

STATEMENT ON RESEARCH

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1. INTRODUCTION

Dynamical systems, or chaos theory, is the study of the long-term behavior of systems that evolve over time. We analyze, for example, the movements of the solar system (this is the n -body problem, which is still unsolved), or the state of the atmosphere and the weather, or the population of the state of Georgia, or the expansion and contraction of heart muscle, or the strategic behavior of economic agents. Mathematically speaking, a dynamical system is just a set X of possible states (the *state space*), along with a rule, or function, f which, given the current state, tells you what the next state will be. For example, if we know the positions and velocities of all the bodies in the solar system right now, then Newton's laws of motion tell us where they will be a year from now. What we are learning is that even very simple rules can lead to extremely complicated, or chaotic, behavior. A famous example is the butterfly effect – even a tiny little change, like a butterfly flapping its wings, can create big consequences, like a hurricane the next year. Another is the Mandelbrot set (Fig. 1), a fractal which arises from the rule $f(z) = z^2 + c$.

In the presence of chaotic dynamics, it is impossible to predict exactly how a system will behave. Instead, we try to answer questions such as, What kinds of behavior are possible? Will the system ever return to its current state? How chaotic is the behavior likely to be? (We can't predict the high temperature in Atlanta on August 1, 2013, but we can say that it will probably be at least 60°.) In my field, topological dynamics, we study systems that have a certain structure (namely, the rule f governing the system must be continuous), and use that structure to try

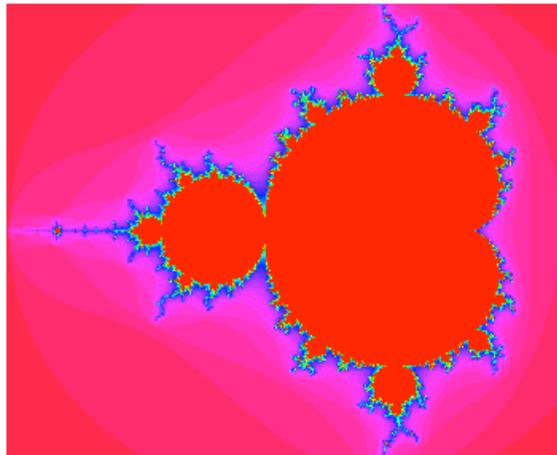


FIGURE 1. The Mandelbrot set. (<http://mathworld.wolfram.com/MandelbrotSet.html>)

to understand the behavior. For example, in [7] David Richeson and I show that certain kinds of dynamical systems must have a *fixed point*, or *equilibrium*, that is, a point that never changes as the system evolves. (A population is at a fixed point when the rate of birth and immigration exactly equals the rate of death and emigration.)

A voting procedure is essentially a rule for combining the opinions of all the individual voters into one opinion for the whole group. The procedure can be very simple (majority vote, for example), but just as with dynamical systems, even a simple voting procedure can lead to complicated and surprising election outcomes. In voting theory, we study what can happen in elections and legislative roll calls, and try to understand the implications. These are important questions in a democracy, and they're also very interesting mathematically.

2. TOPOLOGICAL DYNAMICS

In what follows, let $f : X \rightarrow X$ be a continuous map of a locally compact metric space. A point x in X represents a specific state of the system, and $f(x)$ the next state. The *orbit* of x is $\{x, f(x), f^2(x) = f(f(x)), f^3(x), \dots\}$, the set of all future states of the system starting from state x . I'm interested in the dynamics of f – what are the possible orbits, and how do they behave? We will occasionally assume additional hypotheses (for example, that f is a homeomorphism or that X is compact), but we will not assume any non-topological structure (for example, we will not assume that f is ergodic, or measure preserving, or differentiable).

Since the orbits are often extremely complicated, we sometimes try to study slightly simpler related systems. For example, we can divide the space up into two pieces, labeled with the symbols 1 and 2, and keep track of a point's itinerary. So if a point x begins in piece 1, then moves to 2, and then is in 2 again, its itinerary begins $(1, 2, 2, \dots)$. By examining the set of possible itineraries (the *symbolic dynamics*), we can get information about the original map f .

