ENTROPY FOR SYMBOLIC DYNAMICS WITH OVERLAPPING ALPHABETS

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Abstract. We consider shift spaces in which elements of the alphabet may overlap nontransitively. We define a notion of entropy for such spaces and show that it is equal to a limit of entropies of standard (non-overlapping) shifts when the underlying shift is of finite type. When a shift space with overlaps arises as a model for a discrete dynamical system with a finite set of overlapping neighborhoods, the entropy gives a lower bound for the topological entropy of the dynamical system.

1. Introduction

There is a long history of using symbolic dynamics to model more complicated dynamical systems. Suppose, for example, that we want to understand the dynamics of a continuous dynamical system $f : X \to X$ on a compact metric space X. If we have compact sets $N_1, \ldots, N_r \subset X$ and an f-orbit $(x, f(x), f^2(x), \ldots)$ that remains in $\bigcup_i N_i$, then there is a symbol sequence $(i_0, i_1, \ldots) \in \{1, \ldots, r\}^{\mathbb{N}}$ called an *itinerary* such that $f^j(x) \in N_{i_j}$ for all $j \geq 0$ (the itinerary may not be unique). We can then use the properties of the set of itineraries (a shift space) to tell us about the dynamics of f.

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One example of this approach is a Markov partition, in which the sets $N_1, \ldots, N_r \subset X$ cover the entire space, overlap only on their boundaries, and map across each other in topologically simple ways ([19, §9.6],[1]). In this case the resulting shift space of itineraries is a subshift of finite type. Another example is when the N_i's are pairwise disjoint Conley index pairs, resulting in a cocyclic subshift ([13, 14, 21]).

While these approaches have been very fruitful, there are limitations with each of them. Markov partitions can be very difficult to construct in practice, and the Conley index pairs must be disjoint, making it impossible to cover a connected space.

In this article we present an approach that enables us to cover more of the topological space: we allow the N_i's to have nontrivial intersections. It is well understood how to extract useful dynamical information about f from the shift space when the N_i's are disjoint, but these methods cannot be directly applied in our setting. The complication that arises is that if the N_i's intersect, then an orbit may have multiple itineraries. For example, a fixed point in $N_1 \cap N_2$ could have itineraries $(1, 1, 1, \ldots)$, or $(1, 2, 1, 2, \ldots)$, or any sequence of 1's and 2's. Hence, we cannot obtain a lower bound for the entropy of f in the usual way: by computing the growth rate of the number of words of length n in the shift space.

In this paper we show how to obtain meaningful dynamical information about f when the neighborhoods intersect. In particular, we define a notion of entropy for the space of itineraries that gives a lower bound for the topological entropy of f and show that when the underlying shift is of finite type the entropy is equal to a limit of the entropies of standard (non-overlapping) shifts.

We begin with the definition of our object of study—a shift space with overlaps.

Definition 1. A *shift space with overlaps* is a pair $(\Sigma, \mathcal{I})$ in which Σ is a one-sided shift space on the alphabet $\{1, \ldots, r\}$ and $\mathcal{I}$ is a simple graph with vertex set $\{1, \ldots, r\}$. We call $\mathcal{I}$ an *intersection graph.* Two words $(a_0, a_1, \ldots), (b_0, b_1, \ldots) \in \Sigma$ (finite or infinite) are *indistinguishable* iff there is an $\mathcal{I}$-edge between the vertices a_i and b_i for all i.

Shift spaces with overlaps are interesting objects to study on their own, but most often they are a model for an existing dynamical system. Thus we have the following definition.

Definition 2. A compact metric space X, a continuous map $f : X \to X$, and a collection of nonempty compact sets $N_1, \ldots, N_r \subset X$ is called a *dynamical realization* of $(\Sigma, \mathcal{I})$ provided:

(1) If $N_i \cap N_j \neq \emptyset$ and $i \neq j$, then there is an edge from i to j in $\mathcal{I}$.

(2) If $(a_0, a_1, \ldots)$ is a (finite or infinite) word in Σ, then there is a point $x \in X$ such that $f^i(x) \in \operatorname{Int}(N_{a_i})$ for all i.

In practice, we use topological methods to verify that (2) holds. The methods may be elementary, such as boxes that stretch across other boxes, as in the case of Markov partitions or their topological generalizations shown in Figure 1. Or they may be more sophisticated; for example, in [18] we use techniques from Conley index theory to verify the property.

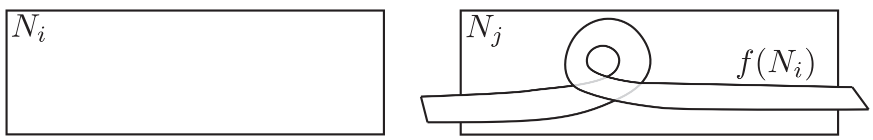

FIGURE 1

For much of this article we assume that Σ is a subshift of finite type. When Σ is the vertex shift associated to a transition graph $\mathcal{T}$ we superimpose the graphs $\mathcal{T}$ and $\mathcal{I}$ with the edges of $\mathcal{T}$ being solid arrows and the edges of $\mathcal{I}$ dashed segments; we will refer to this merged graph as $\mathcal{TI}$. The graph $\mathcal{T}$ has an associated *transition matrix*, $A_{\mathcal{T}}$, in which the (i, j)-entry is 1 if there is an edge from the ith vertex to the jth vertex and is 0 otherwise.

Example 3. Let S^1 be the circle, viewed as $\mathbb{R}/\mathbb{Z}$ and let $f : S^1 \to S^1$ be a map that is C^0-close to the doubling map ($x \mapsto 2x$). Let $N_1 = [-0.1, 0.35]$, $N_2 = [0.15, 0.6]$, $N_3 = [0.4, 0.85]$, and $N_4 = [0.65, 1.1]$. The transition matrix and the graph $\mathcal{TI}$ are shown in Figure 2. The map f is a dynamical realization of $(\Sigma, \mathcal{I})$. We will return to this example in the subsequent sections.

$$A_{\mathcal{T}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

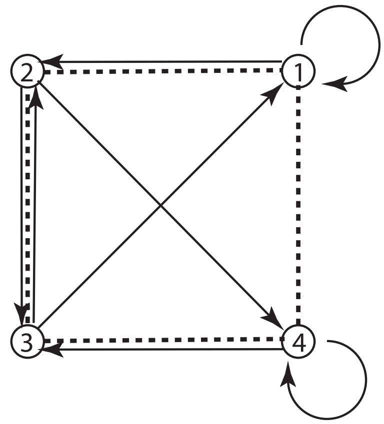

FIGURE 2

We are using $(\Sigma, \mathcal{I})$ as an abstract model of our dynamical system $f : X \to X$. As such, two indistinguishable elements in Σ could be itineraries of the same point in X. Thus in a sense we must treat two indistinguishable elements in Σ as the same point. However, what makes this scenario interesting is that indistinguishability is not a transitive property, and is thus not an equivalence relation. In general, $(\Sigma, \mathcal{I})$ is not a topological space. (It is a tolerance space—see, for example, [20].)

