

# SPECTRAL DECOMPOSITION FOR TOPOLOGICALLY ANOSOV HOMEOMORPHISMS ON NONCOMPACT AND NON-METRIZABLE SPACES

TARUN DAS, KEONHEE LEE, DAVID RICHESON, AND JIM WISEMAN

ABSTRACT. We discuss topological definitions of expansivity, shadowing, and chain recurrence for homeomorphisms. They generalize the usual definitions for metric spaces. We prove various theorems about topologically Anosov homeomorphisms (maps that are expansive and have the shadowing property) on noncompact and non-metrizable spaces that generalize theorems for such homeomorphisms on compact metric spaces. The main result is a generalization of Smale's spectral decomposition theorem to topologically Anosov homeomorphisms on first countable, locally compact, paracompact, Hausdorff spaces.

## 1. INTRODUCTION

The goal of this paper is to extend the following result, Smale's spectral decomposition theorem applied to Anosov diffeomorphisms of compact manifolds, to more general topological spaces.

**Theorem 1** ([15]). *Let  $M$  be a compact manifold and  $f : M \rightarrow M$  be an Anosov diffeomorphism. Then the non-wandering set  $\Omega(f)$  can be written as a finite union of disjoint closed invariant sets on which  $f$  is topologically transitive.*

Recall that a diffeomorphism is *Anosov* if it has a hyperbolic structure on the entire manifold and that every Anosov diffeomorphism is expansive and has the shadowing property (see, e.g., [14, Sect. 9.2]). Theorem 1 has been extended to homeomorphisms on compact metric spaces:

**Theorem 2** ([1, Theorem 11.13], [2, Theorem 3.4.4]). *Let  $X$  be a compact metric space and  $f : X \rightarrow X$  be an expansive homeomorphism with the shadowing property. Then  $\Omega(f)$  can be written as a finite union of disjoint closed invariant sets on which  $f$  is topologically transitive.*

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*Date:* June 5, 2012.

*2000 Mathematics Subject Classification.* Primary 37B20; Secondary 37B25.

*Key words and phrases.* Expansiveness, shadowing, chain recurrent set.

Yang extended Theorem 2 to noncompact metric spaces ([18, Theorem 4]), with the additional (strong) requirement that the chain recurrent set be compact.

It is well known that some dynamical properties of homeomorphisms on compact spaces (i.e., properties that are conjugacy invariant), may not be dynamical properties on noncompact spaces. For example, a dynamical system on a noncompact metric space may be expansive or have the shadowing property with respect to one metric, but not with respect to another metric that induces the same topology (see Examples 6 and 7). Of course, it is preferable to have a theory that is independent of any change of (compatible) metric. In this article we give topological definitions of expansiveness and shadowing that are equivalent to the usual metric definitions for homeomorphisms on compact metric spaces and are dynamical properties for any metric space. Moreover, these definitions apply to non-metrizable spaces as well. For related work see [3, 6, 7, 8, 11, 12, 13].

Then we extend the spectral decomposition theorem to dynamical systems on spaces that are not necessarily metrizable and not necessarily compact. The only concession is that in the noncompact case, the collection of basic sets in the decomposition need not be finite.

**Theorem 3.** *Let  $X$  be a first countable, locally compact, paracompact, Hausdorff space and  $f : X \rightarrow X$  an expansive homeomorphism with the shadowing property. Then  $\Omega(f)$  can be written as a union of disjoint closed invariant sets on which  $f$  is topologically transitive. If  $X$  is compact, then this decomposition is finite.*

*Remark 4.* Every metric space is first countable, paracompact, and Hausdorff ([16, §5]), so in particular the preceding theorem applies to any locally compact metric space. It also applies to non-metrizable spaces such as the unit square with the lexicographic ordering.

The paper is organized as follows. We discuss definitions and preliminaries in the next section. In Section 3, we show that topologically Anosov homeomorphisms have a local product structure, and obtain topological stability as a consequence. In Section 4 we investigate the properties of the non-wandering and chain recurrent sets, and prove our main result, Theorem 3. Finally, we discuss the relationships between the topological definitions of expansiveness and shadowing and the usual metric definitions.

## 2. PRELIMINARIES

The following are the standard definitions for expansiveness and shadowing, which we will refer to as metric expansiveness and metric shadowing.

**Definition 5.** Let  $(X, d)$  be a metric space and  $f : X \rightarrow X$  be a homeomorphism.

- (1)  $f$  is *metric expansive* if there is an  $e > 0$  such that for any distinct  $x, y \in X$ , there exists  $n \in \mathbb{Z}$  such that  $d(f^n(x), f^n(y)) > e$ . The number  $e$  is called an *expansive constant*.
- (2) For  $\delta > 0$ , a  $\delta$ -*chain* is a sequence  $\{x_0, x_1, \dots, x_n\}$  ( $n \geq 1$ ) such that  $d(f(x_{i-1}), x_i) < \delta$  for  $i = 1, \dots, n$ . A  $\delta$ -*pseudo-orbit* is a bi-infinite  $\delta$ -chain.
- (3)  $f$  has the *metric shadowing property* if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that every  $\delta$ -pseudo-orbit  $\{x_i\}$  is  $\varepsilon$ -traced by a point  $y$ ; that is,  $d(f^i(y), x_i) < \varepsilon$  for all  $i$ .
- (4) A point  $x$  is *metric chain-recurrent* if for any  $\delta > 0$ , there is a  $\delta$ -chain from  $x$  to itself.

These definitions clearly require that  $X$  be a metric space. Furthermore, as we see in the following two examples, if  $X$  is noncompact, these properties depend on the choice of metric; a homeomorphism that is metric expansive for  $d$  may not be for  $d'$ , even if  $d$  and  $d'$  induce the same topology. Thus on noncompact spaces, these properties are not invariant under topological conjugacy. In addition, if  $f$  has these properties, its iterates may not ([4, Example 1], [13, Example 9]).

**Example 6.** Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear automorphism induced by the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

and consider the stereographic projection  $P : S^2 - \{(0, 0, 1)\} \rightarrow \mathbb{R}^2$  defined by  $f(x, y, z) = \frac{(x, y)}{(1-z)}$ . Then  $T$  is not expansive if  $\mathbb{R}^2$  has the metric induced by  $P$ , but  $T$  is expansive when  $\mathbb{R}^2$  has the usual metric. Moreover, both metrics induce the same topology on  $\mathbb{R}^2$ .

**Example 7.** Let  $X \subset \mathbb{R}^2$  be the subset  $\bigcup_{n \in \mathbb{Z}} X_n = \{n\} \times [0, 2^{-|n|}]$ , with the metric inherited from  $\mathbb{R}^2$ , and define  $f : X \rightarrow X$  by

$$f(n, y) = \begin{cases} (n+1, 2y), & \text{if } n < 0, \\ (n+1, \frac{1}{2}y), & \text{if } n \geq 0. \end{cases}$$

Choose  $\delta > 0$ . Since  $\text{diam}(X_n) < \delta$  for  $n$  with  $|n| > n_0 = \lceil \frac{1}{\delta} \rceil$ , the uniform continuity of  $f$  and  $f^{-1}$  on the compact set  $X_{-n_0} \cup \dots \cup$

$X_{n_0}$  shows that  $f$  has the metric shadowing property. However,  $f$  is topologically conjugate to the map on the space  $\bigcup_{n \in \mathbb{Z}} (\{n\} \times [0, 1])$  given by translation on the first coordinate and identity on the second, which clearly does not have the metric shadowing property.

