), then g may not be.

It turns out that positive expansiveness is a dynamical property when it is restricted to compact spaces ([2, §2.2], or Theorem 5 below). It is only when we consider noncompact spaces that it is no longer a dynamical property. However,

* Corresponding author.
E-mail addresses: richesod@dickinson.edu (D. Richeson), jwiseman@agnesscott.edu (J. Wiseman).

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the expanding property is not a dynamical property—not even in the compact case. Thus, in this paper we focus on positively expansive dynamical systems, not expanding ones.

A related, weaker notion (which we will not address in this paper) is that of an expansive homeomorphism, under which points move apart either in forward or in backward time. The standard example of an expansive homeomorphism is the full two-sided shift on two symbols, which is not positively expansive. However, the one-sided shift is a positively expansive map.

We propose two topological generalizations of PE, both of which are dynamical properties on compact and noncompact spaces—weak positive expansiveness (wPE) for homeomorphisms or continuous maps and strong positive expansiveness (sPE) for homeomorphisms. Both are candidates for the “right” topological generalization of PE. On a compact space all three notions—PE, sPE, and wPE—are equivalent (Theorem 5). However, on noncompact spaces they are all distinct (Examples 7, 20, and 21). As the names imply, every sPE homeomorphism is wPE. We will show that a map is wPE if and only if it is PE with respect to some metric compatible with the topology (Theorem 4).

The paper is organized as follows. In Section 2 we define all key terms and we state theorems that we will need later. In Section 3 we explore the relationships between dynamical systems that are PE, wPE, and sPE and we prove some general properties. We look at both compact and noncompact spaces. In Section 4 we restrict our attention to homeomorphisms. We will see that the dynamics of PE, wPE, and sPE homeomorphisms are severely limited (Theorems 10, 14 and Proposition 17). Finally, we classify sPE homeomorphisms (Theorems 18 and 19).

We wish to express our deep gratitude for the referee’s extremely insightful and helpful report. It gave us a different perspective on our own work, and helped us clarify the concepts and focus the paper. We are especially grateful to the referee for all or part of Theorems 6, 18, and 19, and the outlines of their proofs.

2. Definitions and background information

In much of this paper we study a map $f : X \to X$ by looking at the associated map $F : X \times X \to X \times X$ given by $F(x, y) = (f(x), f(y))$. We do so because saying that nearby points in X separate under iterates of f is the same as saying that points near the *diagonal* of $X \times X$, $\Delta = \{(x, x)\colon x \in X\}$, move away from Δ under F. For instance, if f is PE with expansive constant ρ, then the only points that remain in the set $V_\rho = \{(x, y) \in X \times X\colon d(x, y) \leqslant \rho\}$ for all positive iterates of F are the points in Δ.

This motivates the following definitions. An *expansivity neighborhood* for f is a closed neighborhood $N \subset X \times X$ of Δ such that for any distinct $x, y \in X$ there exists $n \geqslant 0$ such that $F^n(x, y) \notin N$. If $M \subset X \times X$, let $M(x) = \pi_2(\pi_1^{-1}(x) \cap M) = \{y \in X\colon (x, y) \in M\}$ denote the cross section of $M \subset X \times X$ for a fixed x in the first coordinate. We say that a set $M \subset X \times X$ is *proper* if for any compact subset A, the set $M(A) = \bigcup_{x\in A} M(x)$ is compact. A set N is called *overflowing* if $N \subset f(N)$. We propose the following generalizations of PE.

Definition 1. Let $f : X \to X$ be a continuous map.

(1) f is *weakly positively expansive* if it has an expansivity neighborhood.
(2) A homeomorphism f is *strongly positively expansive* if it has a proper overflowing expansivity neighborhood.

Before proceeding with our discussion we define some technical terms that we use throughout the paper. Suppose $f : X \to X$ is a continuous map and $x \in X$. We define the *ω-limit set* of x to be

$$\omega(x) = \omega(x, f) = \bigcap_{N>0} \operatorname{cl}\biggl(\bigcup_{n>N} f^n(x)\biggr).$$

If f is a homeomorphism we define the *α-limit set* of x to be

$$\alpha(x) = \alpha(x, f) = \omega\bigl(x, f^{-1}\bigr) = \bigcap_{N>0} \operatorname{cl}\biggl(\bigcup_{n>N} f^{-n}(x)\biggr).$$

Thus y is in $\omega(x, f)$ when there exists a sequence k_i tending to infinity such that $f^{k_i}(x)$ tends to y. We say that y is in the *prolongation* of x, $\Omega f(x)$, if there is a sequence k_i tending to infinity and a sequence (x_i, y_i) tending to (x, y) such that $y_i = f^{k_i}(x_i)$ (see [1]). If f is a homeomorphism, then clearly y is in $\Omega f(x)$ if and only if x is in $\Omega f^{-1}(y)$. It is also clear that $\omega(x) \subset \Omega f(x)$ and $\alpha(x) \subset \Omega f^{-1}(x)$.

