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THE SQUARE OF A MAP, SYMBOLIC DYNAMICS AND THE CONLEY INDEX

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ABSTRACT. We consider a map f from a locally compact metric space to itself, and use the discrete Conley index to study the difference between the local dynamics of f and f^2. In particular, we present a method, based on work by Mischaikow, Szymczak, et al., for detecting positive entropy symbolic dynamics by measuring the difference between Conley indices for f and f^2.

1. Introduction. Let $f : X \to X$ be a continuous map of a locally compact metric space and N a compact subset of X. Any point that stays in N under all forward and backward iterates of f certainly does so for f^2 as well, but the converse is not true; thus, the maximal invariant set in N under f^2 contains the corresponding set under f, see Section 2 for exact definitions. In this paper we use the discrete Conley index to study the extent to which the two sets differ.

In particular, we present a method, based on work by Mischaikow, Szymczak, et al. [**2, 16**], for detecting symbolic dynamics by measuring the difference between Conley indices for f and f^2. We see that the nonnilpotence of certain products of the induced maps on homology corresponds to the existence of positive entropy renewal systems. A consequence is that if an invariant set satisfies certain decomposability assumptions and a homology map on the Conley index for f has a nonzero eigenvalue whose square is not an eigenvalue for the corresponding map for f^2, then f has positive topological entropy.

Sections 2 and 3 contain background information, Section 2 on the Conley index and Section 3 on renewal systems. In Section 4 we discuss some basic results on the differences between the local dynamics for f and f^2. Finally, in Section 5 we discuss the method for detecting symbolic dynamics.

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2. The discrete Conley index. The discrete Conley index is a powerful topological tool for studying isolated invariant sets of a given map f. Roughly speaking, it assigns to each such set a pointed space P and a base-point preserving map f_P, which is defined up to an equivalence relation. By studying the simpler map f_P we can draw conclusions about the original map f.

The Conley index was originally developed for flows, see [**3, 15**], and was later extended to the discrete-time case [**4, 11, 13, 17**]. See [**7**] for a good introduction to the Conley index and its history.

An important feature of the discrete Conley index is that, under suitable hypotheses, if two maps f and g are C^0-close to each other, then they have the same index. Thus we can use it to obtain rigorous results from numerical approximations of, for example, Poincaré sections of flows arising from ODE's. (See [**8–10, 12, 22**].)

Our discussion of the discrete Conley index is based on that in [**4**], where one can find more details and proofs of the theorems below.

Let U be an open subset of a locally compact metric space X.

Definition 1. For any set $N \subset U$ and any continuous map $f : U \to X$ we define $\mathrm{Inv}\,(N, f)$, the *maximal f-invariant subset*, to be the set of $x \in N$ such that there exists an f-orbit $\{x_n\}_{n\in\mathbf{Z}} \subset N$ with $x_0 = x$ and $f(x_n) = x_{n+1}$ for all n. A compact set N is called an *f-isolating neighborhood* if $\mathrm{Inv}\,(N, f) \subset \mathrm{Int}\, N$. A set S is called an *isolated f-invariant set* if there exists an f-isolating neighborhood N with $S = \mathrm{Inv}\,(N, f)$. If N is an f-isolating neighborhood, we define the *f-exit set* of N to be

$$N^-(f) := \{x \in N : f(x) \notin \mathrm{Int}\, N\}.$$

A compact set N is an *f-isolating block* if $f^{-1}(N) \cap N \cap f(N) \subset \mathrm{Int}\, N$. Every neighborhood of an isolated f-invariant set S contains an f-isolating block N with $\mathrm{Inv}\,(N, f) = S$.

Definition 2. Let S be an isolated f-invariant set and suppose $L \subset K$ is a compact pair contained in the interior of the domain of f. The pair (K, L) is called an *f-filtration pair* for S provided K and L are each the closures of their interiors and

(1) $\mathrm{cl}\,(K\backslash L)$ is an f-isolating neighborhood of S,

(2) L is a neighborhood of $K^-(f)$ in K, and

(3) $f(L) \cap \mathrm{cl}\,(K\backslash L) = \varnothing$.