To address these issues, we make purely topological definitions for these notions. These definitions are equivalent to the metric ones in the case that  $X$  is a compact metric space; we discuss the relationships among the definitions in general in Section 5.

Recently, the third and fourth authors studied two topological generalizations of positive expansivity ([13]). For a given dynamical system  $f : X \rightarrow X$ , they use the product map  $F \equiv f \times f$  on  $X \times X$  and neighborhoods of the diagonal  $\Delta_X = \{(x, x) : x \in X\}$  to extend the notion of positive expansivity. The idea is that  $x$  is close to  $y$  in  $X$  if and only if the point  $(x, y)$  is close to the diagonal  $\Delta_X$  in  $X \times X$ . Thus, instead of requiring that  $d(x, y)$  be less than  $\varepsilon$ , we can require that  $(x, y)$  be in a given neighborhood of  $\Delta_X$ . This approach is useful for extending dynamical properties from the compact setting to the noncompact setting. (In other words, the approach involves a uniform structure on the space  $X$ ; the neighborhoods of the diagonal are entourages. Any Tychonoff space has a uniform structure.)

We make the standing assumption that all topological spaces are first countable, locally compact, paracompact, and Hausdorff and that all maps are homeomorphisms although some of the results apply more generally. (Every locally compact, Hausdorff space is Tychonoff, and so has a uniform structure.) To avoid confusion, we denote subsets of the product space  $X \times X$  by  $A, M, U$ , etc., and subsets of the base space  $X$  by  $\dot{A}, \dot{M}, \dot{U}$ , etc.

Let  $U$  be a neighborhood of  $\Delta_X$ , and let  $U[x] = \{y \in X : (x, y) \in U\}$  be the cross section of  $U$  at  $x \in X$ . For any point  $x \in X$  and any neighborhood  $\dot{G}$  of  $x$ , we can find a neighborhood  $U$  of  $\Delta_X$  such that  $U[x] \subset \dot{G}$ . A set  $M \subset X \times X$  is *proper* if for any compact subset  $S$ , the set  $M[S] = \bigcup_{x \in S} M[x]$  is compact.  $M$  is *symmetric* if  $M$  is equal to its transpose,  $M^T = \{(y, x) : (x, y) \in M\}$ . Note that if  $M$  is a neighborhood of  $\Delta_X$ , then  $M \cap M^T$  is a symmetric neighborhood of  $\Delta_X$ ; thus we can often work with symmetric neighborhoods without loss of generality.

Let

$$U^n = \{(x, y) : \text{there exists } z_0 = x, z_1, \dots, z_n = y \in X \\ \text{such that } (z_{i-1}, z_i) \in U \text{ for } i = 1, \dots, n\}.$$

The intuition behind this definition is that if we want  $U$  to be the topological equivalent of having an  $(\varepsilon/n)$ -ball at each point, then  $U^n$  would take the place of an  $\varepsilon$ -ball at each point. Observe that if  $U$  is symmetric, then the order of the elements within each ordered pair does not matter. Also, for any neighborhood  $U$  of  $\Delta_X$  we can find a neighborhood  $V$  of  $\Delta_X$  such that  $F(V^2) \subset U$ .

**Definition 8.** A homeomorphism  $f : X \rightarrow X$  is (*topologically expansive*) if there is a closed neighborhood  $N$  of  $\Delta_X$  such that for any distinct  $x, y \in X$  there exists  $n \in \mathbb{Z}$  such that  $F^n(x, y) \notin N$ . Such a neighborhood  $N$  is called an *expansive neighborhood* for  $f$ . Let  $\mathcal{E} = \{(f, X) : X \text{ is first countable, locally compact, paracompact, Hausdorff, } f : X \rightarrow X \text{ is an expansive homeomorphism}\}$ .

**Lemma 9.** *If  $(f, X) \in \mathcal{E}$ , then there is a proper expansive neighborhood.*

*Proof.* Let  $N$  be an expansive neighborhood for  $f$ . Since  $X$  is locally compact, each  $x \in X$  has an open, relatively compact neighborhood  $\dot{U}_x$ . Since  $X$  is paracompact, the open cover  $\{\dot{U}_x\}$  has a closed (and hence compact) locally finite refinement  $\{\dot{V}_\alpha\}$  ([9, Chapter 5, Theorem 28]). Then  $A = N \cap \left(\bigcup_\alpha (\dot{V}_\alpha \times \dot{V}_\alpha)\right)$  is a proper expansive neighborhood.  $\square$

Let  $D$  and  $E$  be neighborhoods of  $\Delta_X$ . A *D-chain* is a sequence  $\{x_0, x_1, \dots, x_n\}$  ( $n \geq 1$ ) such that  $(f(x_{i-1}), x_i) \in D$  for  $i = 1, \dots, n$ . A *D-pseudo-orbit* is a bi-infinite  $D$ -chain. A  $D$ -pseudo-orbit  $\{x_i\}$  is *E-traced* by a point  $y \in X$  if  $(f^i(y), x_i) \in E$  for all  $i \in \mathbb{Z}$ .

**Definition 10.** A homeomorphism  $f : X \rightarrow X$  has the (*topological shadowing property*) if for every neighborhood  $E$  of  $\Delta_X$ , we can find a neighborhood  $D$  of  $\Delta_X$  such that every  $D$ -pseudo-orbit is  $E$ -traced by some point  $y \in X$ . Let  $\mathcal{S} = \{(f, X) : X \text{ is first countable, locally compact, paracompact, Hausdorff, } f : X \rightarrow X \text{ is a homeomorphism with the shadowing property}\}$ .

*Remark 11.* If  $(X, d)$  is a compact metric space, then for any neighborhood  $U$  of  $\Delta_X$ , we can find  $\delta > 0$  such that  $U_\delta = d^{-1}[0, \delta] \subset U$ . On the other hand, every  $U_\delta$  is a neighborhood of  $\Delta_X$ . Thus, the above definitions coincide with the usual notions of expansivity and shadowing on compact metric spaces. However, the following example shows that this argument does not hold if  $X$  is not compact.

**Example 12.** If  $X$  is a noncompact metric space, then a neighborhood  $U$  of  $\Delta_X$  may not contain an open set  $U_\delta$  (defined as above) for any

$\delta > 0$ . For instance, consider the neighborhood  $U = \{(x, y) \in \mathbb{R}^2 : |x - y| < e^{-x^2}\}$  of  $\Delta_{\mathbb{R}}$ . There is no  $\delta > 0$  with  $U_\delta \subset U$ .

As noted earlier, the following properties do not hold for the metric definitions in the noncompact case.