A set S is *invariant* if $f(S) = S$. Given a set N, let $\operatorname{Inv} N$ denote the maximal invariant subset of N. For continuous maps

$$\operatorname{Inv} N = \{x \in N\colon \exists \ldots, x_{-1}, x_0, x_1, \ldots \in N \text{ such that } x = x_0 \text{ and } f(x_k) = x_{k+1} \text{ for all } k \in \mathbb{Z}\}.$$

An invariant set S is called an *isolated invariant set* provided that it has a compact neighborhood N with $S = \operatorname{Inv} N$. The set N is called an *isolating neighborhood* for S. The set S is an *attractor* if there is an isolating neighborhood N for S with $f(N) \subset \operatorname{int}(N)$; in this case N is called an *attracting neighborhood*. Likewise, if f is a homeomorphism, then a set S is a *repeller* if it has a *repelling neighborhood*, an isolating neighborhood N for which $f^{-1}(N) \subset \operatorname{int}(N)$.

We denote the open ε-ball about a point x by $B_\varepsilon(x)$. The ε-neighborhood about a set A is given by $B_\varepsilon(A) = \bigcup_{x \in A} B_\varepsilon(x)$.

A continuous map $f : X \to X$ is *bounded* if there exists a compact set W with the property that for any $x \in X$ there is an $n \geqslant 0$ such that $f^n(x) \in W$. Such a set, W, is called a *window*. Obviously every dynamical system on a compact space X is bounded, but this is not so for noncompact spaces.

Below we state several properties that are equivalent to boundedness; the theorem was first proved, as far as we can ascertain, in [6], and was reproved in [14] (see [14–16] for applications of boundedness).

Theorem 2. [6,14] *If X is a locally compact metric space and $f : X \to X$ is a continuous map, then the following are equivalent*:

(1) *f is bounded.*
(2) *There is a compact set V such that $\emptyset \neq \omega(x) \subset V$ for all $x \in X$.*
(3) *There exists a forward invariant window.*
(4) *There is a compact global attractor Λ (that is, there is an attractor Λ with the property that $\emptyset \neq \omega(x) \subset \Lambda$ for every $x \in X$).*

A set $S \subset X$ is called *forward minimal* if it is a closed set with $f(S) \subset S$, such that no proper closed subset of S has this property. Notice that S is a forward minimal set if and only if $\omega(x) = S$ for every $x \in S$. A dynamical system on a space X is *forward minimal* if X is a forward minimal set. A short proof of the following corollary, which we use in Section 4, is found in [14].

Corollary 3. [3,7,9,14] *A noncompact, locally compact space does not admit a forward minimal dynamical system.*

3. General properties of PE, wPE, and sPE dynamical systems

In the previous section we introduced two topological generalizations of positive expansiveness: weak and strong positive expansiveness. In this section we show how these three properties are related. We will show that on a compact space the notions are the same, but that they are different on a noncompact space.

Our first theorem relates PE maps to wPE maps. It holds for both compact and noncompact spaces, although when the space is compact, we will have a stronger result (Theorem 5).

Theorem 4. *Let $f : X \to X$ be a continuous map on a locally compact metrizable space. Then f is wPE if and only if f is PE with respect to some metric compatible with the topology.*

Proof. First, suppose that f is PE with respect to the metric d, with expansive constant ρ. Then the set $V_\rho = \{(x, y)\colon d(x, y) \leqslant \rho\}$ is an expansivity neighborhood, so f is wPE.

Now suppose that f is wPE. Let $N \subset X \times X$ be an expansivity neighborhood. Pick any $\varepsilon > 0$. We will create a metric δ on X compatible with the topology such that for $(x, y) \notin N$, $\delta(x, y) > \varepsilon$. Then, since for every distinct pair of points $x, y \in X$, there is an $n \geqslant 0$ such that $(f^n(x), f^n(y)) \notin N$, we can conclude that f is PE with respect to δ, with expansive constant ε.

We may assume that N is proper (if not, since X is locally compact we may replace N by a subset of N with this property). Also, we may assume that N is symmetric about the diagonal; that is, $(x, y) \in N$ if and only if $(y, x) \in N$ (if not, replace N by $N \cap N'$, where $N' = \{(x, y) \in X \times X\colon (y, x) \in N\}$). Let $g : X \times X \to \mathbb{R}$ be a continuous function that is equal to ε on Δ and equal to 0 on $(X \times X)\setminus \mathrm{int}(N)$. Define the function $h : X \times X \to \mathbb{R}$ by $h(x, y) = g(x, y) + g(y, x)$. Then h is a continuous function satisfying the following properties:

(1) $h(x, x) = 2\varepsilon$ for all $(x, x) \in \Delta$,
(2) $h(x, y) = 0$ for all $(x, y) \notin N$, and
(3) $h(x, y) = h(y, x)$ for all $(x, y) \in X \times X$.