Our goal is to extract information about the dynamical complexity of f from $(\Sigma, \mathcal{I})$, and most often in this article, from the graph $\mathcal{TI}$. We define a notion of topological entropy for shifts with overlap, $h_{\mathcal{I}}(\Sigma)$, in Section 2 and show that it is a lower bound for the entropy of any dynamical realization f. We provide background information and definitions in Section 3. In Section 4 we prove that in the case of shifts of finite type, $h_{\mathcal{I}}(\Sigma)$ can be computed as a limit of the entropies of ordinary (nonoverlapping) embedded shifts.

Finally, in Section 5 we discuss the relationship between shifts of finite type with overlap and sofic shifts.

One of the motivations for undertaking this study is its potential application to computational dynamical systems. There is an active research program of using the Conley index and computational topology to give rigorous entropy bounds for dynamical systems. (An incomplete list of references to Conley index theory and its applications to symbolic dynamics and computation is [2, 4–12, 16, 17, 22–28].) In particular, one can use rigorous computation to obtain an index system (as described in [18]) and from that a shift space with overlaps. In order to make use of the resulting information, it is necessary to understand the dynamics of shifts with overlaps.

2. Topological entropy of shift spaces with overlap

We are primarily interested in using $(\Sigma, \mathcal{I})$ to help us obtain a lower bound for the entropy of the dynamical realization f. Since $(\Sigma, \mathcal{I})$ is not a topological space, we cannot apply the usual definition of topological entropy. Even if we could, it might overstate the complexity of f, since multiple itineraries might represent the same f-orbit. In this section, we define a notion of entropy for shifts with overlap, and show that it gives a lower bound for the entropy of f (Theorem 4).

If we ignore the intersections, then Σ is a shift space and we can compute $h(\Sigma)$ by computing the growth rate of the number of words of length n as n goes to infinity ([15, §4.1]). Let $\mathcal{B}_n(\Sigma)$ be the set of words of length n; then the topological entropy of Σ is

$$h(\Sigma) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{B}_n(\Sigma)|.$$

Recall that if the shift space Σ is a subshift of finite type with transition graph $\mathcal{T}$, then there is a particularly easy way to compute the entropy:

$$h(\Sigma) = \log(\lambda),$$

where λ is the Perron eigenvalue of $A_{\mathcal{T}}$—the unique largest real eigenvalue of $A_{\mathcal{T}}$.

In our setting, we want to count the number of words of length n that are mutually distinguishable, which in general is less than $|\mathcal{B}_n(\Sigma)|$. (For example, if vertices 1 and 2 overlap, then we would not be able to distinguish the words $(1,3)$ and $(2,3)$.) Let B be a set of words in Σ of length n. We say that B is *n-separated* if no two words in B are indistinguishable. Let $\mathcal{B}_n^{\text{sep}}(\Sigma,\mathcal{I})$ be an n-separated set of maximum cardinality. We define $h_{\mathcal{I}}(\Sigma)$, the *topological entropy of a shift space with overlaps*, to be

$$h_{\mathcal{I}}(\Sigma) = \limsup_{n\to\infty} \frac{1}{n} \log |\mathcal{B}_n^{\text{sep}}(\Sigma,\mathcal{I})|.$$

The following theorem provides the motivation for studying the entropy of shifts with overlaps.

Theorem 4. *If f is a dynamical realization of $(\Sigma,\mathcal{I})$, then $h(f) \geq h_{\mathcal{I}}(\Sigma)$.*

Proof. Let $\varepsilon = \min d(N_i, N_j)$, where the minimum is over all disjoint pairs N_i and N_j. Then an n-separated word in Σ corresponds to an (n,ε)-separated orbit for f. Since $h(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log r(n,\varepsilon)$, where $r(n,\varepsilon)$ is the maximum cardinality of a set of (n,ε)-separated orbits ([19, §8.1]), the result follows. □

3. Background and definitions

3.1. **Graph theory.** In this section we review some basic definitions from graph theory and linear algebra. (See [15, ch. 4] and linear algebra and graph theory texts for more details.)

3.1.1. *Independent vertex sets.* Let $V(\mathcal{G})$ and $E(\mathcal{G})$ denote the vertex and edge sets of a graph $\mathcal{G}$, respectively. Given a (nondirected) graph $\mathcal{G}$, a subset $V \subset V(\mathcal{G})$ is *independent* if there are no edges between any pair of vertices in V. The maximum cardinality of an independent subset of $\mathcal{G}$ is the *independence number* of $\mathcal{G}$ and is denoted $\mathrm{ind}(\mathcal{G})$.

3.1.2. *Irreducibility and primitivity.* Let $\mathcal{G}$ be a directed graph. A *vertex path* $(a_0, a_1, \ldots, a_{l-1})$ of length l is a sequence of vertices in $\mathcal{G}$ such that there is an edge from vertex a_i to a_{i+1} for each i, $0 \le i \le l-2$. In what follows, we assume that $\mathcal{G}$ has r vertices and no parallel edges and that every vertex has at least one edge leaving it and one edge entering it. Then the adjacency matrix for $\mathcal{G}$, $A_{\mathcal{G}}$, is an $r \times r$ (0,1)-matrix with no row or column containing all zeros. Notice that for $k > 1$ the matrix $A_{\mathcal{G}}^k$ need not be a $(0,1)$-matrix. However, if the (i,j)-entry of $A_{\mathcal{G}}^k$ is nonzero, then there exists a vertex path of length $k+1$ from vertex i to vertex j (actually, the value of the (i, j) gives the number of such vertex paths).