**Proposition 13.** (1)  $(f, X) \in \mathcal{E}$  [resp.  $\mathcal{S}$ ] if and only if  $(f^k, X) \in \mathcal{E}$  [resp.  $\mathcal{S}$ ] for all nonzero  $k \in \mathbb{Z}$ .

(2) If  $h : X \rightarrow Y$  is a homeomorphism, then  $(f, X) \in \mathcal{E}$  [resp.  $\mathcal{S}$ ] if and only if  $(h \circ f \circ h^{-1}, Y) \in \mathcal{E}$  [resp.  $\mathcal{S}$ ].

*Proof.* (1) It is clear that  $f$  is expansive (or has the shadowing property) if and only if the same is true for  $f^{-1}$  ( $N$  is an expansivity neighborhood for  $f$  if and only if it is an expansivity neighborhood for  $f^{-1}$ ; a sequence is a  $D$ -chain for  $f$  if and only if it is an  $F^{-1}(D)$ -chain for  $f^{-1}$ ). Thus we will assume without loss of generality that  $k$  is positive.

If  $N$  is an expansive neighborhood for  $f$ , then  $\bigcap_{i=0}^{k-1} F^{-i}(N)$  is an expansive neighborhood for  $f^k$ . Conversely, if  $N$  is an expansive neighborhood for  $f^k$ , then it is *a fortiori* an expansive neighborhood for  $f$ .

Suppose  $f$  has the shadowing property and  $E$  is any neighborhood of  $\Delta_X$ . Then there exists a neighborhood  $D$  of  $\Delta_X$  such that every  $(D, f)$ -pseudo-orbit is  $(E, f)$ -traced by some point  $y$ . Let  $\{\dots, x_{-1}, x_0, x_1, \dots\}$  be a  $(D, f^k)$ -pseudo-orbit. Then

$$\{\dots, x_{-1}, f(x_{-1}), f^2(x_{-1}), \dots, f^{k-1}(x_{-1}), x_0, f(x_0), \dots, f^{k-1}(x_0), x_1, \dots\}$$

is a  $(D, f)$ -pseudo-orbit, and hence there is an  $(E, f)$ -tracing point,  $y$ . It follows that  $y$  is an  $(E, f^k)$ -tracing point for the  $(D, f^k)$ -pseudo-orbit as well. So  $f^k$  has the shadowing property.

Conversely, assume that  $f^k$  has the shadowing property, and let  $E$  be any neighborhood of  $\Delta_X$ . Let  $B$  be a symmetric neighborhood of  $\Delta_X$  such that  $B^k \subset E$ . Then  $E' = \bigcap_{i=0}^{k-1} F^{-i}(B)$  is also a neighborhood of  $\Delta_X$ ; so there exists a neighborhood  $D$  (which we may assume is contained in  $E'$ ) of  $\Delta_X$  such that every  $(D, f^k)$ -pseudo-orbit is  $(E', f^k)$ -traced by some point  $y$ . Let  $C$  be a symmetric neighborhood of  $\Delta_X$  such that  $C^k \subset D$ , and define  $D' = \bigcap_{i=0}^{k-1} F^{-i}(C)$ . Let  $\{\dots, x_{-1}, x_0, x_1, \dots\}$  be a  $(D', f)$ -pseudo-orbit. Then  $\{\dots, x_{-k}, x_0, x_k, x_{2k}, \dots\}$  is a  $(D, f^k)$ -pseudo-orbit; hence there is an  $(E', f^k)$ -tracing point,  $y$ . Then  $y$  is an  $(E, f)$ -tracing point for  $\{\dots, x_{-1}, x_0, x_1, \dots\}$ . So  $f$  has the shadowing property.

(2) Define  $H : X \times X \rightarrow Y \times Y$  to be  $h \times h$ . Then  $N$  is an expansivity neighborhood for  $f$  if and only if  $H(N)$  is an expansivity neighborhood for  $h \circ f \circ h^{-1}$ . The  $(D, f)$ -pseudo-orbit  $\{\dots, x_{-1}, x_0, x_1, \dots\}$  is  $(E, f)$ -traced by the point  $y$  if and only if the  $(H(D), h \circ f \circ h^{-1})$ -pseudo-orbit

$\{\dots, h(x_{-1}), h(x_0), h(x_1), \dots\}$  is  $(H(E), h \circ f \circ h^{-1})$ -traced by the point  $h(y)$ .  $\square$

**Definition 14.** A homeomorphism  $f : X \rightarrow X$  is *topologically Anosov* if  $(f, X) \in \mathcal{E} \cap \mathcal{S}$ .

**Lemma 15.** *Let the homeomorphism  $f : X \rightarrow X$  be topologically Anosov. If  $E$  is a neighborhood of  $\Delta_X$  such that  $E^2$  is an expansive neighborhood for  $f$ , then there is a neighborhood  $D$  of  $\Delta_X$  such that every  $D$ -pseudo-orbit is  $E$ -traced by exactly one orbit of  $X$ . If the  $D$ -pseudo-orbit is periodic, then so is the orbit that  $E$ -traces it.*

*Proof.* Let  $E$  be a symmetric neighborhood of  $\Delta_X$  such that  $E^2$  is an expansive neighborhood for  $f$ . Because  $f$  has the shadowing property, there exists a neighborhood  $D$  of  $\Delta_X$  such that any  $D$ -pseudo-orbit is  $E$ -traced by some point in  $X$ . We must now show uniqueness. Suppose  $\{x_i : i \in \mathbb{Z}\}$  is a  $D$ -pseudo-orbit and that  $x, y \in X$  both  $E$ -trace it. Then for all  $i$ ,  $(x_i, f^i(x)), (x_i, f^i(y)) \in E$ . So  $(f^i(x), f^i(y)) \in E^2$  for all  $i$ . Since  $E^2$  is an expansive neighborhood for  $f$ ,  $(x, y) \in \Delta_X$ , and hence  $x = y$ .

Now suppose  $\{y_k\}$  is a periodic  $D$ -pseudo-orbit with period  $n$  and let  $x$  be the unique  $E$ -tracing point. Then  $\{y_{k+n}\}$  is a periodic  $D$ -pseudo-orbit and since  $f^n(x)$  is an  $E$ -tracing point, it must be the unique one. But  $\{y_k\} = \{y_{k+n}\}$ , so it must be the case that  $x = f^n(x)$ , and hence  $x$  is periodic.  $\square$

### 3. STABLE AND UNSTABLE SETS AND TOPOLOGICAL STABILITY

We now introduce notions of stable and unstable sets.