It is easy to check that $D((x, y), (z, w)) = d(x, z) + d(y, w) + |h(x, y) - h(z, w)|$ is a metric compatible with the topology of $X \times X$. Furthermore, for $(x, x) \in \Delta$ and $(z, w) \notin N$ we have $D((x, x), (z, w)) = d(x, z) + d(x, w) + |h(x, x) - h(z, w)| = d(x, z) + d(x, w) + 2\varepsilon > \varepsilon$.

Now, define a metric δ on X by $\delta(x, y) = \sup_{a\in X} D((a, x), (a, y))$. Note that $\delta(x, y)$ is actually $\max_{a\in S} D((a, x), (a, y))$, where the maximum is over the set $S = \{a \in X\colon (a, x) \in N \text{ or } (a, y) \in N\}$, since if neither (a, x) nor (a, y) is in N, then $D((a, x), (a, y)) = d(a, a) + d(x, y) + |h(a, x) + h(a, y)| = d(x, y) \leqslant D((b, x), (b, y))$ for any $b \in N$. Since N has compact cross sections, S is compact, so δ is a well-defined metric compatible with the topology of X.

Finally, for any $(x, y) \notin N$ we have $\delta(x, y) = \sup_{a\in X} d((a, x), (a, y)) \geqslant d((x, x), (x, y)) > \varepsilon$. □

The following theorem shows that all of our notions are equivalent when the space is compact.

Theorem 5. *Let X be a compact metrizable space and $f : X \to X$ be a continuous map. Then the following are equivalent*:

(1) *f is PE with respect to some compatible metric.*
(2) *f is PE with respect to every compatible metric.*
(3) *f is expanding with respect to some compatible metric.*
(4) *f is wPE.*

If f is a homeomorphism, these properties are also equivalent to:

(5) *f is sPE.*

Proof. The proof that (1), (2), and (3) are equivalent can be found in [2, pp. 40–44] (with the result (1) ⇒ (3) due to Reddy [13]). By Theorem 4 we know that (1) ⇔ (4).

Now, assume that f is a homeomorphism. By definition (5) ⇒ (4). So, it suffices to show that (3) ⇒ (5). Suppose f is expanding with respect to the metric d. Thus, there exist $\varepsilon > 0$ and $\lambda > 1$ such that whenever $d(x, y) < \varepsilon$, $d(f(x), f(y)) > \lambda d(x, y)$. Give $X \times X$ the taxicab metric, that is, $D((x, y), (z, w)) = d(x, z) + d(y, w)$. There is a $\delta > 0$ such that $F(B_\delta(\Delta)) \subset B_\varepsilon(\Delta)$. Let $M = \mathrm{cl}(B_\delta(\Delta))$. For each $z \in M\setminus\Delta$ there exists $n > 0$ such that $F^n(z) \notin M$, so M is a proper expansivity neighborhood. Since f is expanding and $B_\delta(\Delta) \subset F^{-1}(B_\varepsilon(\Delta))$, $M \subset F(M)$. Thus f is sPE. □

Observe that if f is PE, wPE, or sPE, then clearly the restriction of f to any closed invariant subset of X has the same property. We also have the following theorem that relates the properties of f to those of iterates of f.

Theorem 6. *Let X be a locally compact space, $f : X \to X$ a continuous map, and $n > 0$.*

(1) *f is wPE if and only if f^n is wPE.*
(2) *f is sPE if and only if f^n is sPE* (*if f is a homeomorphism*).
(3) *If X is a compact metric space, then f is PE if and only if f^n is PE.*
(4) *If X is a noncompact metrizable space, then f is PE with respect to some metric if and only if f^n is PE with respect to some* (*possibly different*) *metric.*

Proof. (1) Suppose f is wPE with expansivity neighborhood N. Consider the set $N_n := \bigcap_{i=0}^{n-1} F^{-i}(N)$. Clearly N_n is a closed neighborhood of Δ. Let $(x, y) \in N_n\setminus\Delta$. Since $N_n \subset N$, there is an $m > 0$ such that $F^m(x, y) \notin N$.

By construction, $F^{m-k}(x,y) \notin N_n$ for all $0 \leqslant k < n$. We can write $m = qn + p$ where $q \geqslant 0$ and $0 \leqslant p < n$. So, $(F^n)^q(x,y) = F^{m-p}(x,y) \notin N_n$. Thus N_n is an expansivity neighborhood for f^n, and f^n is wPE.

Conversely, it is clear that if f^n is wPE with an expansivity neighborhood N, then N is also an expansivity neighborhood for f.

(2) Let f is be a homeomorphism. If f is sPE, it is clear that f^n is sPE. Conversely, suppose f^n is sPE. Then there is a proper overflowing expansivity neighborhood N for f^n. Let $\tilde{N} = \bigcup_{i=0}^{n-1} F^i(N)$. Clearly $\tilde{N}$ is a proper expansivity neighborhood for f. We will show that it is overflowing. We see that $F(\tilde{N}) = \bigcup_{i=1}^{n} F^i(N) = \bigcup_{i=0}^{n} F^i(N) = \tilde{N} \cup F^n(N) \supset \tilde{N}$, where the second equality holds because $N \subset F^n(N)$. Thus, f is sPE.