A graph is *irreducible* provided there is a vertex path between any two vertices, and a matrix is *irreducible* if it is the adjacency matrix for an irreducible graph. It is always possible to decompose a graph into its irreducible components; the corresponding matrix is (after rearranging the order of the vertices, or equivalently, conjugating by a permutation matrix) in block-triangular form with each of the blocks on the diagonal the adjacency matrix for one of the irreducible components of the graph.

An irreducible graph $\mathcal{G}$ may exhibit cyclic behavior. In particular for each vertex i,

$$\mathrm{Per}(i) = \gcd\{n : \text{there is a vertex path of length } n+1 \text{ from } i \text{ to itself}\}$$

exists and is the same value for all i. This common value is called the *period* of $\mathcal{G}$, denoted $\mathrm{Per}(\mathcal{G})$. Define the *period* of an irreducible $(0, 1)$-matrix A, $\mathrm{Per}(A)$, to be the period of the associated graph. If the period of the graph or the matrix is 1, then we call it *primitive.* If A is an irreducible matrix with $p = \mathrm{Per}(A)$, then (after reordering vertices) A^p is a block triangular matrix with primitive matrices along the diagonal. Finally, if A is a primitive matrix, then A^k is eventually positive. The *index of primitivity* of A, $\gamma(A)$, is the least integer k such that $A^k > 0$.

Now suppose our shift space with overlaps has an associated graph $\mathcal{TI}$; that is, we consider $(\Sigma, \mathcal{I})$ where Σ is a subshift of finite type with transition graph $\mathcal{T}$. We make the standing assumption that every vertex of $\mathcal{T}$ has at least one edge leaving it and one edge entering it; if not, we can remove the "stranded" vertices without affecting the dynamics. Recall that the graph-theoretical notions of irreducibility and primitivity have dynamical interpretations for the shift: $\mathcal{T}$ is irreducible iff Σ is *topologically transitive*, and $\mathcal{T}$ is primitive iff Σ is *topologically mixing*.

3.2. **Higher shifts.** Higher shifts are a well-known way of altering the transition graph for a subshift of finite type to obtain a new graph which generates a topologically conjugate subshift of finite type. The higher vertex shift allows us to subdivide the set of vertices, thus presumably giving us a greater number of disjoint vertices. We show how to implement this notion for shift spaces with overlap generated from a graph $\mathcal{TI}$. We refer the reader to [15, Sect. 2.3] for details on higher shifts for edge-labeled graphs; the constructions can be modified easily for vertex-labeled graphs, which is what we need.

For $m \geq 1$ we define the *mth higher vertex graph* $\mathcal{T}_{[m]}$ to have vertex set equal to the collection of all vertex paths of length m in $\mathcal{T}$, with an edge from a vertex $(i_0, \ldots, i_{m-1})$ to a vertex $(i_1, \ldots, i_{m-1}, i_m)$ provided there is an edge in $\mathcal{T}$ from i_{m-1} to i_m. Note that the first higher vertex shift $\mathcal{T}_{[1]}$ is simply $\mathcal{T}$.

The intersection graph $\mathcal{I}$ induces an intersection graph $\mathcal{I}_{[m]}$ for the vertices of $\mathcal{T}_{[m]}$. There is an (undirected) edge between the vertices $(i_0, \ldots, i_{m-1})$

and $(j_0, \ldots, j_{m-1})$ if the words are indistinguishable; that is, if there is an edge in $\mathcal{I}$ between i_k and j_k for all k.

Example 5. We return to Example 3. The vertices of $\mathcal{T}_{[2]}$ are $(1,1)$, $(1,2)$, $(2,3)$, $(2,4)$, $(3,1)$, $(3,2)$, $(4,3)$, and $(4,4)$—one for each edge (or vertex path of length 2) in $\mathcal{T}$. There are twelve edges in $\mathcal{I}_{[2]}$. They can be seen in the graph $\mathcal{T}_{[2]}\mathcal{I}_{[2]}$ in Figure 3.

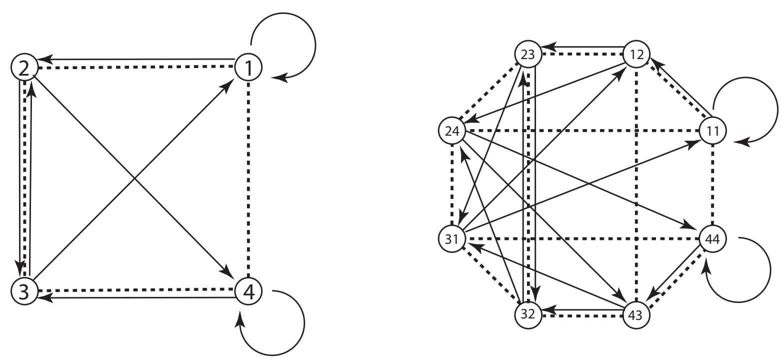

FIGURE 3

4. ENTROPY OF SHIFTS OF FINITE TYPE WITH OVERLAP

We are now able to state our main results, which state that the entropy of a shift with overlap defined by a primitive transition graph is given by a limit (actually a supremum) involving the independence numbers of the higher shifts. The supremum result allows us to obtain a lower bound for the entropy without taking m to infinity. (If the transition graph is not primitive, we can use these results to find the entropy by decomposing the graph into irreducible components, and further into primitive components; the entropy is the maximum over all irreducible components.)

Theorem 6. *If $(\Sigma, \mathcal{I})$ is the shift space with overlaps associated to the graph $\mathcal{T}\mathcal{I}$, then*

$$h_{\mathcal{I}}(\Sigma) = \limsup_{m\to\infty} \frac{\log(\operatorname{ind}(\mathcal{I}_{[m]}))}{m}.$$

If $\mathcal{T}$ is primitive, then

$$h_{\mathcal{I}}(\Sigma) = \lim_{m\to\infty} \frac{\log(\operatorname{ind}(\mathcal{I}_{[m]}))}{\gamma(\mathcal{T}_{[m]})} = \lim_{m\to\infty} \frac{\log(\operatorname{ind}(\mathcal{I}_{[m]}))}{m}.$$

(In particular, both of these limits exist.)