Theorem 3. *Let S be an isolated f-invariant set. For every neighborhood V of S, there exists an f-filtration pair (K, L) for S with $L \subset K \subset V$. Moreover there is a neighborhood of f in the C^0 topology such that, for any $\tilde{f}$ in this neighborhood, $\widetilde{S} = \mathrm{Inv}\,(K \setminus L, \tilde{f})$ is an isolated $\tilde{f}$-invariant set and (K, L) is an $\tilde{f}$-filtration pair for $\widetilde{S}$.*

Theorem 4. *Let $P = (K, L)$ be an f-filtration pair for f, and let K_L denote the quotient space K/L where the collapsed set L is denoted $[L]$ and is taken as the base-point. Then f induces a continuous base-point preserving map $f_P : K_L \to K_L$ with the property $[L] \subset \mathrm{Int}\, {f_P}^{-1}([L])$.*

Given a point $x \in K$, we denote by $[x]$ its image in K_L.

Remark 5. Observe that we can identify the set $\mathrm{Inv}\,(K_L \setminus \{[L]\}, f_P)$ with $S = \mathrm{Inv}\,(\mathrm{cl}\,(K \setminus L), f)$.

Theorem 3 tells us that we can find an f-filtration pair for any isolated f-invariant set S. Our choice of filtration pairs is not unique, even up to homotopy equivalence. Any two f-filtration pairs for S will, however, be shift equivalent, as we now discuss.

Suppose $\mathcal{K}$ is a category. Let X, X' be objects in $\mathcal{K}$ and $f : X \to X$, $g : X' \to X'$ be endomorphisms. We say that (X, f) and (X', g) are *shift equivalent* [**19**], or write $f \sim_s g$, if there exist $m \in \mathbf{Z}^+$, $r : X \to X'$ and $s : X' \to X$ such that the diagrams

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "r"'] & X \arrow[d, "r"'] \\ X' \arrow[r, "g"] & X' \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "s"'] & X' \arrow[d, "s"'] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

commute and $r \circ s = g^m$ and $s \circ r = f^m$. The integer m is called the *lag*.

Again, see [**4**] for proofs of the theorems in this section.

Theorem 6. *Suppose $P = (K, L)$ and $P' = (K', L')$ are f-filtration pairs for S. Then the induced maps, f_P and $f_{P'}$, on the corresponding pointed spaces, are shift equivalent.*

Let S be an isolated f-invariant set, and consider the homotopy class of base-point preserving maps on K_L with f_P as a representative. We let $h_P(S)$ denote this collection, and $[\![f_P]\!]$ the shift equivalence class of $h_P(S)$. We may now make the following definition.

Definition 7. Let S be an isolated f-invariant set for a continuous map f. Then define the *discrete homotopy Conley index* of S, $\mathrm{Con}(S)$, to be $[\![f_P]\!]$, the shift equivalence class of $h_P(S)$, where $P = (K, L)$ is an f-filtration pair for S. We apply the singular homology functor (for the purposes of this paper, we will consider only real coefficients, i.e., in what follows $H_*(-) = H_*(-; \mathbf{R})$) to obtain the *homological Conley index*, $\mathrm{Con}_*(S)$. In other words, $\mathrm{Con}_*(S)$ is the shift equivalence class of $(f_P)_* : H_*(K_L, [L]) \to H_*(K_L, [L])$ where $P = (K, L)$ is an f-filtration pair for S.

3. Renewal systems. Let $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_k) \in \{0, 1\}^{k+1}$ and $\beta = (\beta_0, \beta_1, \dots, \beta_l) \in \{0, 1\}^{l+1}$ be two finite words, and define $\Gamma(\alpha, \beta)$, the *renewal system* generated by α and β, to be the subshift of (Σ_2^+, σ) (the full one-sided shift on the symbols 0 and 1) generated by all infinite concatenations of the words α and β. For example, $\Gamma((0), (1))$ is the full shift, and $\Gamma((0), (1, 1))$ is the even shift. (In general, a renewal system (a generalization of subshifts of finite type introduced by Adler) can have more than two generating words, see [**6**, Section 13.1], but we will be concerned primarily with those generated by only two).

We are interested in the entropy of $\Gamma(\alpha, \beta)$. We first make the following definitions.

Definition 8. Define the finite word $\alpha * \beta$ by setting

$$\alpha * \beta := (\alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_l).$$

For n a positive integer, define $n \cdot \alpha$ by setting $1 \cdot \alpha = \alpha$ and

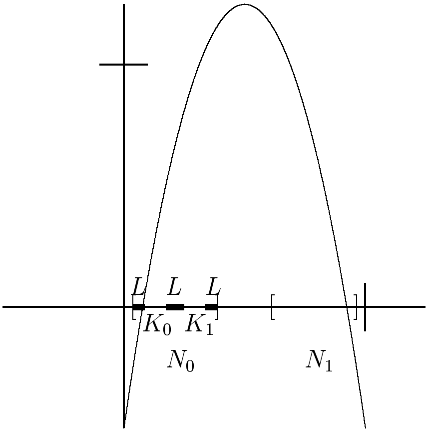

FIGURE 1. Quadratic map.