(3) See [2, p. 40].

(4) This part follows from part (1) and Theorem 4. □

In Theorem 5 and in Theorem 6(3) we obtain very strong results when we assume the underlying space is compact. The reason is that in this case PE is a dynamical property. The same is not true for noncompact spaces. We conclude this section with examples that show that the compactness assumptions are crucial. First, we see that a map may be PE with respect to one metric, but not another compatible metric. Thus, it is not a dynamical property.

Example 7. Consider the homeomorphism $f:\mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$. Clearly f is PE with respect to the usual metric. However, if we give $\mathbb{R}$ the metric inherited from $S^1 \backslash \{\text{north pole}\}$, then f is no longer PE.

It follows from the definitions that on any space an sPE homeomorphism is wPE. In the next section we show that the converse is not true. In Examples 20 and 21 we present homeomorphisms (on noncompact spaces) that are wPE (and PE), but not sPE.

We know from Theorem 4 that a homeomorphism that is sPE or wPE is PE with respect to some metric. In the following example we see that such a homeomorphism may not be PE with respect to every metric.

Example 8. Let $T:\mathbb{R} \to \mathbb{R}$ be the translation $T(t) = t + 1$. Clearly this function is not PE with respect to the standard metric. However, it is wPE and sPE. The neighborhood $N = \{(t_1, t_2) \in \mathbb{R}^2\colon\ |t_1 - t_2| \leqslant e^{-t_1}\}$ is a proper overflowing expansivity neighborhood. In fact, we can generalize this example. If Y is any locally compact space, then the translation map $f = \mathrm{Id}_Y \times T$ on $Y \times \mathbb{R}$ is sPE. To see this, take M to be a proper neighborhood of the diagonal in $Y \times Y$. Then the set $\{((y_1, t_1), (y_2, t_2))\colon\ (y_1, y_2) \in M,\ d(y_1, y_2) \leqslant e^{-t_1},\ |t_1 - t_2| \leqslant e^{-t_1}\}$ is a proper overflowing expansivity neighborhood. Furthermore, since T is clearly conjugate to T^{-1}, it follows that f^{-1} is sPE as well.

Finally we show that the Theorem 6(3) does not hold for noncompact spaces. That is, we give an example of a map f and a metric such that f is PE, but f^k $(k > 0)$ is not.

Example 9. Consider the following subsets of $\mathbb{R}^2$ (with the usual metric).

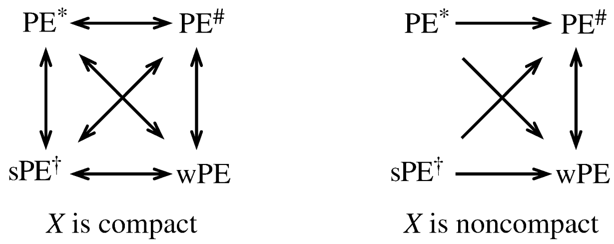
$$X_1 = \{(n, 0)\colon\ n \in \mathbb{Z},\ n \neq 0\},$$

$$X_2 = \{(n, n)\colon\ n \in \mathbb{Z},\ n \text{ odd}\} \cup \{(n, 1/n)\colon\ 0 \neq n \in \mathbb{Z},\ n \text{ even}\}.$$

Let $X = X_1 \cup X_2$ and define $f: X \to X$ to be the function that shifts elements to the right. That is, for $(n, 0) \in X_1$, $f(n, 0) = (n + 1, 0)$, except $f(-1, 0) = (1, 0)$. In X_2 we have $f(n, n) = (n + 1, 1/(n + 1))$ for n odd, except $f(-1, -1) = (1, 1)$, and $f(n, 1/n) = (n + 1, n + 1)$ for n even. Then f is PE, but f^2 is not (for n even $d(f^{2i}(n, 1/n), f^{2i}(n, 0)) \to 0$ as $i \to \infty$).

Better still, Bryant and Coleman give a pathological example of a homeomorphism $f:[0, \infty) \to [0, \infty)$ with the property that f is PE, but f^k is not PE for any $k > 1$ [4]. By acting by f on each ray from the origin we obtain a homeomorphism $g:\mathbb{R}^n \to \mathbb{R}^n$ with the same property.

We conclude this discussion with Fig. 1, which presents the relationships between PE, sPE, and wPE on both compact and noncompact metrizable spaces.



*PE with respect to **any** metric compatible with the topology
#PE with respect to **some** metric compatible with the topology
†Homeomorphisms only

Fig. 1.

4. Homeomorphisms

In this section we restrict our attention to homeomorphisms. From Theorem 5 we know that on a compact space all three notions, PE, wPE, and sPE, are equivalent. In fact, as the following theorem states, most compact spaces do not admit such a dynamical system (see [16] for a proof and for further references).