Corollary 7. *If $\mathcal{T}$ is primitive, then*

$$h_{\mathcal{I}}(\Sigma) = \sup_{m>0} \left\{ \frac{\log(\operatorname{ind}(\mathcal{I}_{[m]}))}{\gamma(\mathcal{T}_{[m]})} \right\}.$$

In particular, $h_{\mathcal{I}}(\Sigma) \geq \frac{\log(\operatorname{ind}(\mathcal{I}_{[m]}))}{\gamma(\mathcal{T}_{[m]})}$ for any m.

We first give some results on the relationship of independence, irreducibility, and higher shifts to entropy. In Section 4.4 we apply these results to prove Theorem 6 and Corollary 7.

4.1. **Independence and entropy.** In this section we exploit the fact, formalized in the following proposition, that if all words are distinguishable (there are no overlaps), then we can treat our shift space with overlaps as an ordinary shift space.

Proposition 8. *Let $(\Sigma, \mathcal{I})$ be a shift space with overlaps in which the edge set of $\mathcal{I}$ is empty. Then $h_{\mathcal{I}}(\Sigma) = h(\Sigma)$.*

Corollary 9. *Let $(\Sigma, \mathcal{I})$ be a shift space with overlaps. If $\Sigma' \subset \Sigma$ is a shift space in which all elements are pair-wise distinguishable, then $h_{\mathcal{I}}(\Sigma) \geq h_{\mathcal{I}}(\Sigma') = h(\Sigma')$.*

For any $V \subset V(\mathcal{I})$, let $\Sigma_V = \{(a_0, a_1, \ldots) \in \Sigma : a_i \in V \ \forall i\}$. Notice that if Σ is a shift space, then Σ_V is also a shift space. A *vertex-induced subgraph* of a graph $\mathcal{G}$ is a subgraph $\mathcal{G}'$ with the property that any $\mathcal{G}$-edge whose endpoints are in $V(\mathcal{G}')$ is in $E(\mathcal{G}')$. If $V \subset V(\mathcal{I})$, let $\mathcal{I}_V$ denote the vertex-induced subgraph of $\mathcal{I}$ with vertex set V.

Corollaries 10 and 11 give us a tactic for finding a lower bound for $h_{\mathcal{I}}(\Sigma)$. Remove enough vertices of $\mathcal{I}$ so that we have an independent set of vertices; then the corresponding shift space with overlaps has no overlaps—it is just a shift space. The entropy of this shift space is a lower bound for $h_{\mathcal{I}}(\Sigma)$.

Corollary 10. *If $V \subset V(\mathcal{I})$ is an independent set, then $h_{\mathcal{I}}(\Sigma) \geq h_{\mathcal{I}_V}(\Sigma_V) = h(\Sigma_V)$.*

In the following corollary we assume that Σ is a shift of finite type with transition graph $\mathcal{T}$ and intersection graph $\mathcal{I}$.

Corollary 11. *Suppose $\mathcal{T}$ is a complete digraph. Then $h_{\mathcal{I}}(\Sigma) \geq \log(\mathrm{ind}(\mathcal{I}))$.*

Proof. Let V be a maximum cardinality independent subset of $\mathcal{I}$. The vertex-induced subgraph of $\mathcal{T}$ with vertex set V is a complete digraph with $\mathrm{ind}(\mathcal{I})$ vertices. By Corollary 10, $h_{\mathcal{I}}(\Sigma) \geq h(\Sigma_V) = \log(\mathrm{ind}(\mathcal{I}))$. □

4.2. **Irreducibility and entropy.** In this section we use the simple structure of primitive graphs to obtain an estimate for entropy.

Theorem 12. *Suppose $\mathcal{T}$ is primitive with index of primitivity γ. Then* $h_{\mathcal{I}}(\Sigma) \geq \dfrac{\log(\mathrm{ind}(\mathcal{I}))}{\gamma}$.

Proof. Let $\{b_1, \dots, b_{\mathrm{ind}(\mathcal{I})}\}$ be an independent set of vertices. Since $A^\gamma > 0$, for any pair b_i, b_j, there is a word $(b_i, a_2, a_3, \dots, a_\gamma, b_j)$ in Σ. More generally, for any n and any $b_{i_1}, \dots, b_{i_n}$, there is a word

$$(b_{i_1}, a_2, \dots, a_\gamma, b_{i_2}, a_{\gamma+2}, \dots, a_{2\gamma}, \dots, b_{i_n}, a_{(n-1)\gamma+2}, \dots, a_{n\gamma})$$

in Σ. The collection of all $\mathrm{ind}(\mathcal{I})^n$ such words is $n\gamma$-separated, so we have that $|\mathcal{B}^{\mathrm{sep}}_{n\gamma}(\Sigma, \mathcal{I})| \geq \mathrm{ind}(\mathcal{I})^n$, and thus

$$h_{\mathcal{I}}(\Sigma) = \limsup_{n\to\infty} \frac{1}{n} \log |\mathcal{B}^{\mathrm{sep}}_{n}(\Sigma, \mathcal{I})| \geq \limsup_{n\to\infty} \frac{1}{n\gamma} \log |\mathcal{B}^{\mathrm{sep}}_{n\gamma}(\Sigma, \mathcal{I})| \geq \frac{\log(\mathrm{ind}(\mathcal{I}))}{\gamma}.$$

□

Corollary 13. *If $\mathcal{T}$ is primitive with N vertices, then* $h_{\mathcal{I}}(\Sigma) \geq \dfrac{\log(\mathrm{ind}(\mathcal{I}))}{N^2 - 2N + 2}$.

Proof. This follows from the fact that $\gamma(\mathcal{T}) \leq N^2 - 2N + 2$ ([3, Thm. 4.14]).

□

In general $\mathcal{T}$ is not primitive. In this case we break down $\mathcal{T}$ into its irreducible components. Each irreducible component has some period p, and can thus be decomposed into p primitive components. We then apply Theorem 12 to obtain the following corollary.

Corollary 14. *Suppose $\mathcal{T}' \subset \mathcal{T}$ is a primitive component with index of primitivity γ that resides in an irreducible component of period p and that $\mathcal{I}'$ is the associated intersection graph. Then*

$$h_{\mathcal{I}}(\Sigma) \geq \frac{\log(\operatorname{ind}(\mathcal{I}'))}{p\gamma}.$$

Corollary 15. *If there are two vertices of $\mathcal{T}$ that are in the same primitive component and are not joined by an $\mathcal{I}$-edge, then $(\Sigma, \mathcal{I})$ has positive entropy.*

Example 16. Let $U \subset \mathbb{R}^2$ be an open set and $f : U \to \mathbb{R}^2$ be a continuous function that maps the sets $N_1, \ldots, N_{11} \subset U$ as shown in Figure 4. The associated graph $\mathcal{TI}$ is shown on the right.