$n \cdot \alpha = \alpha * ((n-1) \cdot \alpha)$. We say that two finite words ω^1 and ω^2 are *independent* if there do not exist a word α and positive integers n_1 and n_2 such that $\omega^1 = n_1 \cdot \alpha$ and $\omega^2 = n_2 \cdot \alpha$.

It can be difficult in general to compute the entropy of a subshift, as it involves finding the roots of the characteristic polynomial of a (possibly large) related matrix. For a renewal system, however, it is much easier. (Thanks to Michał Misiurewicz for bringing the following to my attention.)

Theorem 9. *Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_0, \beta_1, \ldots, \beta_l)$ be two independent words. The topological entropy of $\Gamma(\alpha, \beta)$ is equal to the log of the absolute value of the largest root of the polynomial $X^{l+k+1} - X^l - X^k$.*

Proof. This follows from the fact that the entropy is equal to the log of the spectral radius of a (suitably chosen) adjacency matrix, see [**6**, Chapter 4], and from the very useful formula for the characteristic polynomial of such a matrix given in [**1**, Theorem 4.4.14]. □

If we don't want to go to the trouble of computing the entropy exactly, we can get an easy estimate by noting that, if α and β are independent, then the Nth power of $\Gamma(\alpha, \beta)$ (where N is the least common multiple of k and l) contains the full shift, so the entropy is at least $(\log 2)/N$. (Of course, if α and β are not independent, then the entropy is zero.)

4. Invariant sets under f and f^2. We are interested generally in the differences between the dynamics of f and f^2 on invariant sets. We begin with the following observations.

Proposition 10. *Let $f : X \to X$ be a continuous map of a locally compact metric space.*

(1) *Any f^2-isolating neighborhood is an f-isolating neighborhood.*

(2) *Any isolated f-invariant set is also an isolated f^2-invariant set.*

(3) *If S^2 is an isolated f^2-invariant set, then $S^2 \cup f(S^2)$ is f-invariant, but not necessarily isolated under f.*

(4) *If f is a homeomorphism, then for any set N, $\operatorname{Inv}(f(N), f^2) = f(\operatorname{Inv}(N, f^2))$.*

(5) *Let N be an attracting neighborhood for f^2, i.e., $f^2(N) \subset \operatorname{Int} N$, with $S^2 = \operatorname{Inv}(N, f^2)$, and let M be a small compact neighborhood of $f(N)$. Then $N \cup M$ is an attracting neighborhood for f, and $\operatorname{Inv}(N \cup M, f) = S^2 \cup f(S^2)$.*

Proof. (1) Let N be an f^2-isolating neighborhood. Since $\operatorname{Inv}(N, f) \subset \operatorname{Inv}(N, f^2)$, and $\operatorname{Inv}(N, f^2) \subset \operatorname{Int} N$, N is an f-isolating neighborhood.

(2) Let N be an f-isolating neighborhood for S. It is easy to verify that $N \cap f^{-1}(N)$ is an f^2-isolating neighborhood for S.

(3) It is clear that $S^2 \cup f(S^2)$ is f-invariant. To see that it is not necessarily f-isolated, let $f : \mathbf{R} \to \mathbf{R}$ be the quadratic map whose graph is pictured in Figure 1, with the intervals N_0 and N_1 as marked (we can ignore the subsets K_0, K_1, and L for now). It is well known, see [**14**], for example, that $\operatorname{Inv}(N_0 \cup N_1, f)$ is a Cantor set and f restricted to $\operatorname{Inv}(N_0 \cup N_1, f)$ is topologically conjugate to (Σ_2^+, σ), the full one-sided shift on the symbols 0 and 1; let $h : \Sigma_2^+ \to \operatorname{Inv}(N_0 \cup N_1, f)$ be the conjugacy homeomorphism. Then $S = \operatorname{Inv}(N_0, f)$ is the (fixed)

singleton set $h(\{0,0,0,\dots\})$ and $S^2 = \operatorname{Inv}(N_0, f^2)$ is the set

$$S^2 = h(\{(0,\alpha_1,0,\alpha_2,0,\alpha_3,\dots) : \alpha_i \in \{0,1\}\ \forall\, i\}).$$