Theorem 10. [10,8,16] *Let X be a compact metric space. A homeomorphism $f : X \to X$ is PE (or wPE or sPE) if and only if X is finite.*

Before investigating noncompact spaces we prove a few corollaries of Theorem 10, the first of which we will use later.

Corollary 11. *Suppose X is a compact metrizable space and $f : X \to X$ is a PE map such that $f : X \to f(X)$ is a homeomorphism. Then X is finite and hence $f : X \to X$ is a homeomorphism.*

Proof. The set $\Lambda = \bigcap_{n \geqslant 0} f^n(X)$ is a compact global attractor. Then $f : \Lambda \to \Lambda$ is a PE homeomorphism of a compact set. By Theorem 10, Λ is finite. So, for some large m each point of Λ is a fixed point for $g = f^m$. By Theorem 6, $g : X \to X$ is PE. Let $x \in X$. Since $\omega(x, g) \subset \Lambda$, and Λ consists of a finite number of fixed points, $\omega(x, g)$ is a single fixed point. Since g is PE, $x = \omega(x, g)$, and it follows that $X = \Lambda$. □

There is a long history of trying to determine which spaces admit forward minimal (invertible or noninvertible) dynamical systems (see [11] for references and results). Kolyada, Snoha, and Trofimchuk proved that if X is a compact Hausdorff space and $f : X \to X$ is a continuous open map that is forward minimal, then f must be a homeomorphism [11, Theorem 2.4]. In the following corollary we see that for wPE maps on manifolds, the possibilities are even more limited.

Corollary 12. *If M is a topological manifold without boundary and $f : M \to M$ is a continuous map that is both forward minimal and wPE (or PE with respect to some compatible metric), then M is a periodic orbit.*

Proof. By Corollary 3, M must be compact. Thus, this corollary will follow from Theorem 10 provided we can show that f is a homeomorphism. First we will show that f is locally one-to-one; i.e., that each point $x \in X$ has an open neighborhood U_x such that $f|_{U_x}$ is one-to-one. Let N be an expansivity neighborhood for f. Take U_x to be any open neighborhood of x such that $U_x \times U_x \subset N$. If $y, z \in U_x$ and $y \neq z$, then $(y, z) \in N \backslash \Delta$. Thus, $F(y, z) \notin \Delta$, or equivalently, $f(y) \neq f(z)$.

Next, Brouwer's invariance of domain theorem [12, p. 206] tells us that since f is locally one-to-one, it is an open mapping. Thus the result of Kolyada, Snoha, and Trofimchuk cited above [11, Theorem 2.4] implies that f is a homeomorphism, so M is a periodic orbit. □

Proposition 13. *Let X be a locally compact metrizable space and suppose $f : X \to X$ is a wPE homeomorphism (equivalently, f is PE with respect to some compatible metric). Then every compact positively invariant set is a finite collection of repelling periodic orbits.*

Proof. Choose a metric such that $f : X \to X$ is PE with expansive constant ρ, and let $\Lambda \subset X$ be a compact, positively invariant set. Since $f : \Lambda \to \Lambda$ is PE, Corollary 11 states that Λ is finite. Thus, to prove the proposition it suffices to show that each periodic orbit is repelling.

Suppose Λ is a single periodic orbit (the general case is proved similarly). Let ε be a constant such that $0 < \varepsilon < \rho$ and such that $f(B_\varepsilon(x)) \cap B_\varepsilon(y) = \emptyset$ for all $y \in \Lambda \backslash \{f(x)\}$. Let $U = \bigcup_{x \in \Lambda} B_\varepsilon(x)$. Since f is PE, the only orbit that remains in U for all forward time is the periodic orbit. That is, for each $x \in U \backslash \Lambda$ there is an $m > 0$ such that $f^m(x) \notin U$.

Let X^* be the one-point compactification of X and let $Y = X^* \backslash \Lambda$. Then f extends to a homeomorphism on the noncompact space Y and, by the discussion above, $W = Y \backslash U$ is a window for $f : Y \to Y$. By Theorem 2, there is a window W_0 such that $f(W_0) \subset \operatorname{int}(W_0)$. Thus, the set $N = X^* \backslash \operatorname{int}(W_0)$ is a repelling neighborhood for Λ. □

We know from Theorem 10 that being PE is a strong restriction for homeomorphisms on compact spaces. The following theorem shows that this is also true on noncompact spaces.

Theorem 14. *Let X be a locally compact metrizable space and let $f : X \to X$ be a wPE homeomorphism* (*equivalently, f is PE with respect to some compatible metric*). *Then, for each $x \in X$ one of the following holds*:

(1) *x is a repelling periodic point,*
(2) *$\omega(x) = \emptyset$, or*
(3) *$\omega(x)$ is noncompact.*

If X has no isolated points, then the set of periodic points in X is nowhere dense.