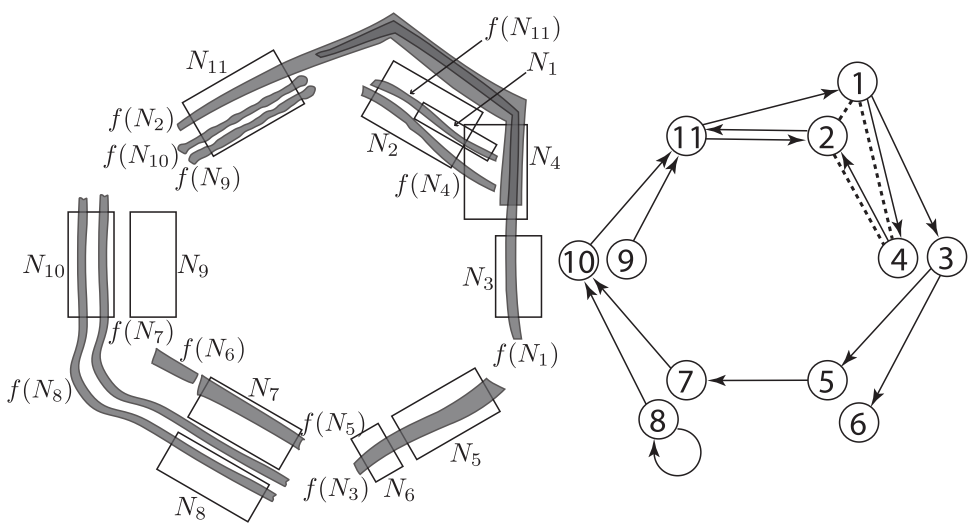

FIGURE 4

First we remove vertices 6 and 9 because they lack an outgoing and an incoming edge in $\mathcal{T}$, respectively. (Note: if we also removed two of the vertices 1, 2, and 4 to obtain an independent set, then we could apply

Corollary 10, but the corresponding shift space would have zero entropy.) The resulting transition graph has two irreducible components, with vertex sets $\{8\}$ and $\{1,2,3,4,5,7,9,10,11\}$. The second of these has period 2 and it decomposes into the primitive components $\{3,4,7,11\}$ and $\{1,2,5,10\}$. It is straightforward to show that both of these have index of primitivity 4 and that the first of these is an independent set. Thus we conclude that $h(f) \geq h_{\mathcal{I}}(\Sigma) \geq \frac{1}{2\cdot 4}\log(4) \approx 0.173$.

Example 17. We return to the doubling map in Example 3. Since $A_{\mathcal{T}}^2$ consists of all 1's, $A_{\mathcal{T}}$ is irreducible with $\mathrm{Per}(A_{\mathcal{T}}) = 1$ and primitive with $\gamma(A_{\mathcal{T}}) = 2$. The maximal independent sets in $\mathcal{I}$ are $\{1,3\}$ and $\{2,4\}$, so $\mathrm{ind}(\mathcal{I}) = 2$. Thus $h(f) \geq h_{\mathcal{I}}(\Sigma) \geq \frac{1}{2}\log(2)$, which is half the entropy of the doubling map.

4.3. **Higher shifts.** In this section we relate the entropy, independence number, and index of primitivity, respectively, of a shift with overlap given by $\mathcal{T}\mathcal{I}$ to those of the higher shifts given by $\mathcal{T}_{[m]}\mathcal{I}_{[m]}$.

Theorem 18. *If $\mathcal{T}_{[m]}\mathcal{I}_{[m]}$ is the graph associated to the mth higher shift of $(\Sigma_{\mathcal{T}}, \mathcal{I})$, then $h_{\mathcal{I}_{[m]}}(\Sigma_{\mathcal{T}_{[m]}}) = h_{\mathcal{I}}(\Sigma_{\mathcal{T}})$.*

Proof. Let $m \geq 1$ be fixed. For each n, there exists a natural bijective function $\psi_n : \mathcal{B}_{n+m-1}(\Sigma_{\mathcal{T}}) \to \mathcal{B}_n(\Sigma_{\mathcal{T}_{[m]}})$. Specifically, $\psi_n(b_0, \ldots, b_{n+m-2}) = (w_0, \ldots, w_{n-1})$, where $w_i = (b_i, \ldots, b_{m+i-1})$. Moreover, $\psi_n(b_0, \ldots, b_{n-1})$ and $\psi_n(c_0, \ldots, c_{n-1})$ are distinguishable if and only if $(b_0, \ldots, b_{n-1})$ and

$(c_0, \ldots, c_{n-1})$ are distinguishable. Thus, $|\mathcal{B}^{\text{sep}}_{n+m-1}(\Sigma_{\mathcal{T}}, \mathcal{I})| = |\mathcal{B}^{\text{sep}}_{n}(\Sigma_{\mathcal{T}_{[m]}}, \mathcal{I}_{[m]})|$, and hence

$$\begin{aligned} h_{\mathcal{I}_{[m]}}(\Sigma_{\mathcal{T}_{[m]}}) &= \limsup_{n\to\infty} \frac{1}{n} \log |\mathcal{B}^{\text{sep}}_{n}(\Sigma_{\mathcal{T}_{[m]}}, \mathcal{I}_{[m]})| \\ &= \limsup_{n\to\infty} \frac{1}{n} \log |\mathcal{B}^{\text{sep}}_{n+m-1}(\Sigma_{\mathcal{T}}, \mathcal{I})| \\ &= h_{\mathcal{I}}(\Sigma_{\mathcal{T}}). \end{aligned}$$

□

Proposition 19. $\text{ind}(\mathcal{I}_{[m]}) \geq \text{ind}(\mathcal{I})$.

Proof. (Recall that we are assuming that each vertex has at least one edge leaving it.) If $\{b_1, \ldots, b_{\text{ind}(\mathcal{I})}\}$ is an independent set of vertices of $\mathcal{I}$, and $w_i = (b_i, a_{i_1}, \ldots, a_{i_{m-1}})$ is, for each b_i, any word of length m, then $\{w_1, \ldots, w_{\text{ind}(\mathcal{I})}\}$ is an independent set of vertices of $\mathcal{I}_{[m]}$.