**Definition 16.** Let  $x \in X$  and  $B$  be a neighborhood of  $\Delta_X$ .

- (1) The *local stable set* of  $x$  relative to  $B$  is

$$W_B^s(x) = \{y \in X : F^i(x, y) \in B, \forall i \geq 0\},$$

- (2) the *local unstable set* of  $x$  relative to  $B$  is

$$W_B^u(x) = \{y \in X : F^i(x, y) \in B, \forall i \leq 0\},$$

- (3) the *stable set* of  $x$  is

$$W^s(x) = \{y \in X : \forall \text{ neighborhood } B \text{ of } \Delta_X, \exists n \in \mathbb{N} \text{ such that } F^i(x, y) \in B, \forall i \geq n\}, \text{ and}$$

- (4) the *unstable set* of  $x$  is

$$W^u(x) = \{y \in X : \forall \text{ neighborhood } B \text{ of } \Delta_X, \exists n \in \mathbb{N} \text{ such that } F^i(x, y) \in B, \forall i \leq -n\}.$$

A neighborhood  $U \subset X \times X$  of the diagonal  $\Delta_X$  is *wide* if there is a compact set  $\dot{S} \subset X$  such that  $U \cup (\dot{S} \times X) = X \times X$ . In other words,  $U$  is wide if for any  $x$  not in the compact set  $\dot{S}$ , the cross section  $U[x]$  is  $X$ .

**Lemma 17** (Continuity Lemma). *Let  $Y$  be locally compact and Hausdorff. A map  $f : X \rightarrow Y$  is continuous if for any wide neighborhood  $U$  of  $\Delta_Y$ , there exists a neighborhood  $V$  of  $\Delta_X$  such that  $F(V) \subset U$ .*

*Proof.* Let  $x$  be any point in  $X$  and let  $\dot{B} \subset Y$  be any neighborhood of  $f(x)$ ; to prove continuity we will find a neighborhood  $\dot{A}$  of  $x$  such that  $f(\dot{A}) \subset \dot{B}$ . Let  $U$  be any wide neighborhood of  $\Delta_Y$  such that  $U[f(x)] \subset \dot{B}$  (for example,  $U = (\dot{B} \times \dot{B}) \cup ((Y - \text{cl}(\dot{B}')) \times Y)$ , where  $\dot{B}'$  is a neighborhood of  $f(x)$  such that  $\text{Cl}(\dot{B}') \subset \dot{B}$ ). Then, by hypothesis, there exists a neighborhood  $V$  of  $\Delta_X$  such that  $F(V) \subset U$ . So  $F(x, V[x]) = \{f(x)\} \times f(V[x]) \subset \{f(x)\} \times U[f(x)]$ , so  $f(V[x]) \subset U[f(x)] \subset \dot{B}$ . Thus we can take  $\dot{A} = V[x]$ .  $\square$

**Lemma 18.** *Let  $(f, X) \in \mathcal{E}$  and let  $A$  be a proper expansive neighborhood for  $f$ . For each  $N \in \mathbb{N}$ , define  $V_N(A) := \{(x, y) \in X \times X : F^n(x, y) \in A \text{ for all } |n| \leq N\}$ . Then for any wide neighborhood  $U$  of  $\Delta_X$  there exists  $N \in \mathbb{N}$  such that  $V_N(A) \subset U$ . Conversely, for every  $N$  there exists a neighborhood  $U$  of  $\Delta_X$  such that  $U \subset V_N$ .*

*Proof.* Suppose there is a wide neighborhood  $U$  of  $\Delta_X$  such that for each  $N \in \mathbb{N}$  there exists  $(x_N, y_N) \in V_N(A) \cap ((X \times X) - U)$ . Let  $L = \{(x_N, y_N) : N \in \mathbb{N}\}$ . Because  $U$  is a wide neighborhood of  $\Delta$ , there exists a compact set  $\dot{S} \subset X$  such that  $U \cup (\dot{S} \times X) = X \times X$ . Then  $L \subset (X \times X - U) \cap A \subset \dot{S} \times A[\dot{S}]$ , which is compact, so  $\text{cl } L$  has a limit point,  $(p, q)$ . Clearly  $p \neq q$ . On the other hand, choose a subsequence  $(x_{N_k}, y_{N_k})$  in  $L$  converging to  $(p, q)$  as  $k \rightarrow \infty$ . Observe that for any integer  $i$ ,  $F^i(p, q) = \lim_{k \rightarrow \infty} F^i(x_{N_k}, y_{N_k}) \in A$ , since  $F^i(x_{N_k}, y_{N_k}) \in A$  for  $|i| \leq N_k$ . Since  $F^i(p, q) \in A$  for all  $i$  and  $A$  is an expansive neighborhood, we must have that  $p = q$ , which is a contradiction.

For the converse, take  $U = \bigcap_{|n| \leq N} F^{-n}(\text{int } A)$ .  $\square$

**Lemma 19.** *Let  $(f, X) \in \mathcal{E}$ . If  $B$  is a proper expansive neighborhood for  $f$  and  $x$  is a periodic point, then  $W_B^\sigma(x) \subset W^\sigma(x)$ , where  $\sigma = s, u$ .*

*Proof.* We will prove the case  $\sigma = s$ ; the case  $\sigma = u$  is similar. For the sake of contradiction, suppose there is a periodic point  $x \in X$  and a proper expansive neighborhood  $B$  such that  $W_B^s(x) \not\subset W^s(x)$ . Say that  $x$  has period  $p$  and the periodic orbit is  $\{x_0, \dots, x_{p-1}\}$ . Because  $W_B^s(x) \not\subset W^s(x)$ , there is a  $y \in W_B^s(x)$  and a neighborhood  $C$  of



$\Delta_X$  such that  $F^n(x, y) \notin C$  for infinitely many  $n > 0$ . So, there exist infinitely many points of the form  $(x_j, f^{n_k}(y))$  for some  $0 \leq j < p-1$ . Because  $B[x_j]$  is compact, the sequence  $\{(x_j, f^{n_k}(y))\}$  has a limit point  $(x_j, y_0) \in B - \text{int}(C)$  (without loss of generality, assume that the sequence is convergent). Clearly,  $F^n(x_j, y_0) \in B$  for all  $n \geq 0$ . For each  $i > 0$  consider the sequence  $\{(f^{-i}(x_j), f^{n_k-i}(y))\}$ . It has a limit  $(x_{j-i}, y_{-i}) \in \{x_{j-i}\} \times B[x_{j-i}]$  (where the subscript of  $x$  is taken mod  $p$ ). Then  $f(y_{-i}) = y_{1-i}$  for all  $i > 0$ . Thus,  $F^n(x_j, y_0) = (f^n(x_j), y_n) \in B$  for all  $n < 0$ . By expansiveness, this implies that  $(x_j, y_0) \in \Delta_X$ . But  $(x_j, y_0) \notin \text{int}(C)$ . So this is a contradiction.  $\square$

The results in the following proposition are generalizations of the ones for compact spaces ([2, Theorem 4.1.1, Lemma 2.4.1(1)]).