Thus the set $S^2 \cup f(S^2)$ is

$$\begin{aligned} S^2 \cup f(S^2) = h(&\{(0,\alpha_1,0,\alpha_2,0,\alpha_3,\dots) : \alpha_i \in \{0,1\}\ \forall i\} \\ &\cup \{(\beta_1,0,\beta_2,0,\beta_3,0,\dots) : \beta_i \in \{0,1\}\ \forall i\}), \end{aligned}$$

which is not isolated under f. (For any positive ε, there exists a positive integer n such that the point

$$h((\overbrace{0,0,\dots,0}^{2n \text{ times}},1,\overbrace{0,0,\dots,0}^{2n \text{ times}},1,0,0,\dots)),$$

which is not in $S^2\cup f(S^2)$, stays within ε of $S^2\cup f(S^2)$ under all iterates of f.)

(4) It is clear that $f(\operatorname{Inv}(N,f^2))$ is contained in $\operatorname{Inv}(f(N),f^2)$. Conversely, let x be an element of $\operatorname{Inv}(f(N),f^2)$. This means that there exists an f^2-orbit $\{x_n\}_{n\in\mathbf{Z}} \subset f(N)$ with $x_0 = x$ and $f^2(x_n) = x_{n+1}$ for all n. Then the f^2-orbit $\{f^{-1}(x_n)\}_{n\in\mathbf{Z}}$ is contained in N, so $f^{-1}(x)$ is in $\operatorname{Inv}(N,f^2)$.

(5) We assume that M is small enough that $f(M) \subset \operatorname{Int} N$. Let x be a point in $N \cup M$. If x is in N, then $f(x)$ is in $\operatorname{Int} M$, by definition of M, and if x is in M, then $f(x)$ is in $\operatorname{Int} N$. Thus $f(N\cup M)$ is in $\operatorname{Int}(N\cup M)$, i.e., $N\cup M$ is an attracting neighborhood for f.

Observe further that M is an attracting neighborhood for f^2. Let x be a point in $\operatorname{Inv}(N\cup M, f)$, so that there is an f-orbit $\{x_n\}_{n\in\mathbf{Z}} \subset N\cup M$ with $x_0 = x$. Of the elements of the set $\{x_{-2j}\}_{j=1}^{\infty}$, either infinitely many are in N, or infinitely many are in M. Assume the former. Since N is an f^2-attracting neighborhood, if x_{2n} is in N, then x_{2m} is in N as well for all $m \ge n$. Thus x_{2n} is in N for every $n \in \mathbf{Z}$, so x is in $\operatorname{Inv}(N,f^2)$.

Similarly, if infinitely many elements of the set $\{x_{-2j}\}_{j=1}^{\infty}$ are in M, then x is in $\operatorname{Inv}(M,f^2) = \operatorname{Inv}(f(N),f^2)$. Since N is an f^2-attractor, we have that $\operatorname{Inv}(f(N),f^2) = f(\operatorname{Inv}(N,f^2)$, so we have shown that $\operatorname{Inv}(N\cup M,f) \subset S^2 \cup f(S^2)$. Since $\operatorname{Inv}(N\cup M,f)$ clearly contains $S^2\cup f(S^2)$, we are done. $\square$

Remark 11. The converse of Proposition 10.1 is not true. Consider the map $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = -x$. Then the interval $N = [1, 2]$ is an f-isolating neighborhood ($\mathrm{Inv}\,(N, f) = \varnothing$), but $\mathrm{Inv}\,(N, f^2) = N$.

The converse of Proposition 10.2 is also false. To see this, consider again the set $S^2 = h(\{(0, \alpha_1, 0, \alpha_2, 0, \alpha_3, \dots) : \alpha_i \in \{0, 1\}\ \forall\, i\})$ from the quadratic map pictured in Figure 1. This set is clearly not invariant under f.

Finally, Proposition 10.4 is not true if f is only a continuous map instead of a homeomorphism. Consider the map $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 0$. Then $\mathrm{Inv}\,([1, 2], f^2) = \varnothing$, but $\mathrm{Inv}\,(f([1, 2]), f^2) = \mathrm{Inv}\,(\{0\}, f^2) = \{0\}$.

We want to use the discrete Conley index to detect differences in the dynamics of f and f^2. The following proposition allows us to make a fairly coarse comparison.