□

Proposition 20. *Assume that $\mathcal{T}$ has at least two vertices. If $\mathcal{T}$ is primitive, then so is $\mathcal{T}_{[m]}$, and $\gamma(\mathcal{T}_{[m]}) = \gamma(\mathcal{T}) - 1 + m$. (If $\mathcal{T}$ has only one vertex, then $\mathcal{T}_{[m]}$ is isomorphic to $\mathcal{T}$.)*

Proof. A graph is primitive with exponent γ if and only if from any vertex to any other there is a vertex path of length $\gamma + 1$, and γ is the smallest number with this property. Observe that $(i_{m-1}, k_1, \ldots, k_p, j_0)$ is a vertex path in $\mathcal{T}$ (of length $p + 2$, with $p \geq 0$) if and only if

$$((i_0, \ldots, i_{m-1}), (i_1, \ldots, i_{m-1}, k_1), \ldots, (k_p, j_0, \ldots, j_{m-2}), (j_0, \ldots, j_{m-1}))$$

is a vertex path in $\mathcal{T}_{[m]}$ (of length $p+m+1$) for any vertices $(i_0,\ldots,i_{m-1})$ and $(j_0,\ldots,j_{m-1})$. Between any two vertices of $\mathcal{T}$ there is a vertex path of length $p+2=\gamma(\mathcal{T})$, so between any two vertices of $\mathcal{T}_{[m]}$ there is a vertex path of length $p+m+1=\gamma(\mathcal{T})-1+m$. Thus $\gamma(\mathcal{T}_{[m]})\leq\gamma(\mathcal{T})-1+m$.

To prove the opposite inequality, first assume that $\gamma(\mathcal{T})>1$. Then there exist two vertices of $\mathcal{T}$ such that there is no vertex path from the first to the second of length $\gamma(\mathcal{T})$, and thus two vertices of $\mathcal{T}_{[m]}$ with no vertex path of length $\gamma(\mathcal{T})-1+m$ between them, so $\gamma(\mathcal{T}_{[m]})\geq\gamma(\mathcal{T})-1+m$.

Finally, if $\gamma(\mathcal{T})=1$, then any vertex of $\mathcal{T}$ can follow any other vertex. For $i\neq j$, it is clear that the shortest vertex path in $\mathcal{T}_{[m]}$ from $(i,\ldots,i)$ to $(j,\ldots,j)$ has length $m+1$, so $\gamma(\mathcal{T}_{[m]})\geq m=\gamma(\mathcal{T})-1+m$.

□

Example 21. We return again to Example 3. Since $\gamma(\mathcal{T})=2$, Proposition 20 says that $\gamma(\mathcal{T}_{[2]})=3$, as we can check by observing that $A_{[2]}^2$ has zero entries while $A_{[2]}^3$ does not (where $A_{[2]}$ is the adjacency matrix for $\mathcal{T}_{[2]}$).

Notice that the set of vertices $\{(1,1),(2,3),(3,1),(4,3)\}$ is a maximum cardinality independent set in $\mathcal{I}_{[2]}$, so $\operatorname{ind}(\mathcal{I}_{[2]})=4$. Theorems 4, 12, and 18 tells us that

$$h(f)\geq h_{\mathcal{I}}(\Sigma_{\mathcal{T}})=h_{\mathcal{I}_{[2]}}(\Sigma_{\mathcal{T}_{[2]}})\geq\frac{\log 4}{3}=\frac{2\log 2}{3},$$

which is greater than $\frac{1}{2}\log 2$, the lower bound we obtained in Example 17 using $\mathcal{T}$. In this particular example we see that $\gamma(\mathcal{T}_{[m]})=m+1$ and

$\mathrm{ind}(\mathcal{I}_{[m]}) = 2^m$. So

$$h(f) \geq \frac{\log 2^m}{m+1} = \frac{m \log 2}{m+1}$$

for all m. This implies that $h(f) \geq \log 2$, the entropy of the doubling map.

4.4. **Proofs.** In Example 21 we saw explicitly that we were able to compute the entropy by looking at the sequence of higher shifts. We are now able to prove our main results, which state that the same is true for any primitive $\mathcal{T}$.

Proof of Theorem 6. Observe that, by definition, the set of words of length m is equal to the vertex set of $\mathcal{T}_{[m]}\mathcal{I}_{[m]}$, that is, $\mathcal{B}_m(\Sigma) = V(\mathcal{T}_{[m]}\mathcal{I}_{[m]})$. Moreover, an m-separated set B in Σ corresponds to an independent set in $\mathcal{I}_{[m]}$, and hence $|\mathcal{B}_m^{\mathrm{sep}}(\Sigma, \mathcal{I})| = \mathrm{ind}(\mathcal{I}_{[m]})$. Thus the limit superior equality follows from the definition of entropy.

Now suppose that $\mathcal{T}$ is primitive with index of primitivity $\gamma(\mathcal{T})$. Then $\mathcal{T}_{[m]}$ is also primitive and $\gamma(\mathcal{T}_{[m]}) = \gamma(\mathcal{T}) + m - 1$. So it follows that

$$h_{\mathcal{I}}(\Sigma) = \limsup_{m\to\infty} \frac{\log(\mathrm{ind}(\mathcal{I}_{[m]}))}{m} = \limsup_{m\to\infty} \frac{\log(\mathrm{ind}(\mathcal{I}_{[m]}))}{\gamma(\mathcal{T}_{[m]})}$$

Clearly

$$\limsup_{m\to\infty} \frac{\log(\mathrm{ind}(\mathcal{I}_{[m]}))}{m} \geq \liminf_{m\to\infty} \frac{\log(\mathrm{ind}(\mathcal{I}_{[m]}))}{m} = \liminf_{m\to\infty} \frac{\log(\mathrm{ind}(\mathcal{I}_{[m]}))}{\gamma(\mathcal{T}_{[m]})}.$$

So to complete the proof it is enough to show that

$$\liminf_{m\to\infty} \frac{\log(\mathrm{ind}(\mathcal{I}_{[m]}))}{m} \geq h_{\mathcal{I}}(\Sigma).$$