Proof. Let $f : X \to X$ be a PE homeomorphism of a locally compact metrizable space. By Theorem 10 the theorem holds for compact spaces, so assume that X is noncompact. Suppose $\omega(x)$ is nonempty and compact for some $x \in X$. It suffices to show that $\omega(x)$ is a repelling periodic orbit. Consider the set

$$Y = \{f^n(x) \colon n \geqslant 0\} \cup \omega(x).$$

Y is compact and $f : Y \to f(Y) \subset Y$ is a PE homeomorphism onto its image. By Corollary 11, Y is finite, thus x is periodic, and by Proposition 13 it is repelling. From this it is easy to conclude that if X has no isolated points, then the set of periodic points is nowhere dense. □

The following corollary follows from Theorems 10, 2, and 14.

Corollary 15. *Let X be an infinite, locally compact metrizable space, and $f : X \to X$ be a homeomorphism. If f is bounded, then it is not PE with respect to any compatible metric.*

We now investigate sPE homeomorphisms. Define an f-invariant equivalence relation $\sim$ on X by setting $x \sim y$ if $\alpha(x) = \alpha(y)$. Let $E_x = \{y \in X \colon \alpha(x) = \alpha(y)\}$ denote the equivalence class containing x. Let $E_0 \subset X$ denote the equivalence class consisting of all points x with $\alpha(x) = \emptyset$.

In order to understand the dynamics of sPE homeomorphisms it suffices to understand the dynamics on each E_x. In Propositions 16 and 17 we describe some basic properties of α-limit sets and the associated equivalence classes. Then, in Theorems 18 and 19 we classify all sPE homeomorphisms. Theorem 18 describes the dynamics on the equivalence classes with nonempty α-limit sets, and Theorem 19 describes the dynamics on the equivalence classes with empty α-limit sets.

Proposition 16. *Let $f : X \to X$ be an sPE homeomorphism. Then*

(1) *each equivalence class E_x is closed and open,*
(2) *for any $x \in X$, $\alpha(x) = \Omega f^{-1}(x)$, and*
(3) *if N is a proper overflowing expansivity neighborhood and $(x, y) \in \operatorname{int}(N)$, then $\alpha(x) = \alpha(y)$.*

Proof. First we will show that if N is a proper overflowing expansivity neighborhood and $(x, y) \in \mathrm{int}(N)$, then $\Omega f^{-1}(x) \subset \alpha(y)$. Let $z \in \Omega f^{-1}(x)$. Then there are sequences $\{k_i\}_{i=1}^{\infty}$ tending to infinity and $\{x_i\}_{i=1}^{\infty}$ tending to x such that $\lim_{i\to\infty} f^{-k_i}(x_i) = z$. Let A be the compact set $\{f^{-k_i}(x_i)\}_{i=1}^{\infty} \cup \{z\}$. We will show that $f^{-k_i}(y)$ converges to z, which implies that z is in $\alpha(y)$.

Since $N \subset F(N)$, it follows that $F^{-n}(N) \subset N$ for every $n > 0$, and since N is an expansivity neighborhood, $\Omega F^{-1}(x, y) \subset \Delta$. We can assume that for all i, the point (x_i, y) is in N, so that the sequence $\{f^{-k_i}(y)\}_{i=1}^{\infty}$ is contained in the compact set $N(A)$. If w is a limit point of $\{f^{-k_i}(y)\}_{i=1}^{\infty}$, then (z, w) is in $\Omega F^{-1}(x, y)$, which is contained in the diagonal, so $w = z$. Thus, $\Omega f^{-1}(x) \subset \alpha(y)$.

Finally, recall that for any x, $\alpha(x) \subset \Omega f^{-1}(x)$. So, for any $(x, y) \in \mathrm{int}(N)$ we have $\alpha(x) \subset \Omega f^{-1}(x) \subset \alpha(y) \subset \Omega f^{-1}(y) \subset \alpha(x)$. Thus all of these sets are identical. This proves (2) and (3).

We may assume that N is a symmetric, proper, overflowing expansivity neighborhood. Then by (3), $\mathrm{int}(N(x)) \subset E_x$. Thus the equivalence class is open and, since its complement is the union of the other equivalence classes, it is also closed. □

Proposition 17. *Let $f : X \to X$ be an sPE homeomorphism, and let E_z be the equivalence class with $A = \alpha(z)$. If $x \in E_z \setminus A$ then $\omega(x) = \emptyset$. If $A \neq \emptyset$ then A is a single repelling periodic orbit and E_z is exactly its basin of repulsion.*

Proof. Suppose $x \in E_z$ and $\omega(x) \neq \emptyset$. Let $y \in \omega(x) \subset \Omega f(x)$. Then $x \in \Omega f^{-1}(y) = \alpha(y)$. Since E_z is closed, $y \in E_z$, and so $\alpha(y) = A$. Thus, if $\omega(x) \neq \emptyset$, then $x \in A$.