My dissertation and much of my subsequent work involve the use of the Conley index to detect symbolic dynamics in isolated invariant sets. An isolated invariant set is, roughly, a set which has its own dynamics, independent of the rest of the system. For example, a sealed dome might have its own weather, or the population of mammals on a desert island might grow in isolation from the rest of the world. The discrete Conley index is a functor that, to each isolated invariant set S for a map f , assigns a simpler space P_S and map $f_S : P_S \rightarrow P_S$. The Conley index is constant under small perturbations of the map f . This "continuation property" allows us, if F is a sufficiently good approximation to f , to define a Conley index for F that will be the same as the index for f . Thus any conclusions we can draw about the dynamics of F from its Conley index will also hold for f . In particular, we can use the Conley index to study computer approximations of dynamical systems. (See [3] for an introduction to Conley index theory.)

In [14] and [15], I study the case when an invariant set S for f can be decomposed into disjoint pieces, S_1 and S_2 , so that we can relate the dynamics on S to symbolic dynamics. In particular, we can get an estimate for the topological entropy of f . The topological entropy is a measure of the complexity of the dynamics of f . Positive topological entropy implies chaotic behavior. My research gives an algorithm for detecting positive entropy symbolic dynamics, by examining the algebraic topology of the Conley index and the relationships among the indices for

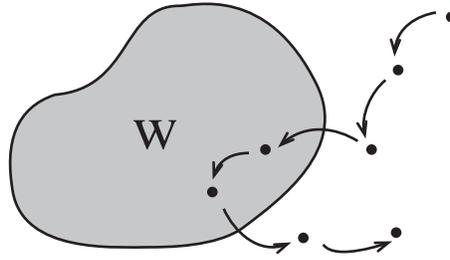


FIGURE 2. A bounded map.

S , S_1 , and S_2 . In [16], I use similar techniques to detect chaos by comparing the local dynamics of f and f^2 .

I have been collaborating with David Richeson of Dickinson College since 2001. We began by studying bounded maps on noncompact spaces. A map is *bounded* if there exists a compact set W such that the forward orbit of every point intersects W (Fig. 2). Clearly, every map on a compact space is bounded. In fact, roughly speaking, bounded maps behave like maps on compact spaces. In [7], we derive some of the properties of bounded maps and use them to prove that every bounded map on \mathbb{R}^n has a fixed point. Using the ideas of boundedness, we considered the Poincaré-Birkhoff theorem, which concerns the existence of fixed points for certain homeomorphisms of the annulus arising from the three-body problem. In [8], we prove some generalizations for bounded homeomorphisms of the open annulus. We also use boundedness to study positively expansive maps, as we now discuss.

An important property of many chaotic systems is that they are expansive, which means that any two distinct orbits must separate in either forward or backward time. Richeson and I have studied systems in which this separation must occur in forward time. A dynamical system is *positively expansive* if no two points can stay close together forever under iteration. More precisely, f is positively expansive if there exists an $\varepsilon > 0$ such that if $d(f^n(x), f^n(y)) < \varepsilon$ for all $n \geq 0$, then $x = y$.

In [10], we give a new, short proof of the fact that there are no positively expansive homeomorphisms on an infinite compact space. We show that the diagonal in the product space $X \times X$ is an attractor for $f^{-1} \times f^{-1}$ and then apply some of our earlier results on bounded dynamical systems. Thus, while there are noninvertible positively expansive maps of infinite compact spaces, positively expansive homeomorphisms are of interest only on noncompact spaces. However, on a noncompact space, the property of being positively expansive depends on the choice of metric - f can be positively expansive with respect to one metric and not with respect to another. To address this, in [11] we define topological generalizations of positive expansiveness and discuss the dynamical structure of homeomorphisms with these properties.