**Proposition 20.** *Let  $f : X \rightarrow X$  be topologically Anosov. Then we can find neighborhoods  $B$  and  $D$  of  $\Delta_X$  and a continuous map  $t : D \rightarrow X$  such that*

- (1)  $W_B^s(x) \cap W_B^u(y)$  contains at most one point for any  $x, y \in X$ ,
- (2)  $W_B^s(x) \cap W_B^u(y) = \{t(x, y)\}$  if  $(x, y) \in D$ ,
- (3)  $W_B^s(x) \cap D[x] = \{y : y = t(x, y), \text{ where } (x, y) \in D\}$ ,
- (4)  $W_B^u(x) \cap D[x] = \{y : y = t(y, x), \text{ where } (x, y) \in D\}$ ,

*Proof.* Let  $A$  be a proper expansive neighborhood for  $f$ . Let  $B$  be a symmetric neighborhood of  $\Delta_X$  such that  $B^3 \subset A$ , and let  $E = B \cap F^{-1}(B)$ , which is also a symmetric neighborhood of  $\Delta_X$ . Since  $f$  has the shadowing property, there exists a neighborhood  $D$  of  $\Delta_X$  such that every  $D$ -pseudo-orbit is  $E$ -traced by some orbit in  $X$ . We claim that  $E^2$  is an expansive neighborhood for  $f$ , and hence by Lemma 15 the tracing point is unique. Let  $(x, y) \in E^2$ . Then there is a  $z \in X$  such that  $(x, z), (z, y) \in E$ . So  $(f(x), f(z)), (f(z), f(y)) \in B$  and hence  $(f(x), f(y)) = F(x, y) \in B^2 \subset B^3 \subset A$ . This implies that  $F(E^2)$  is an expansive neighborhood for  $f$ , and hence so is  $E^2$ .

For each point  $(x, y) \in D$ , define a  $D$ -pseudo-orbit  $\{x_i\}$  in  $X$  by

$$x_i = \begin{cases} f^i(x) & \text{if } i \geq 0 \\ f^i(y) & \text{if } i < 0. \end{cases}$$

Let  $t(x, y)$  denote the unique  $E$ -tracing point. This defines a map  $t : D \rightarrow X$  (we postpone the proof of continuity to the end of the proof).

Let  $(x, y) \in D$ . Then  $F^n(x, t(x, y)) \in E \subset B$  for all  $n \geq 0$ , so  $t(x, y) \in W_B^s(x)$ . Likewise,  $F^n(y, t(x, y)) \in E \subset B$  for all  $n < 0$  and since  $F(E) \subset B$ ,  $F^{-1}(y, t(x, y)) \in E$  implies  $(y, t(x, y)) \in B$ . So

$t(x, y) \in W_B^u$ . In particular,

$$t(x, y) \in W_B^s(x) \cap W_B^u(y).$$

Furthermore, the expansivity of  $f$  does not allow  $W_B^s(x) \cap W_B^u(y)$  to have more than one point. This proves (1) and (2).

We prove (3); the proof of (4) is similar. To prove equality we prove containment in both directions. ( $\supset$ ): Let  $y = t(x, y)$  and  $(x, y) \in D$ . Then  $y$   $E$ -traces  $x$  in forward time. So  $F^n(x, y) \in E \subset B$  for all  $n \geq 0$ . Thus  $y \in W_B^s(x) \cap D[x]$ . ( $\subset$ ): Suppose  $y \in W_B^s(x) \cap D[x]$ . Then,  $F^n(x, y) \in B$  for all  $n \geq 0$ . Clearly  $F^n(y, y) \in B$  for all  $n < 0$ , so  $y$   $B$ -traces the pseudo-orbit  $\{x_i\}$  (defined as above). By definition, the point  $t(x, y)$   $E$ -traces this pseudo-orbit, and because  $E \subset B$ , it also  $B$ -traces it. So, for all  $n \in \mathbb{Z}$ ,  $(f^n(t(x, y), x_n)) \in B$  and  $(f^n(y), x_n) \in B$ , which implies that  $F^n(t(x, y), y) \in B^2 \subset A$ . Thus, by expansiveness,  $y = t(x, y)$ .

We now show that the tracing map  $t$  is continuous. Let  $U$  be a wide neighborhood of  $\Delta_X$ . By Lemma 17 we must find a neighborhood  $V$  of  $\Delta_X$  such that  $T(V) \subset U$ , where  $T = t \times t$ . By Lemma 18, we can find  $N$  such that  $V_N(A) \subset U$ . Define neighborhoods of  $\Delta_X$

$$W_1 = \bigcap_{n=0}^N F^{-n}(B) \text{ and } W_2 = \bigcap_{n=0}^N F^n(B).$$

Let  $g : X^4 \rightarrow X^4$  be the homeomorphism given by

$$g(x, y, x_1, y_1) = (x, x_1, y, y_1),$$

and take

$$V = g^{-1}(W_1 \times W_2) \cap (D \times D).$$

Clearly  $V$  is a neighborhood of  $\Delta_D$ . Let  $(x, y, x_1, y_1) \in V$ . Because  $(x, y), (x_1, y_1) \in D$ ,  $F^n(t(x, y), x), F^n(x_1, t(x_1, y_1)) \in B$  for  $0 \leq n \leq N$ , and because  $(x, x_1) \in W_1$ ,  $F^n(x, x_1) \in B$  for  $0 \leq n \leq N$ . So  $F^n(t(x, y), t(x_1, y_1)) \in B^3 \subset A$  for  $0 \leq n \leq N$ . Similarly, we can show that  $F^n(t(x, y), t(x_1, y_1)) \in A$  for  $-N \leq n \leq 0$ , and hence  $T(V) \subset V_N(A) \subset U$ . Thus  $t$  is continuous.  $\square$

**Definition 21.** A homeomorphism  $f : X \rightarrow X$  is called *topologically stable* if for any neighborhood  $B$  of  $\Delta_X$  there exists a neighborhood  $D$  of  $\Delta_X$  such that for any homeomorphism  $g : X \rightarrow X$  satisfying  $(f(x), g(x)) \in D$ , for all  $x \in X$  there exists a continuous self map  $h$  of  $X$  satisfying  $(h(x), x) \in B$  for all  $x \in X$  and  $f \circ h = h \circ g$ .

We close this section by proving the following extension of a main result in [17].

**Theorem 22.** *If  $X$  is a first countable, locally compact, paracompact, Hausdorff space and the homeomorphism  $f : X \rightarrow X$  is topologically Anosov, then  $f$  is topologically stable.*

*Proof.* Let  $B$  be a neighborhood of  $\Delta_X$ . Let  $A$  be a proper expansive neighborhood for  $f$  and  $E$  be a symmetric neighborhood of  $\Delta_X$  such that  $E^3 \subset A \cap B$ . By Lemma 15 there is a neighborhood  $D$  of  $\Delta_X$  such that every  $D$ -pseudo-orbit is uniquely  $E$ -traced by some point of  $X$ . Now take a homeomorphism  $g : X \rightarrow X$  which is  $D$ -close to  $f$ . Let  $x \in X$ . Because  $(f(g^{n-1}(x)), g^n(x)) \in D$  for all  $n \in \mathbb{Z}$ ,  $\{g^n(x) : n \in \mathbb{Z}\}$  is a  $D$ -pseudo-orbit for  $f$ . By the shadowing property, this defines a map  $h : X \rightarrow X$  sending  $x$  to the unique point  $h(x)$  that  $E$ -traces the pseudo-orbit.