Since we have $\mathrm{ind}(\mathcal{I}_{[m]})$ m-separated words and $\mathcal{T}$ is primitive, we can make from them $(\mathrm{ind}(\mathcal{I}_{[m]}))^{\lfloor M/\gamma(\mathcal{T}_{[m]})\rfloor}$ M-separated words by concatenating, as in the proof of Theorem 12. Thus

$$\mathrm{ind}(\mathcal{I}_{[M]}) \geq \mathrm{ind}(\mathcal{I}_{[m]})^{\lfloor M/\gamma(\mathcal{T}_{[m]})\rfloor},$$

and therefore

$$\liminf_{M\to\infty} \frac{\log(\mathrm{ind}(\mathcal{I}_{[M]}))}{M} \geq \frac{\log(\mathrm{ind}(\mathcal{I}_{[m]}))}{\gamma(\mathcal{T}_{[m]})}.$$

Since this holds for all m, we have that

$$\liminf_{M\to\infty} \frac{\log(\mathrm{ind}(\mathcal{I}_{[M]}))}{M} \geq \limsup_{m\to\infty} \frac{\log(\mathrm{ind}(\mathcal{I}_{[m]}))}{\gamma(\mathcal{T}_{[m]})} = h_{\mathcal{I}}(\Sigma).$$

□

Proof of Corollary 7. It is obvious that

$$h_{\mathcal{I}}(\Sigma) \leq \sup_{m>0} \left\{ \frac{\log(\mathrm{ind}(\mathcal{I}_{[m]}))}{\gamma(\mathcal{T}_{[m]})} \right\}.$$

We must prove the reverse inequality. To do this, observe that since we have $\mathrm{ind}(\mathcal{I}_{[m]})$ m-separated words and $\mathcal{T}$ is primitive, we can make them into $\mathrm{ind}(\mathcal{I}_{[m]})^k$ words of length $k\gamma(\mathcal{T}_{[m]})$ (for any k) by concatenating. Thus $\mathrm{ind}(\mathcal{I}_{[k\gamma(\mathcal{T}_{[m]})]}) \geq \mathrm{ind}(\mathcal{I}_{[m]})^k$, so

$$\frac{\log(\mathrm{ind}(\mathcal{I}_{[k\gamma(\mathcal{T}_{[m]})]}))}{k\gamma(\mathcal{T}_{[m]})} \geq \frac{\log((\mathrm{ind}(\mathcal{I}_{[m]}))^k)}{k\gamma(\mathcal{T}_{[m]})} = \frac{\log(\mathrm{ind}(\mathcal{I}_{[m]}))}{\gamma(\mathcal{T}_{[m]})}.$$

By taking the limit of both sides as $k \to \infty$, we obtain

$$h_{\mathcal{I}}(\Sigma) \geq \frac{\log(\mathrm{ind}(\mathcal{I}_{[m]}))}{\gamma(\mathcal{T}_{[m]})};$$

since this holds for any m, we are done. □

Example 22. Recall from Example 5 that $\mathcal{T}_{[m]}$ is a primitive graph with index of primitivity $m+1$ and independence number 2^m. By Corollary 7,

$$h(f) \geq h_{\mathcal{I}}(\Sigma) = \sup\left\{\frac{\log(\operatorname{ind}(\mathcal{I}_{[m]}))}{\gamma(\mathcal{T}_{[m]})}\right\} = \sup\left\{\frac{\log 2^m}{m+1}\right\} = \sup\left\{\frac{m\log 2}{m+1}\right\} = \ln 2.$$

Theorem 6 gives us the same bound,

$$h(f) \geq h_{\mathcal{I}}(\Sigma) = \lim_{m\to\infty}\frac{\log(\operatorname{ind}(\mathcal{I}_{[m]}))}{m} = \lim_{m\to\infty}\frac{\log 2^m}{m} = \ln 2,$$

and happens to be the limit of a constant sequence.

The following example shows that the sequence $\left\{\frac{\log(\operatorname{ind}(\mathcal{I}_{[m]}))}{\gamma(\mathcal{T}_{[m]})}\right\}$ may not be nondecreasing.

Example 23. Consider the graph $\mathcal{TI}$ shown in Figure 5.

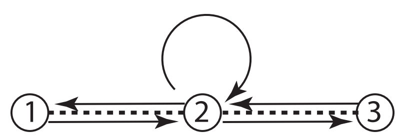

FIGURE 5

It is easy to check by hand that

$$\frac{\log(\operatorname{ind}(\mathcal{I}_{[1]}))}{\gamma(\mathcal{T}_{[1]})} = \frac{1}{2}\ln(2) > \frac{1}{3}\ln(2) = \frac{\log(\operatorname{ind}(\mathcal{I}_{[2]}))}{\gamma(\mathcal{T}_{[2]})}.$$

Thus the sequence is not nondecreasing.

We used a computer to find the first several terms of the sequence; based on this, the pattern appears to be $\left\{\frac{1}{n+1}\ln(2^m)\right\}_{n=1}^{\infty}$ where $m = n/2$ for n even and $m = (n+1)/2$ for n odd. Simplified, the nth term becomes

$\dfrac{n \ln 2}{2(n+1)}$ when n is even and $\frac{1}{2}\ln(2)$ when n is odd. The two values given by the theorem (the supremum and the limit superior) are equal and the limit superior is equal to the limit. In particular, this common value, $\frac{1}{2}\ln(2)$ was the first term in the sequence.

5. Relationship to sofic shifts

A sofic shift is constructed from a transition graph, as with a shift of finite type, but with the difference that two or more vertices may share the same label (see [15, Chapter 3] for details). We can view a sofic shift as a special case of a shift with overlaps, in which vertices are adjacent if and only they share the same label. In this case the adjacency relation is clearly transitive, so not every shift of finite type with overlap arises in this way. We can, however, turn a shift of finite type with overlap into a sofic shift by extending the adjacency relation to make it transitive.

More precisely, let $(\Sigma, \mathcal{I})$ be a shift of finite type with overlaps, and let $A_1, \ldots, A_r$ be the connected components of the intersection graph $\mathcal{I}$. Define the *associated sofic shift* $\Sigma_{\mathcal{I}}$ on the alphabet $\{A_1, \ldots, A_r\}$ as follows. The element $(A_{i_0}, A_{i_1}, \ldots) \in \Sigma_{\mathcal{I}}$ if and only if there is an element $(a_0, a_1, \ldots) \in \Sigma$ such that $a_k \in A_{i_k}$ for all k.