Our most recent work deals with recurrence times. In [5], we study rotations on the circle. For a fixed interval (or intervals), we consider the sequence of times for which the orbit of a given point lies in the interval, and then ask the question, Given only that sequence, to what extent is it possible to recover the interval and the point? In [6], we study recurrence times for ε -chains (or pseudo-orbits) and their dependence on ε , and use them to estimate topological entropy.

I am currently interested in two projects that tie together earlier work. One involves numerical approximations to dynamical systems. By dividing the state

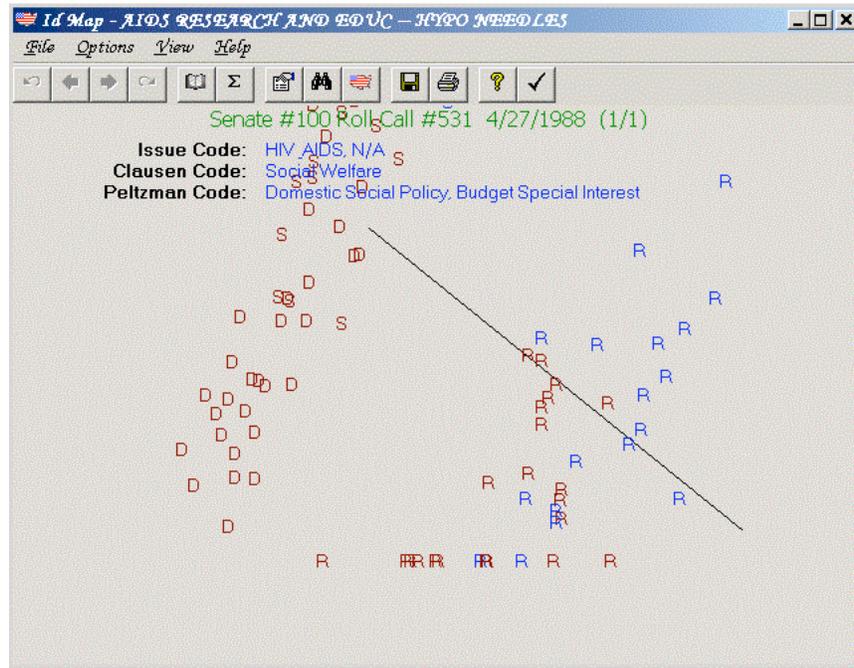


FIGURE 3. Spatial voting (<http://www.princeton.edu/~voteview/>)

space up into a grid, and seeing how the grid elements map to each other, we get a discrete map that approximates the original map. (See [2], for example.) I am interested in applying the Conley index to understand the relationship between the recurrence times for the two maps.

I am also studying topologically hyperbolic systems, that is, systems that are expansive and have the shadowing property. Several different kinds of Conley indices arise in a natural way for such systems, and the relationships among the indices provide information about the dynamics.

3. VOTING THEORY AND UNDERGRADUATE RESEARCH

I began studying voting theory in graduate school, under the direction of Don Saari. In [13], I show that there are no restrictions on the set of possible election outcomes under approval voting when one or more candidates drop out. I returned to voting theory several years later, working with Phil Everson and Rick Valelly (a statistician and a political scientist, respectively, both at Swarthmore) to understand and to explain to political scientists the results on spatial voting in [4]. Spatial voting is a model of legislator behavior in which each legislator has an ideal outcome point in the space of legislative issues; he or she will vote yea or nay on a bill depending on which outcome is closer to his or her ideal point (Fig. 3). Poole and Rosenthal show that much of American congressional voting history can be explained with a spatial model; in [1] we discuss the mathematics behind their results and suggest possible applications. Everson, Valelly, and I are currently working on one such application: a study of the dimensionality of congressional voting. We hope to show, roughly, that the dividing lines for votes on, say, social security and voting rights, lie at an angle to each other, thus providing evidence that legislators' issue space is at least two-dimensional.