Thus for any  $x \in X$ , we have  $(f^n(h(x)), g^n(x)) \in E$  for  $n \in \mathbb{Z}$ . In particular,  $(h(x), x) \in E \subset E^3 \subset B$  for all  $x \in X$ . Furthermore, substituting  $g(x)$  for  $x$  yields

$$(f^n(h(g(x))), g^n(g(x))) = (f^n((h \circ g)(x)), g^n(g(x))) \in E$$

and

$$(f^{n+1}(h(x)), g^{n+1}(x)) = (f^n((f \circ h)(x)), g^n(g(x))) \in E$$

for all  $n \in \mathbb{Z}$ . This implies that

$$(f^n((f \circ h)(x)), f^n((h \circ g)(x))) \in E^2 \subset A$$

for all  $n \in \mathbb{Z}$ , and so  $(f \circ h)(x) = (h \circ g)(x)$ .

Now we will use Lemma 17 to show that  $h$  is continuous. Let  $U$  be a wide neighborhood of  $\Delta_X$ . By Lemma 18 there exists  $N \in \mathbb{N}$  such that  $V_N(A) \subset U$ . Let  $W = \bigcap_{|k| \leq N} G^k(E)$ , where  $G = g \times g$ . We must show that  $H(W) \subset U$ , where  $H = h \times h$ . Let  $(x, y) \in W$ . If  $|n| \leq N$ , then  $(h(g^n(x)), g^n(x))$ ,  $(g^n(x), g^n(y))$ , and  $(h(g^n(y)), g^n(y))$  are in  $E$ , and hence

$$(f^n(h(x)), f^n(h(y))) = (h(g^n(x)), h(g^n(y))) \in E^3 \subset A.$$

Thus,  $H(x, y) \in V_N(A) \subset U$ . □

*Remark 23.* In fact, we can prove something slightly stronger than the conclusion of the previous theorem. If the neighborhood  $B$  is symmetric and  $B^2$  is an expansive neighborhood for  $f$ , then  $h$  is unique. Suppose  $h'$  is another continuous map satisfying the same properties as  $h$ . Then for all  $n \in \mathbb{Z}$ ,  $(h(g^n(x)), g^n(x))$ ,  $(h'(g^n(x)), g^n(x)) \in B$ , and hence  $(f^n(h(x)), f^n(h'(x))) = (h(g^n(x)), h'(g^n(x))) \in B^2$ . So,  $h(x) = h'(x)$ .

*Remark 24.* If the homeomorphism  $g$  in the above theorem is also expansive with  $B^2$  an expansive neighborhood and  $B$  is symmetric,

then  $h$  is injective. Indeed, suppose  $h(x) = h(y)$  for some  $x, y \in X$ . Then for any  $n \in \mathbb{Z}$ ,

$$h(g^n(x)) = f^n(h(x)) = f^n(h(y)) = h(g^n(y)).$$

In particular,  $(h(g^n(x)), g^n(x)), (h(g^n(y)), g^n(y)) = (h(g^n(x)), g^n(y)) \in B$ , so  $(g^n(x), g^n(y)) \in B^2$  for all  $n \in \mathbb{Z}$ . Thus  $x = y$ .

#### 4. CHAIN RECURRENCE AND DECOMPOSITION THEOREM

Chain recurrence is an important notion in dynamical systems defined on metric spaces. In this section we introduce a topological generalization of chain recurrence. We then prove our main theorem, Theorem 3.

**Definition 25.** Let  $x, y \in X$ . If there is an  $A$ -chain from  $x$  to  $y$  and one from  $y$  to  $x$  for every neighborhood  $A$  of  $\Delta_X$ , then we write  $x \sim y$ . The set  $\text{CR}(f) = \{x \in X : x \sim x\}$  is called the *chain recurrent set* for  $f$ . The relation  $\sim$  induces an equivalence relation on  $\text{CR}(f)$ ; the equivalence classes are called *chain components* of  $f$ .

**Definition 26.** The *non-wandering set* of  $f$  is  $\Omega(f) = \{x \in X : \text{for any open neighborhood } \dot{G} \text{ of } x, f^n(\dot{G}) \cap \dot{G} \neq \emptyset \text{ for some } n > 0\}$ .

**Proposition 27.** *If  $(f, X) \in \mathcal{S}$ , then  $\Omega(f) = \text{CR}(f)$ .*

*Proof.* It is clear from the definitions that  $\Omega(f) \subset \text{CR}(f)$ , so we must show the opposite inclusion. Let  $x \in \text{CR}(f)$  and  $\dot{U}$  be an open neighborhood of  $x$ . Let  $E$  be a neighborhood of  $\Delta_X$  such that  $E[x] \subset \dot{U}$ . By the shadowing property there exists a neighborhood  $D$  of  $\Delta_X$  such that every  $D$ -pseudo-orbit is  $E$ -traced by some point. Since  $x$  is chain recurrent, there exists a  $D$ -chain  $\{x_0 = x, x_1, \dots, x_n = x\}$ . We can extend this  $D$ -chain (in any way we want) to a full  $D$ -pseudo-orbit. This pseudo-orbit is  $E$ -traced by a point  $y \in X$ . This means that  $y, f^n(y) \in E[x] \subset \dot{U}$ , and so  $x \in \Omega(f)$ .  $\square$

We have the following generalization of a result proved by Yang ([18, Lemma 1(2)]); Yang uses the metric definitions of expansivity, shadowing, and chain recurrence.

**Proposition 28.** *If the homeomorphism  $f : X \rightarrow X$  is topologically Anosov, then  $\text{Per}(f)$  is dense in  $\text{CR}(f)$ .*

*Proof.* Let  $x \in \text{CR}(f)$  and  $\dot{U}$  be an open set containing  $x$ . Let  $E$  be a neighborhood of  $\Delta_X$  such that  $E[x] \subset \dot{U}$  and  $E^2$  is an expansive neighborhood for  $f$ . It suffices to show that there is a periodic point in  $E[x]$ .

By Lemma 15, we can find a symmetric neighborhood  $D$  of  $\Delta_X$  such that every periodic  $D$ -pseudo-orbit is  $E$ -traced by a periodic point. Let  $V = D \cap F^{-1}(D)$ . By Proposition 27,  $\text{CR}(f) = \Omega(f)$ , so  $x$  is non-wandering. Thus, there is a  $y \in V[x]$  and an  $n \in \mathbb{N}$  such that  $f^n(y) \in V[x]$ . Then  $(x, y), (x, f^n(y)) \in V$  and hence  $(f(x), f(y)), (f^n(y), x) \in D$ . This enables us to define a periodic  $D$ -pseudo-orbit  $\{\dots, x_0, x_1, \dots, x_{n-1}, x_0, x_1, \dots\}$  in which  $x_0 = x, x_1 = f(y), \dots, x_{n-1} = f^{n-1}(y)$ . By Lemma 15 there is a periodic point  $p$  that  $E$ -traces this pseudo-orbit. So  $p \in E[x]$ .  $\square$

**Proposition 29.** *Let  $X$  be a first countable, Tychonoff space,  $f : X \rightarrow X$  be a homeomorphism, and  $R$  be a chain component of  $f$ . Then,*

- (1)  $\text{CR}(f)$  is closed in  $X$ ,
- (2)  $R$  is closed in  $X$ , and
- (3)  $R$  and  $\text{CR}(f)$  are  $f$ -invariant.