Definition 12. Let S be an isolated f-invariant set. Define $(\mathrm{Con}(f, S))^2$ to be $[\![(f_P)^2]\!]$, where $P = (K, L)$ is an f-filtration pair for S and $f_P : K_L \to K_L$ is the induced map.

Proposition 13. *Let S be an isolated f-invariant set, and thus also an isolated f^2-invariant set. Then* $\mathrm{Con}(S, f^2) = (\mathrm{Con}(S, f))^2$.

Proof. Let $P = (K, L)$ be an f-filtration pair for S and $f_P : K_L \to K_L$ the induced map. Let $B \subset K_L$ be a compact set contained in $\mathrm{Int}\,({f_P}^{-1}([L]))$ and containing the base-point $[L]$ in its interior. Then $\mathrm{cl}\,(K_L \backslash B)$ is an $(f_P)^2$-isolating neighborhood for S (where we are identifying $\mathrm{Inv}\,(K_L \setminus \{[L]\}, f_P)$ with S). Let L' be a compact set containing $(f_P)^{-2}(B)$ in its interior and sufficiently small that $(f_P)^3(L') = [L]$. Then $P' = (K_L, L')$ is an $(f_P)^2$-filtration pair for S. Let $(f^2)_{P'} : K_L/L' \to K_L/L'$ be the induced map, so that $\mathrm{Con}(S, (f_P)^2) = [\![(f^2)_{P'}]\!]$.

First, we note that the action of $(f_P)^2$ on a neighborhood of S in K_L is the same as that of f^2 on a neighborhood of S in X, so $\mathrm{Con}(S, (f_P)^2) = \mathrm{Con}(S, f^2)$. Therefore, to complete the proof we need to show that $(f^2)_{P'}$ is shift equivalent to $(f_P)^2$.

We construct the shift equivalence as follows. Define the map $r : K_L \to K_L/L'$ to be the projection induced by the inclusion of the pair (K, L) into (K, L'). Define the map $s : K_L/L' \to K_L$ by setting

$$s([x]) = \begin{cases} [L] & \text{if } [x] = [L'], \\ (f_P)^4([x]) & \text{if } [x] \in K_L \backslash L'. \end{cases}$$

(This is continuous because $(f_P)^4(L') = [L]$.) Then r and s give a shift equivalence of lag two between $(f_P)^2$ and $(f^2)_{P'}$, completing the proof. $\square$

Corollary 14. *Let $f : X \to X$ be a continuous map of a locally compact metric space, and N an f^2-isolating neighborhood. If*

$$\mathrm{Con}(\mathrm{Inv}\,(N, f^2), f^2) \neq (\mathrm{Con}(\mathrm{Inv}\,(N, f), f))^2,$$

then $\mathrm{Inv}\,(N, f^2)$ *is strictly larger than* $\mathrm{Inv}\,(N, f)$.

5. Symbolic dynamics. In this section we try to measure more precisely how $\mathrm{Inv}\,(N, f^2)$ differs from $\mathrm{Inv}\,(N, f)$, in order to detect interesting symbolic dynamics associated to f. To make this work, we need to assume that S^2 has an f^2-isolating neighborhood N_0 with certain properties. Specifically, we will assume that N_0 is such that there exists a compact set N_1, disjoint from N_0, such that $f(S^2)$ is contained in $N_0 \cup N_1$. (This is equivalent to the condition that the set $f(S^2) \cap \partial N_0$ is empty.) We will get our symbolic dynamics by keeping track of where the iterates of each point in S^2 lie, N_0 or N_1.

Our method will not work on every isolated f^2-invariant set. First of all, note that N_1 could be empty (for example, if N_0 is an attracting neighborhood for f). In that case, we will of course be unable to detect any symbolic dynamics by studying N_0 and N_1. More seriously, S^2 may have no isolating neighborhood N_0 meeting our requirements. The following property characterizes such invariant sets.

Proposition 15. *Let S^2 be an isolated f^2-invariant set. Assume that S^2 has no f^2-isolating neighborhood N_0 with the property that $f(S^2) \cap \partial N_0$ is empty. Then for every sufficiently small $\varepsilon > 0$, there is a point x of $f(S^2)$ with $\mathrm{dist}\,(x, S^2) = \varepsilon$. Furthermore, no f^2-isolating neighborhood for S^2 is an f-isolating block.*