One nice feature of voting theory as a research area is that the mathematics involved is largely accessible to undergraduate majors (unlike much of topological dynamics). I have some experience doing research with students. At Swarthmore, I supervised senior theses. Most of these were expository, but a few involved original research. In [12], one senior, Danielle Silverman, and I expanded her result into a published paper. I took a short course on supervising undergraduate research at the Joint Mathematics Meetings in New Orleans in January 2007, and I'm eager to apply what I've learned to work with students here at Agnes Scott.

4. OTHER ACTIVITIES

I have co-organized two special sessions at AMS meetings (one, on contemporary dynamical systems, at the joint meetings in San Antonio in January 2006, and one, on dynamical systems, at the sectional meeting at Davidson, NC, in March 2007). The work involved writing a proposal, inviting speakers, collecting abstracts, scheduling talks, chairing sessions, and general trouble-shooting. It's time-consuming, but rewarding, and I'm open to doing it again. I have also reviewed ten papers for *Mathematical Reviews* and served as a referee for *Topology and its Applications*, *The International Journal of Mathematics and Mathematical Sciences*, and *Mathematics Magazine*, and I plan to continue with both activities.

REFERENCES

1. Phil Everson, Rick Valeyly, and Jim Wiseman, *The Poole-Rosenthal scores: a primer for political scientists*, submitted.
2. W. D. Kalies, K. Mischaikow, and R. C. A. M. VanderVorst, *An algorithmic approach to chain recurrence*, *Found. Comput. Math.* **5** (2005), no. 4, 409–449. MR MR2189545 (2006i:37035)
3. Konstantin Mischaikow and Marian Mrozek, *Conley index*, *Handbook of dynamical systems*, Vol. 2, North-Holland, Amsterdam, 2002, pp. 393–460. MR MR1901060 (2003g:37022)
4. Keith Poole and Howard Rosenthal, *Congress: a political-economic history of roll call voting*, Oxford University Press, 1997.
5. David Richeson, Paul Winkler, and Jim Wiseman, *Itineraries of rigid rotations and diffeomorphisms of the circle*, submitted.
6. David Richeson and Jim Wiseman, *Chain recurrence rates and topological entropy*, submitted.
7. ———, *A fixed point theorem for bounded dynamical systems*, *Illinois J. Math.* **46** (2002), no. 2, 491–495. MR MR1936931 (2003h:37013)
8. ———, *Bounded homeomorphisms of the open annulus*, *New York J. Math.* **9** (2003), 55–68 (electronic). MR MR2016180 (2004j:37079)
9. ———, *Addendum to: "A fixed point theorem for bounded dynamical systems"* [*Illinois J. Math.* **46** (2002), no. 2, 491–495, *Illinois J. Math.* **48** (2004), no. 3, 1079–1080. MR MR2114271 (2005k:37024)
10. ———, *Positively expansive homeomorphisms of compact spaces*, *Int. J. Math. Math. Sci.* (2004), no. 53–56, 2907–2910. MR MR2145368 (2005m:37024)
11. ———, *Positively expansive dynamical systems*, *Topology Appl.* **154** (2007), no. 3, 604–613. MR MR2280904
12. Danielle Silverman and Jim Wiseman, *Noting the difference: musical scales and permutations*, *Amer. Math. Monthly* **113** (2006), no. 7, 648–651. MR MR2252936 (2007c:00003)
13. Jim Wiseman, *Approval voting in subset elections*, *Econom. Theory* **15** (2000), no. 2, 477–483. MR MR1789242
14. ———, *Detection of renewal system factors via the Conley index*, *Trans. Amer. Math. Soc.* **354** (2002), no. 12, 4953–4968 (electronic). MR MR1926844 (2003e:37022)
15. ———, *Symbolic dynamics from signed matrices*, *Discrete Contin. Dyn. Syst.* **11** (2004), no. 2–3, 621–638. MR MR2083435 (2005d:37023)
16. ———, *The square of a map, symbolic dynamics and the Conley index*, *Rocky Mountain J. Math.* **37** (2007), no. 1, 327–342. MR MR2316452