Intuitively, the original shift with overlaps $(\Sigma, \mathcal{I})$ factors onto the associated sofic shift $\Sigma_{\mathcal{I}}$, and so should have entropy at least as great. That is not precisely correct, however, since we cannot talk about continuous maps because in general a shift space with overlaps is not a topological space. We do have the following result.

Theorem 24. *Let $\Sigma_{\mathcal{I}}$ be the sofic shift associated to the shift of finite type with overlaps $(\Sigma, \mathcal{I})$. Then $h_{\mathcal{I}}(\Sigma) \geq h(\Sigma_{\mathcal{I}})$.*

Proof. Let $\mathcal{B}_m(\Sigma_{\mathcal{I}})$ be the set of words of length m in $\Sigma_{\mathcal{I}}$. For each element $(A_{i_0}, \ldots, A_{i_{m-1}}) \in \mathcal{B}_m(\Sigma_{\mathcal{I}})$, pick one word $(a_0, \ldots, a_{m-1})$ in Σ such that $a_k \in A_{i_k}$ for $k = 0, \ldots, m-1$. The collection of these words in Σ are m-separated. However, it may not be a maximal m-separated set. Thus $|\mathcal{B}_m^{\text{sep}}(\Sigma, \mathcal{I})| \geq |\mathcal{B}_m(\Sigma_{\mathcal{I}})|$, and hence $h_{\mathcal{I}}(\Sigma) \geq h(\Sigma_{\mathcal{I}})$. □

Making the adjacency relation transitive may render previously distinguishable words indistinguishable. For example, the intersection graph in Example 2 has a single connected component, so the associated sofic shift has only one symbol and therefore zero entropy. However, if the adjacency relation is already transitive, then the associated sofic shift carries the same information as the original.

A set of vertices in a nondirected graph form a *clique* if every pair of vertices in the set are joined by an edge.

Proposition 25. *If the vertices in each connected component of $\mathcal{I}$ form a clique, then $h_{\mathcal{I}}(\Sigma) = h(\Sigma_{\mathcal{I}})$.*

Proof. By Theorem 24, all we must prove is that $h_{\mathcal{I}}(\Sigma) \leq h(\Sigma_{\mathcal{I}})$. Let $\mathcal{B}_m^{\text{sep}}(\Sigma, \mathcal{I})$ be a maximal m-separated set. Consider the function $\psi : \mathcal{B}_m^{\text{sep}}(\Sigma, \mathcal{I}) \to$

$\mathcal{B}_m(\Sigma_{\mathcal{I}})$ given by $\psi(a_0, a_1, \dots, a_{m-1}) = (A_{i_0}, A_{i_1}, \dots, A_{i_{m-1}})$ where $a_k \in A_{i_k}$ for all k. We will prove that this function is injective. Suppose $\psi(a_0, a_1, \dots, a_{m-1}) = \psi(b_0, b_1, \dots, b_{m-1})$ for some $(a_0, a_1, \dots, a_{m-1}), (b_0, b_1, \dots, b_{m-1}) \in \mathcal{B}_m^{\text{sep}}(\Sigma, \mathcal{I})$. Then $a_k, b_k \in A_{i_k}$ for all k. But the subgraphs of $\mathcal{I}$ with vertex sets A_{i_k} are cliques, so a_k and b_k are the same or are indistinguishable. Thus $(a_0, a_1, \dots, a_{m-1})$ and $(b_0, b_1, \dots, b_{m-1})$ are indistinguishable. Since $\mathcal{B}_m^{\text{sep}}(\Sigma, \mathcal{I})$ is m-separated, $(a_0, a_1, \dots, a_{m-1}) = (b_0, b_1, \dots, b_{m-1})$, and ψ is injective. Because ψ is an injective function between finite sets, we conclude that $|\mathcal{B}_m^{\text{sep}}(\Sigma, \mathcal{I})| \le |\mathcal{B}_m(\Sigma_{\mathcal{I}})|$. Hence $h_{\mathcal{I}}(\Sigma) \le h(\Sigma_{\mathcal{I}})$. □

Example 26. Consider the graph $\mathcal{T}\mathcal{I}$ shown on the left in Figure 6. The adjacency relation is clearly transitive. If we give vertices 1 and 3 the same label we obtain the "golden mean" shift shown on the right (a shift that is well-known to be sofic but not of finite type). By Proposition 25 the entropies of the two are equal, and we can use methods from [15, Chapter 3] for computing the entropy of sofic shifts to determine that $h_{\mathcal{I}}(\Sigma) = h(\Sigma_{\mathcal{I}}) = \log(\frac{1}{2}(1 + \sqrt{5}))$.

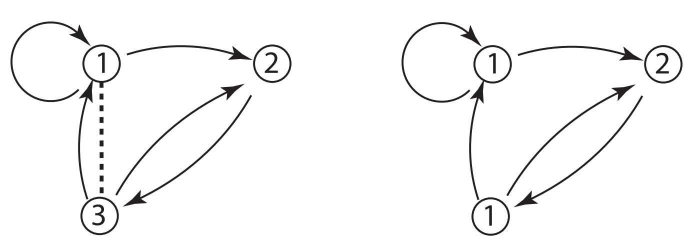

FIGURE 6

Example 27. Finally, consider the sofic shift associated to the shift space with overlaps in Example 16 (with vertices 6, 8, and 9 removed). We give vertices 1, 2, and 4 the same label as in Figure 7. Again, using the methods from [15, Chapter 3], and the fact that all connected components of $\mathcal{I}$ are cliques, we find that

$$h(f) \geq h_{\mathcal{I}}(\Sigma) = \log\left(\frac{1}{6}(108+12\sqrt{69})^{1/3} + 2(108+12\sqrt{69})^{-1/3}\right)) \approx .281,$$

which is a larger lower bound than the one obtained in Example 16.

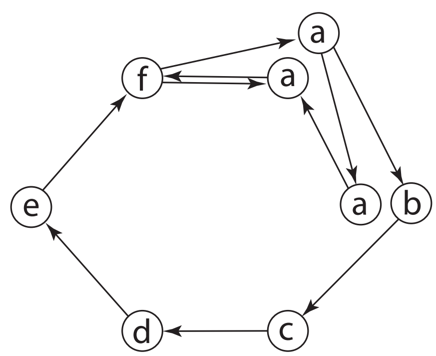

FIGURE 7

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