Proof. By hypothesis, for every f^2-isolating neighborhood N_0 for S^2, there is a point of $f(S^2)$ which lies in ∂N_0. Since every sufficiently small neighborhood of S^2 is an f^2-isolating neighborhood, this proves the first assertion. Since any point of S^2 lies in $N_0 \cap f^{-2}(N_0)$, this also shows that $f^{-1}(N_0) \cap N_0 \cap f(N_0)$ is not contained in $\operatorname{Int} N_0$, i.e., that N_0 is not an f-isolating block. □

Let S^2 be an isolated f^2-invariant set. For the rest of this paper we will assume that there does exist an f^2-isolating neighborhood N_0 for S^2 with the property that $f(S^2)$ is disjoint from ∂N_0. Let N_1 be a compact set disjoint from N_0 and containing the set $f(S^2) \cap \operatorname{cl}(X \backslash N_0)$. Then let K be an f^2-isolating block for S^2 contained in N_0 and sufficiently small that $f(K)$ is contained in $N_0 \cup N_1$ (such a K exists by Theorem 3).

Now, if L is a sufficiently small neighborhood of the f^2-exit set of K, then by [**4**, Theorem 3.6] $P = (K, L)$ is an f^2-filtration pair for S^2, and $\operatorname{Con}(S^2, f^2) = [\![(f^2)_P]\!]$, where $(f^2)_P$ is the map induced by f^2 on K_L.

Since N_0 is an f^2-isolating neighborhood, it is *a fortiori* an f-isolating neighborhood. Let $S = \operatorname{Inv}(N_0, f)$, and note that S is contained in $N_0 \cap f^{-1}(N_0)$. We wish to construct a map $(f^2_{00})_P : K_L \to K_L$ with the property that $[\![(f^2_{00})_P]\!] = (\operatorname{Con}(f, S))^2$. (Unfortunately f does not, in general, induce a continuous map on K_L, or this would be easy.). By measuring the difference between $(f^2)_P$ and $(f^2_{00})_P$, we can measure the difference between S^2 and S and perhaps thereby detect symbolic dynamics. To this end, define K_0 to be the set $K \cap f^{-1}(N_0)$ and K_1 to be $K \cap f^{-1}(N_1)$ and observe that K is the disjoint union of K_0 and K_1. Similarly, L is the disjoint union of L_0 and L_1, where $L_i = L \cap f^{-1}(N_i)$ for $i = 0, 1$.

Now the pointed space K_L is the one-point union of the pointed spaces $(K_0)_{L_0}$ and $(K_1)_{L_1}$. Define maps $e_i : (K_i)_{L_i} \to K_L$ and $r_i : K_L \to (K_i)_{L_i}$, $i = 0, 1$, to be inclusion and projection, respectively, and define the map $(f^2_{00})_P : K_L \to K_L$ to be the composition $(f^2)_P \circ i_0 \circ r_0$. That is, $(f^2_{00})_P$ collapses points in $(K_1)_{L_1}$ to the base-point before applying $(f^2)_P$, so that any point not sent to the base-point must lie in $[N_0 \cup f^{-1}(N_0)]$. Since K_1 is the set of points in K that are mapped out of N_0 by f, we expect that $(f^2_{00})_P$ should give information about S and not S^2, which may contain points of K_1.

Similarly, define the map $(f^2_{10})_P : K_L \to K_L$ to be the composition $(f^2)_P \circ i_1 \circ r_1$, and note that $(f^2_{10})_P$ sends any point not in $[N_0 \cup f^{-1}(N_1)]$ to the base-point. Thus $(f^2_{10})_P$ is giving information about points of S^2 that are not in S.

The following proposition will be useful.

Proposition 16. $[\![(f^2_{00})_P]\!] = (\mathrm{Con}(S, f))^2$.

Proof. Lemma 3.1 of [**18**] tells us that the shift equivalence class of $(f^2_{00})_P$ is equal to $\mathrm{Con}(\mathrm{Inv}\,(K_0, f^2), f^2)$. We observe that $\mathrm{Inv}\,(K_0, f^2)$ is equal to S. (Prove by double inclusion: $S = \mathrm{Inv}\,(N_0, f) = \mathrm{Inv}\,(K, f)$ is clearly contained in $\mathrm{Inv}\,(K \cap f^{-1}(N_0), f^2) = \mathrm{Inv}\,(K_0, f^2)$. Conversely, $\mathrm{Inv}\,(K \cap f^{-1}(N_0), f^2)$ is clearly contained in $\mathrm{Inv}\,(K, f)$.) Thus $[\![(f^2_{00})_P]\!] = \mathrm{Con}(\mathrm{Inv}\,(K_0, f^2), f^2) = \mathrm{Con}(S, f^2) = (\mathrm{Con}(S, f))^2$. ◻