*Proof.* First we will prove that the set

$$S = \{(x, y) \in X \times X : \forall \text{ neighborhood } A \text{ of } \Delta_X, \exists A\text{-chain from } x \text{ to } y\}$$

is closed in  $X \times X$ . Let  $\{(x_i, y_i)\}_{i=0}^{\infty} \subset S$  be a sequence converging to  $(x, y) \in X \times X$ . Let  $A$  be a neighborhood of  $\Delta_X$ ; we must prove that there is an  $A$ -chain from  $x$  to  $y$ .

Choose symmetric neighborhoods  $W$  and  $U$  of  $\Delta_X$  and  $N > 0$  such that  $W^3 \subset A, U \subset W \cap f^{-1}(W)$ , and  $(x_N, x), (y_N, y) \in U$ . Since  $(x_N, y_N) \in S$  there exists a  $W$ -chain  $\{x_N = z_0, z_1, \dots, z_m = y_N\}$ . Suppose  $m = 1$ . Then  $(f(x_N), y_N) \in W$ . Moreover, because  $(x, x_N) \in U \subset f^{-1}(W)$ ,  $(f(x), f(x_N)) \in W$ . Also,  $(y_N, y) \in U \subset W$ . So  $(f(x), y) \in W^3 \subset A$ , and hence  $\{x, y\}$  is an  $A$ -chain. Likewise, if  $m > 1$ , we can show that  $(f(x), z_1), (f(z_{m-1}), y) \in W^2 \subset A$ , and therefore  $\{x, z_1, \dots, z_{m-1}, y\}$  is an  $A$ -chain. Hence  $(x, y) \in S$ .

Now, let  $S^T = \{(x, y) \in X \times X : (y, x) \in S\}$  and  $\tilde{S} = S \cap S^T$ . Then  $\tilde{S} = \{(x, y) \in X \times X : x \sim y\}$ . Since  $S$  and  $S^T$  are closed sets, so is  $\tilde{S}$ .

(1) Note that  $\text{CR}(f) = \pi_1(\Delta_X \cap \tilde{S})$ , where  $\pi_1 : X \times X \rightarrow X$  is the projection onto the first coordinate. Since  $\Delta_X \cap \tilde{S}$  is closed in  $\Delta_X$  and  $\pi_1 : \Delta_X \rightarrow X$  is a homeomorphism,  $\text{CR}(f) = \pi_1(\Delta_X \cap \tilde{S})$  is closed as well.

(2) Given any  $x \in \text{CR}(f)$ ,  $\tilde{S}[x]$  is closed, and  $\tilde{S}[x]$  is the chain component of  $f$  containing  $x$ .

(3) Let  $x \in R$ . We must show that  $f(x) \in R$ . Let  $A$  be a neighborhood of  $\Delta_X$ . Clearly  $\{x, f(x)\}$  is an  $A$ -chain from  $x$  to  $f(x)$ . Thus it remains to find an  $A$ -chain from  $f(x)$  to  $x$ . Choose neighborhoods  $B$  and  $C$  of  $\Delta_X$  such that  $B^2 \subset A$  and  $C \subset B \cap f^{-1}(B)$ . Since  $x$  is chain

recurrent, there is a  $C$ -chain  $\{x = x_0, x_1, \dots, x = x_n\}$  from  $x$  to itself. Then  $\{f(x), x_2, x_3, \dots, x = x_n\}$  is the desired  $A$ -chain.  $\square$

**Proposition 30.** *If the homeomorphism  $f : X \rightarrow X$  is topologically Anosov and  $R$  is a chain component, then  $R$  is both open and closed in  $\text{CR}(f)$ .*

*Proof.* Let  $R$  be a chain component of  $f$ . It follows from Proposition 29 that  $R$  is closed in  $\text{CR}(f)$ , so we must only show that  $R$  is open in  $\text{CR}(f)$ . Let  $D$  be the neighborhood of  $\Delta_X$ , which we may assume is open and symmetric, given by Proposition 20. Because  $D[R]$  is open, it suffices to show that a point  $x \in D[R] \cap \text{CR}(f)$  is in  $R$ . Let  $E$  be a neighborhood of  $\Delta_X$ ; we will find an  $E$ -chain from  $x$  to  $x$  via a point of  $R$ .

Let  $U$  be a neighborhood of  $\Delta_X$  such that  $U^2 \subset E \cap D$ . By Proposition 28, there is a periodic point  $p \in U[x] \subset D[x]$ . Then there exists  $q \in R$  such that  $(p, q), (q, p) \in D$ ; by Proposition 28 we may assume  $q$  is periodic. By Proposition 20(2) and Lemma 19, there exist  $z_1, z_2 \in X$  such that  $z_1 \in W^u(p) \cap W^s(q)$  and  $z_2 \in W^s(p) \cap W^u(q)$ . So, there exist  $L, M, N, K \in \mathbb{N}$  such that  $(f(p), f^{-L}(z_1)), (q, f^M(z_1)), (f(q), f^{-N}(z_2)), (p, f^K(z_2)) \in U$ . Then

$$\{p, f^{-L}(z_1), \dots, f^{M-1}(z_1), q, f^{-N}(z_2), \dots, f^{K-1}(z_2), p\}.$$

is a  $U$ -chain, and hence

$$\{x, f^{-L}(z_1), \dots, f^{M-1}(z_1), q, f^{-N}(z_2), \dots, f^{K-1}(z_2), x\}.$$

is an  $E$ -chain.  $\square$

**Definition 31.** A map  $f : X \rightarrow X$  is *topologically transitive* if for any pair of non-empty open sets  $\dot{G}, \dot{H} \subset X$ , there exists  $n \in \mathbb{N}$  such that  $f^n(\dot{G}) \cap \dot{H} \neq \emptyset$ .

We now show that a map that is expansive and has the shadowing property is topologically transitive on each chain component. This is a generalization of a result of Yang's ([18, Lemma 3]) in which he uses the metric definitions of expansivity, shadowing, and chain recurrence.

**Proposition 32.** *If the homeomorphism  $f : X \rightarrow X$  is topologically Anosov and  $R$  is a chain component, then  $f|_R : R \rightarrow R$  is topologically transitive.*

*Proof.* Let  $\dot{G}$  and  $\dot{H}$  be nonempty open sets in  $R$ . Let  $x \in \dot{G}$  and  $y \in \dot{H}$ . Let  $E$  be a neighborhood of  $\Delta_X$  such that  $x \in E[x] \cap R \subset \dot{G}$ ,  $y \in E[y] \cap R \subset \dot{H}$ ,  $E[x] \cap \text{CR}(f) \subset R$ , and  $E^2$  is an expansive neighborhood for  $f$ . By Lemma 15, there exists a neighborhood  $D$  of  $\Delta_X$  (we may

also assume  $D \subset E$ ) such that every periodic  $D$ -pseudo-orbit is  $E$ -traced by a periodic orbit. Since  $x, y \in R$ , there is a  $D$ -chain from  $x$  to itself through  $y$ ,  $\{x_0 = x, \dots, x_n = y, \dots, x_m = x\}$  in  $X$  which we can extend to a periodic  $D$ -pseudo-orbit,  $\{\dots, x_{m-1}, x_m = x_0, \dots, x_m = x_0, x_1, \dots\}$ . By Lemma 15 there is a periodic point  $p \in X$  that  $E$ -traces this pseudo-orbit. In fact,  $p \in E[x] \cap \text{CR}(f) \subset E[x] \cap R \subset \dot{G}$  and  $f^n(p) \in E[y] \cap R \subset \dot{H}$ . So  $f^n(p) \in f^n(\dot{G}) \cap \dot{H} \neq \emptyset$ .  $\square$

Finally, we have all the ingredients to prove our main theorem.