Next, assume that $A = \alpha(z)$ is nonempty. For the sake of contradiction, assume that $\alpha(z)$ is infinite. Since $\alpha(z)$ is closed, it is locally compact and, by Theorem 10, it is noncompact. We claim that $\alpha(z)$ is backward minimal (that is, forward minimal for f^{-1}), thereby contradicting Corollary 3. It suffices to show that for any two points $x, y \in \alpha(z)$, x is a limit point of the f^{-1}-orbit of y. Because $y \in \alpha(z)$, there is a $k \geqslant 0$ such that $(f^{-k}(z), y) \in \mathrm{int}(N)$. By Proposition 16, $\alpha(y) = \alpha(f^{-k}(z)) = \alpha(z)$. Thus, $x \in \alpha(y)$, and we have a contradiction. Thus $\alpha(z)$ must be finite. Theorem 14 implies that every periodic orbit is repelling, which completes the proof. □

Theorem 18. *Let $f : X \to X$ be a homeomorphism of a locally compact metrizable space. The following are equivalent*:

(1) *f is wPE and f^{-1} is bounded.*
(2) *f has finitely many repelling periodic orbits and X is the union of their repelling basins.*
(3) *f is sPE with finitely many equivalence classes and with the null class E_0 empty.*

Proof. (1) ⇒ (2). Suppose that f is wPE and f^{-1} is bounded. Since f^{-1} is bounded, it has a compact global attractor Λ. By Theorem 10, Λ must be finite, and thus is a finite collection of attracting (with respect to f^{-1}) periodic orbits.

(2) ⇒ (3). We must construct a proper overflowing expansivity neighborhood N. Let $P \subset X$ be the set of periodic points of f. Then $\tilde{P} = \{(p, p)\colon p \in P\}$ is a set of repelling periodic orbits for F. We begin by constructing a compact F-repelling neighborhood N_0 for $\tilde{P}$ with the property that for some $\varepsilon > 0$, the set $\{(x, y)\colon x \in \pi_1(N_0),\ d(x, y) \leqslant \varepsilon\}$ is contained in N_0. Let N' be any compact F-repelling neighborhood for $\tilde{P}$. Then $R' = \pi_1(N')$ is an f-repelling neighborhood for P, and so is $R = f^{-1}(R')$. Define the set $N_0 = \pi_1^{-1}(R) \cap N'$. It is easy to see that N_0 is an F-repelling neighborhood for $\tilde{P}$, and, since $R = f^{-1}(R') \subset \mathrm{int}(R')$, it has the desired property.

Next, for $i = 1, 2, \ldots$, define the compact set $N_i \subset X \times X$ by

$$N_i = \left\{F^i(x, y)\colon x \in \mathrm{cl}\big(R \setminus f^{-1}(R)\big),\ d(x, y) \leqslant \frac{\varepsilon}{i}\right\}.$$

Finally, define the set $N \subset X \times X$ by $N = \bigcup_{i=0}^{\infty} N_i$.

We claim that N is a proper expansivity neighborhood. First, we show that it is a neighborhood of Δ. For any $x \in X$, there is a least nonnegative integer n_x such that $x \in f^{n_x}(R)$. If $x \in \mathrm{int}(f^{n_x}(R))$, then $(x, x) \in \mathrm{int}(N_{n_x})$. If $x \in \mathrm{bd}(f^{n_x}(R))$, then $(x, x) \in \mathrm{int}(N_{n_x} \cup N_{n_x+1})$. Thus N is a neighborhood of Δ.

Next, observe that any compact set in $X \times X$ intersects only finitely many of the N_i's (because any compact set in X is contained in $f^n(R)$ for some n). Thus N is closed and proper. Furthermore, $N_0 \subset F(N_0)$ since N_0 is a repelling neighborhood, and $N_{i+1} \subset F(N_i)$ by definition. Thus N is overflowing.

Finally, we must show that if $F^i(x, y) \subset N$ for all $i \geqslant 0$, then $x = y$. Assume that $(x, y) \in N_k$, so that $F^i(x, y) \subset N_{k+i}$ for all $i \geqslant 0$. Then $d(f^{-k}(x), f^{-k}(y)) \leqslant \frac{\varepsilon}{i}$ for all i, so $x = y$.

It is clear that there are finitely many equivalence classes and that E_0 is empty.

(3) $\Rightarrow$ (1). Since f is sPE, it is certainly wPE. The union over all $x \in X$ of $\alpha(x)$ is a compact (in fact, finite) global attractor for f^{-1}. □

To complete the classification of sPE homeomorphisms, we consider the case $E_0 = X$.

Theorem 19. *Let $f : X \to X$ be a homeomorphism of a locally compact metrizable space. The following are equivalent*:

(1) *f is sPE and $\alpha(x) = \emptyset$ for every $x \in X$.*
(2) *$\Omega f(x) = \emptyset$ for all $x \in X$.*
(3) *There exist a space Y and a set $Z \subset Y \times \mathbb{R}$ such that $f : X \to X$ is topologically conjugate to the translation map $(\mathrm{Id}_Y \times T) : Z \to Z$, where T is the translation $T(t) = t + 1$.*
(4) *f and f^{-1} are both sPE.*

Proof. (1) $\Rightarrow$ (2). Recall that y is in $\Omega f(x)$ if and only if x is in $\Omega f^{-1}(y)$. Thus, since $\Omega f^{-1}(y) = \alpha(y) = \emptyset$ for all y, $\Omega f(x) = \emptyset$ for all x.