We can associate a symbolic dynamical system to the map f restricted to $S^2 \cup f(S^2)$. We define, see [**2**], a continuous map $\Theta : N_0 \cup N_1 \to \{0, 1\}$ by setting

$$\Theta(x) = \begin{cases} 0 & \text{if } x \in N_0, \\ 1 & \text{if } x \in N_1. \end{cases}$$

Recall that (Σ_2^+, σ) is the full one-sided shift on the symbols 0 and 1. We relate the dynamics of f on $S^2 \cup f(S^2)$ to symbolic dynamics via the itinerary map $\rho : S^2 \cup f(S^2) \to \Sigma_2^+$ defined by

$$\rho(x) = (\Theta(x), \Theta(f(x)), \Theta(f^2(x)), \dots).$$

It is clear that ρ is continuous and that $\sigma \circ \rho = \rho \circ f$.

We will use information from the Conley index maps $(f^2_{00})_P$ and $(f^2_{10})_P$ (more specifically, the homology maps that they induce) to detect interesting subshifts of the image shift $\rho(S^2 \cup f(S^2))$.

Assume that, for some positive integer q, $H_q((K_i)_{L_i})$ is nontrivial and finite-dimensional for both $i = 1, 2$. For notational convenience, let A, A_{00} and A_{10} denote the induced homology maps $((f^2)_P)_q : H_q(K_L) \to H_q(K_L)$, $((f^2_{00})_P)_q : H_q(K_L) \to H_q(K_L)$, and $((f^2_{10})_P)_q : H_q(K_L) \to H_q(K_L)$, respectively. Thus the shift equivalence class of A is $\mathrm{Con}_q(S^2, f^2)$ and, by the definitions of $(f^2_{00})_P$ and $(f^2_{10})_P$, $A = A_{00} + A_{10}$.

We will need some more notation. Given a word

$$\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{2k}, \alpha_{2k+1}) \in \{(0,0),(0,1)\}^{k+1}$$

(here we are considering $\{(0,0),(0,1)\}^{k+1}$ as a subset of $\{0,1\}^{2k+2}$), define the map $A_\alpha : H_q(K_L) \to H_q(K_L)$ by setting

$$A_\alpha = A_{\alpha_{2k+1}\alpha_{2k}} \circ A_{\alpha_{2k-1}\alpha_{2k-2}} \circ \cdots \circ A_{\alpha_1\alpha_0},$$

and define $(f^2_\alpha)_P : K_L \to K_L$ similarly.

Proposition 17. *If A_α is nonnilpotent, then there is a point x in S^2 whose itinerary $\rho(x)$ is*

$$(\alpha, \alpha, \alpha, \dots) = (\alpha_0, \alpha_1, \dots, \alpha_{2k}, \alpha_{2k+1}, \alpha_0, \dots, \alpha_{2k+1}, \alpha_0, \dots).$$

Proof. (This proposition is similar to Theorem 4.4 of [**16**] but requires a different proof since we are not assuming that $N_0 \cup N_1$ is an isolating neighborhood for f.)

Assume that there is no such point x in S^2. Then, by compactness, there exists a positive integer M such that

$$\bigcap_{j=0}^{M} \bigcap_{i=0}^{2k+1} f^{-((2k+2)j+i)}(N_{\alpha_i})$$

is empty. But this implies that ${(f^2_\alpha)_P}^M([x])$ is the base-point $[L]$ for every $[x]$ in K_L, which in turn implies that $(A_\alpha)^M$ is the zero map, contradicting the assumption that A_α is nonnilpotent. ◻

Theorem 18. *Let $\alpha \in \{(0,0),(0,1)\}^{k+1}$ and $\beta \in \{(0,0),(0,1)\}^{l+1}$ be two words such that any finite product of A_α and A_β is nonnilpotent. Then $\rho(S^2 \cup f(S^2))$ contains the renewal system $\Gamma(\alpha,\beta)$. Thus, if α and β are independent, then $h_{top}(f) \geq (\log 2)/n$, where n is the least common multiple of $2k+2$ and $2l+2$.*

Proof. Let $\omega = (\omega_0, \omega_1, \dots) \in \{0,1\}^{+\infty}$ be an element of $\Gamma(\alpha,\beta)$. We must show that there exists a point x in $S^2 \cup f(S^2)$ with $\rho(x) = \omega$.