*Proof of Theorem 3.* Proposition 27 tells us that  $\Omega(f) = \text{CR}(f)$ . We know that  $\text{CR}(f)$  has a natural decomposition into chain components,  $\{R_\lambda\}$ , which, by Propositions 29(3) and 30, are open and closed in  $\Omega(f)$  and are  $f$ -invariant. By Proposition 32,  $f$  is topologically transitive on each such component. Finally, notice that if  $X$  is compact, then so is  $\Omega(f)$ , and because  $\{R_\lambda\}$  is a open cover of  $\Omega(f)$  by disjoint sets, it must be finite.  $\square$

## 5. RELATIONSHIPS TO METRIC DEFINITIONS

In this section we discuss the relationships between our topological definitions and the usual metric definitions for expansiveness, shadowing, and chain recurrence. First, we give another generalization of chain recurrence, due to Hurley.

**Definition 33** ([8]). Let  $(X, d)$  be a metric space and  $f : X \rightarrow X$  be a homeomorphism. Let  $\mathcal{P}$  denote the set of continuous functions from  $X$  to  $(0, \infty)$ . For  $\delta \in \mathcal{P}$ , a  $\delta$ -chain is a sequence  $\{x_0, x_1, \dots, x_n\}$  ( $n \geq 1$ ) such that  $d(f(x_{i-1}), x_i) < \delta(f(x_{i-1}))$  for  $i = 1, \dots, n$ . A point  $x$  is *strongly chain-recurrent* if for any  $\delta \in \mathcal{P}$ , there is a  $\delta$ -chain from  $x$  to itself.

In the case that we have a metric space, the various properties are related in the following way.

**Proposition 34.** *Let  $(X, d)$  be a metric space and  $f : X \rightarrow X$  be a homeomorphism.*

- (1) *Metric expansivity implies topological expansivity, but not vice versa.*
- (2) *Metric shadowing does not imply topological shadowing, nor does topological shadowing imply metric shadowing.*
- (3) *Strong chain recurrence and topological chain recurrence are equivalent. They imply metric chain recurrence, but not vice versa.*

*If  $X$  is compact, then metric and topological expansivity are equivalent, metric and topological shadowing are equivalent, and metric, topological, and strong chain recurrence are equivalent.*

*Proof.* (1) If  $e$  is an expansive constant, then  $U_e = d^{-1}[0, e]$  is an expansive neighborhood; thus metric expansivity implies topological expansivity. It is not difficult to see that the homeomorphism in Example 6 is topologically expansive. Thus, the opposite implication does not hold.

(2) The first half follows from Example 7. The given homeomorphism has the shadowing property with respect to one metric, but it does not have the topological shadowing property (this is easiest to see with the second metric).

Let  $f$  be the identity map on the space  $X = \{x_n\}_{n=1}^\infty$ , where  $x_n = \sum_{i=1}^n (1/i)$ , given the metric inherited from  $\mathbb{R}$ . Since  $X$  has the discrete topology, we can pick a neighborhood  $D$  of  $\Delta_X$  such that  $D[x] = \{x\}$  for all  $x$ . Then a  $D$ -pseudo-orbit is an actual  $f$ -orbit (that is, a fixed point). So the topological shadowing property is trivially satisfied. On the other hand, given  $\delta > 0$  and  $N > 1/\delta$ ,  $\{\dots, x_N, x_N, x_N, x_{N+1}, x_{N+2}, \dots\}$  is a  $\delta$ -pseudo-orbit, but clearly such a pseudo-orbit is not traced by any orbit. So  $f$  does not have the metric shadowing property.

(3) Topological chain recurrence clearly implies strong chain recurrence. To prove the opposite implication, we must show that for any open neighborhood  $U$  of  $\Delta_X$ , there exists a function  $\delta \in \mathcal{P}$  such that  $B_\delta \subset U$ , where  $B_\delta = \{(x, y) : d(x, y) < \delta(x)\}$ . We may assume that  $U[x] \subsetneq X$  for all  $x$ . Define a function  $h : X \rightarrow (0, \infty)$  by  $h(x) = d(x, X - U[x])$ . Since  $h$  is lower semicontinuous, the function  $\delta(x) = \inf\{h(y) + d(x, y) : y \in X\}$  is continuous (see [19, Theorem 67.2]), and  $0 < \delta(x) < h(x)$  for all  $x$ . Thus  $B_\delta \subset B_h \subset U$ .

It is clear that topological and strong chain recurrence imply metric chain recurrence. Hurley gives an example ([8, Example 1]) showing that the opposite implication does not hold.

Finally, the equivalences in the compact case are clear, as in Remark 11. □

Hurley has shown ([8]) that strong (or topological) chain recurrence is the right notion to extend Conley's fundamental theorem of dynamical systems to noncompact metric spaces. The second example from (2) is expansive but not metric expansive, and it has the shadowing property but not the metric shadowing property; thus Theorem 3 applies, but a version using the usual metric definitions would not. These examples, together with Proposition 13, suggest that the topological definitions



are a useful way to study the dynamical structure of a homeomorphism on a noncompact space, even if the space is metric and the usual metric definitions could be applied.

## 6. ACKNOWLEDGEMENTS

We are grateful to the referee for helpful corrections and suggestions. This work was supported by BK 21 Maths vision 2013 Project of Korea. The first author would like to thank Professor Keonhee Lee for his warm hospitality and useful discussions during his visit to Chungnam National University, Daejeon in Korea, spring semester 2007. The third and fourth authors would like to thank Professor Tarun Das for the invitation to M. S. University of Baroda for the 2010 ICM satellite conference on Various Aspects of Dynamical Systems and for his invitation to join this research project.

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THE M. S. UNIVERSITY OF BARODA, VADODARA, GUJARAT, INDIA  
*E-mail address:* tarukd@gmail.com

CHUNGNAM NATIONAL UNIVERSITY, DAEJEON, 305-764, KOREA  
*E-mail address:* khlee@cnu.ac.kr

DICKINSON COLLEGE, CARLISLE, PA 17013, USA  
*E-mail address:* richesod@dickinson.edu

AGNES SCOTT COLLEGE, DECATUR, GA 30030, USA  
*E-mail address:* jwiseman@agnesscott.edu