(2) $\Rightarrow$ (3). Antosiewicz and Dugundji showed [5] that if ϕ is a flow on X' and $\Omega\phi(x) = \emptyset$ for all $x \in X'$, then there is a space Y so that ϕ is topologically conjugate to the translation flow on $Y \times \mathbb{R}$. To get our result, we apply their theorem to the suspension flow of f. Then Z is the section in $Y \times \mathbb{R}$ corresponding to X in the suspension space. In order to avoid the lengthy definitions for flows, we omit the details.

(3) $\Rightarrow$ (4). If Y is locally compact, then this follows from Example 8. Even if Y is not locally compact, Z is locally compact, and this is sufficient for the argument in Example 8 to go through.

(4) $\Rightarrow$ (1). If $\alpha(x) \neq \emptyset$, then the restriction of f^{-1} to the equivalence class E_x is bounded. But the restriction is still sPE, and so by Corollary 15 cannot be bounded. □

The previous theorems completely describe the dynamics of sPE homeomorphisms. Given any sPE homeomorphism $f : X \to X$, we can write X as the (possibly infinite) union of equivalence classes that are closed, open, and f-invariant: $X = \bigcup E_{x_\alpha}$. On each E_{x_α} the dynamics can have one of two forms. One possibility is that E_{x_α} is simply a single repelling periodic orbit and its basin of repulsion (Theorem 18). The other possibility is that every point in E_{x_α} has no recurrent dynamics in either forward or backward time and is roughly the same as a translation (as described in Theorem 19).

We conclude with two examples. In both examples we have homeomorphisms that are PE with respect to the usual metric (and are thus wPE). However, because they do not satisfy the properties in Proposition 17 they cannot be sPE.

Example 20. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the time-one map of the flow shown in Fig. 2. The map has a repelling fixed point, p, and an invariant line, l. We may assume that the homeomorphism restricted to l moves points to the right at an exponential rate. Thus, f is PE with respect to the usual metric.

Notice that $\omega(x) \subset l$ is infinite and noncompact for any point in the open upper half plane except $x = p$ (in fact, we may arrange the spiraling so that $\omega(x) = l$) and $\omega(x) = \emptyset$ for any point in the closed lower half plane. Also $\alpha(x) = \emptyset$ or $\alpha(x) = \{p\}$ for every $x \in \mathbb{R}^2$.

Also, notice that although f is PE, f is not sPE. This fact follows from Proposition 17 and the observation that there are nonperiodic points with nonempty ω-limit sets.

Example 21. In this example we construct a homeomorphism on the punctured torus. We begin with a vector $\mathbf{v} \in \mathbb{R}^2$ with irrational slope and let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be translation by $\mathbf{v}$. Let $g : T \to T$ be the associated map on the torus; that is, $g(x) = \pi \circ h(y)$ where $\pi : \mathbb{R}^2 \to T$ is the usual projection and $y \in \pi^{-1}(x)$. Modify g by slowing down the rotation

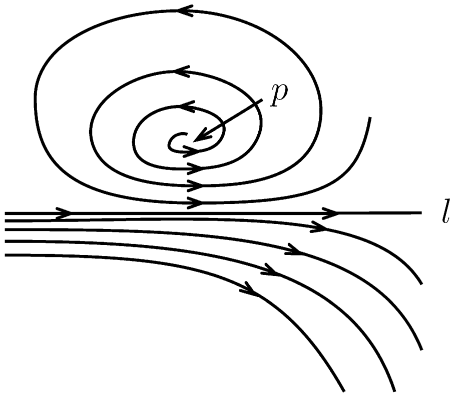


Fig. 2. Nonempty, noncompact ω-limit set.

near the origin so that the origin becomes a fixed point. We may now remove this fixed point to obtain the noncompact space $X = T\backslash\{\mathbf{0}\}$ and a homeomorphism $f : X \to X$. Finally, we may alter the metric on X by allowing it to "blow up" at the puncture. That is, distances on the rays emanating from the puncture go to infinity as one approaches the hole. Visually we may think of this as the metric inherited from $\mathbb{R}^3$ if the puncture is pulled off to infinity. With this metric f is PE. Observe that f satisfies all of the properties given in Theorem 14. Every point $x \in T$ has a dense orbit, however, there are three different behaviors. If $x \neq \pi(\lambda\mathbf{v})$ for any $\lambda \in \mathbb{R}$ then $\alpha(x) = X$ and $\omega(x) = X$. If $x = \pi(\lambda\mathbf{v})$ and $\lambda > 0$ then $\alpha(x) = \emptyset$ and $\omega(x) = X$, and if $x = \pi(\lambda\mathbf{v})$ with $\lambda < 0$ then $\alpha(x) = X$ and $\omega(x) = \emptyset$. Since f does not satisfy the properties given in Proposition 17 we know that it is not sPE.

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