First assume that ω is in $\{(0,0),(0,1)\}^{+\infty}$, so that ω is a concatenation of the words α and β. For every positive integer k, define the finite word ω^k to be $(\omega_0,\dots,\omega_{2k-1})$. By hypothesis, A_{ω^k} is nonnilpotent for all k, so Proposition 17 tells us that there exists an element x_k of S^2 such that $\rho(x_k)=(\omega^k,\omega^k,\omega^k,\dots)$. Any limit point x of the sequence $x_1,x_2,x_3,\dots$ has the property that $\rho(x)=\omega$.

If ω is not in $\{(0,0),(0,1)\}^{+\infty}$, then the word $\omega'=(0,\omega_0,\omega_1,\dots)$ is. Let x' be a point of S^2 with $\rho(x')=\omega'$, and take x to be $f(x')$. □

Thus we have reduced the problem of detecting positive-entropy renewal systems to that of detecting two words with the above property. Methods for determining the existence of such words are discussed in [**2, 21**]; methods for finding them explicitly are discussed in [**20**]. The space of all nonnilpotent words for a given pair of matrices is an example of a cocyclic subshift [**5**].

We need the following result, which is part of [**2**, Proposition 3.4].

Lemma 19 [**2**]. *Let M, M_{00}, and M_{10} be linear self-maps of a finite-dimensional vector space with $M=M_{00}+M_{11}$. If M_{00} has a nonzero eigenvalue that is not an eigenvalue of M, then there exist independent words α and β such that any finite product of M_α and M_β is nonnilpotent.*

Corollary 20. *If $\mathrm{Con}_q(S,f)$ has a nonzero eigenvalue λ such that λ^2 is not an eigenvalue of $\mathrm{Con}_q(S^2,f^2)$, then there exists a positive integer n such that $(\rho(S^2\cup f(S^2)),\sigma^n)$ contains the full two shift Σ_2^+.*

Proof. Proposition 16 tells us that the shift equivalence class of A_{00} is $(\mathrm{Con}_q(S,f))^2$. Since shift equivalent linear maps have the same nonzero spectrum [**2**, Proposition 2.3], the nonzero eigenvalues of A_{00} are exactly the squares of the nonzero eigenvalues of $\mathrm{Con}_q(S,f)$. Therefore, since $A=A_{00}+A_{10}$, the previous lemma tells us that there exist independent words α and β such that any finite product of A_α and A_β is nonnilpotent. The result now follows from Theorem 18. □

Example 21. We again consider the quadratic map $f : \mathbf{R} \to \mathbf{R}$ whose graph is pictured in Figure 1. Let the sets N_0, N_1, K_0, K_1, and L be as marked. Here K_L is homotopy equivalent to the one-point union of two circles, and $H_1(K_L, [L]) = \mathbf{R}^2$. Using the obvious basis, we have that $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, $A_{00} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $A_{10} = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}$. Any finite product of A_{00} and A_{10} is nonnilpotent, so Theorem 18 tells us that $\rho(S^2 \cup f(S^2))$ contains the renewal system generated by $(0,0)$ and $(0,1)$. Thus $(\log 2)/2$ is a lower bound for the topological entropy of f restricted to $S^2 \cup f(S^2)$, which is itself a lower bound for the topological entropy of f.

We can also observe that 1 is an eigenvalue of A_{00}, while A has no nonzero eigenvalues, so Corollary 20 tells us that some power of f restricted to $S^2 \cup f(S^2)$ factors onto the full two-shift. Note that neither Theorem 18 nor Corollary 20 can detect that in fact it is the first power of f restricted to $S^2 \cup f(S^2)$ that factors onto the full two-shift.

6. Conclusion.

The method that we have studied for detecting symbolic dynamics is based on those in [**2**] and [**16**]. The key difference is that they consider only the map f and the case that $N_0 \cup N_1$ is an f-isolating neighborhood, which is a stronger assumption than we need.

Finally, we observe that there are several ways to generalize our results. First of all, there is no mathematical reason that we had to compare f and f^2, rather than f and f^n for some other n. This generalization is straightforward; we restricted our attention here to f^2 for the sole reason that doing so greatly simplified the notation and shortened the exposition. It is possible that we can obtain more information about f by combining the results of the comparisons of f and f^n for a number of different n's, perhaps getting symbolic dynamics on more than two symbols. Another potentially fruitful generalization is to compare the maps f^p and f^q for relatively prime p and q. Again, combining several of these comparisons might yield useful information about f.

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